# Three types of ambiguity

Lars Peter Hansen

University of Chicago and NBER

Thomas J. Sargent

New York University and the Hoover Institution [*]

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### Abstract

For each of three types of ambiguity, we compute a robust Ramsey plan and an associated worst-case probability model. *Ex post*, ambiguity of type I implies endogenously distorted *homogeneous* beliefs, while ambiguities of types II and III imply distorted *heterogeneous* beliefs. Martingales characterize alternative probability specifications and clarify distinctions among the three types of ambiguity. We use recursive formulations of Ramsey problems to impose local predictability of commitment multipliers directly. To reduce the dimension of the state in a recursive formulation, we transform the commitment multiplier to accommodate the heterogeneous beliefs that arise with ambiguity of types II and III. Our formulations facilitate comparisons of the consequences of these alternative types of ambiguity.

KEYWORDS: Robustness, ambiguity, martingales, Ramsey plan, commitment, local predictability, heterogeneous beliefs.

# 1 Introduction

Rational expectations models attribute a unique probability model to diverse agents. Gilboa and Schmeidler (1989) express a single person's ambiguity with a *set* of probability models.

A coherent multi-agent setting with ambiguity must impute possibly distinct sets of models to different agents, and also specify each agent's understanding of the sets of models of other agents.[1] This paper studies three ways of doing this for a Ramsey planner.

We analyze three types of ambiguity, called I, II, and III, that a Ramsey planner might have. In all three, the Ramsey planner believes that private agents experience no ambiguity. This distinguishes our models from others that attribute ambiguity to private agents. For example, in what we shall call the type 0 ambiguity analyzed by Karantounias (2012), the planner has no model ambiguity but believes that private agents do.

To illustrate these distinctions, figure 1 depicts four types of ambiguity within a class of models in which a Ramsey planner faces a private sector. The symbols $x$ and $o$ signify distinct probability models over exogenous processes. (The exogenous process is a cost-push shock in the example that we will carry along in this paper). Circles with either $x$'s or $o$ denote boundaries of sets of models. An $x$ denotes a Ramsey planner's model while an $o$ denotes a model of the private sector. In a rational expectations model, there is one model $x$ for the Ramsey planner and the same model $o = x$ for the private sector, so a graph like figure 1 for a rational expectations model would be a single $x$ on top of a single $o$.

The top left panel of figure 1 depicts the type of ambiguity analyzed by Karantounias (2012).[2] To distinguish it from three other types to be studied in this paper, we call this type 0 ambiguity. A type 0 Ramsey planner has a single model $x$ but thinks that private agents have a set of models $o$ contained in an entropy ball that surrounds the planner's model. Karantounias's Ramsey planner takes into account how its actions influence private agents' choice of a worst-case model along the boundary of the set of models depicted by the $o$'s. Part of the challenge for the Ramsey planner is to evaluate the private agent's Euler equation using the private agent's worst-case model drawn from the boundary of the set.[3]

Models of types I, II, and III differ from the type 0 model because in these three models, the Ramsey planner believes that private agents experience *no* model ambiguity. But the planner experiences ambiguity. The three types differ in what the planner is

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[1]Battigalli et al. (2011) analyze self-confirming equilibria in games where players are ambiguity averse.

[2]Orlik and Presno (2012) expand the space of strategies to study problems in which a Ramsey planner cannot commit and in which the private sector and the Ramsey planner both have sets of probability models. They represent history-dependent strategies in terms of pairs of continuation values and also promised marginal utilities of private consumption.

[3]Through its choice of actions that affect the equilibrium allocation, the planner manipulates private agents' worst-case model.
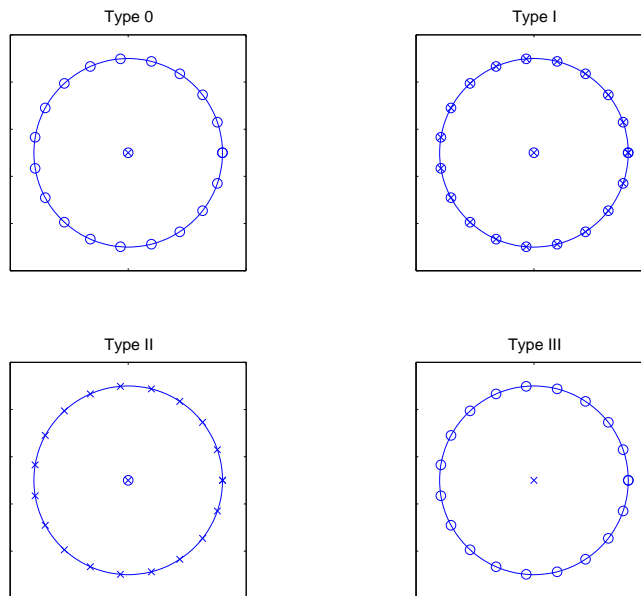
Figure 1: Type 0, top left: Ramsey planner trusts its approximating model (x), knowing private agents (o) don't trust it. Type I, top right: Ramsey planner has set of models (x) centered on an approximating model, while private sector knows a correct model (o) among Ramsey planner's set of models x. Type II, bottom left; Ramsey planner has set of models (x) surrounding its approximating model, which private sector trusts (o). Type III, bottom right: Ramsey planner has single model (x) but private sector has another model in an entropy ball around (x).

ambiguous about. The private sector's response to the Ramsey planner's choices and the private sector's view of the exogenous forcing variables have common structures across all three types of ambiguity. In all three, private agents view the Ramsey planner's history-dependent strategy as a sequence of functions of current and past values of exogenously specified processes. In addition, the private sector has a well specified view of the evolution of these exogenous processes. These two inputs determine the private sector's actions. Although the planner's strategy and the private sector's beliefs differ across our three types of ambiguity, the mapping (i.e., the *reaction function*) from these inputs into private sector responses is identical. We will represent this generalized notion of a reaction function as a sequence of private sector Euler equations. When constructing Ramsey plans under our three types of ambiguity, we will alter how the Ramsey planner views both the evolution of the exogenous processes and the beliefs of the private sector. We will study the consequences of three alternative configurations that reflect differences in what the Ramsey planner is ambiguous about.

The top right panel of figure 1 depicts type I ambiguity. Here the Ramsey planner has a set of models $x$ centered on an approximating model. The Ramsey planner is uncertain about both the evolution of the exogenous processes and how the private sector views these processes. The planner presumes that private sector uses a probability specification that actually governs the exogenous processes. To cope with its ambiguity, the Ramsey planner's alter ego chooses a model on the circle, while evaluating private sector Euler equations using that model.

The bottom left panel of figure 1 depicts type II ambiguity. In the spirit of Hansen and Sargent (2008, ch. 16), the Ramsey planner has a set of models surrounding an approximating model $x$ that the private sector $o$ completely trusts; so the private sector's set of models is a singleton on top of the Ramsey planner's approximating model. The Ramsey planner's probability-minimizing alter ego chooses model on the circle, while evaluating private the agent's Euler equations using the approximating model $o$.

The bottom right panel of figure 1 depicts type III ambiguity. Following Woodford (2010), the Ramsey planner has a single model $x$ of the exogenous processes and thus no ambiguity along this dimension. Nevertheless, the planner faces ambiguity because it knows only that the private sector's model $o$ is within a "ball" around its own model. The Ramsey planner evaluates the private sector's Euler equations using a worst-case model chosen by the Ramsey planner's alter ego.

This figure is just for motivation. Our formal analysis is more complex. There are

many (an infinite number of) dimensions associated with our "entropy balls" of probability specifications. Technically, we do not specify such balls but instead penalize relative entropy as a way to restrain how much concern the Ramsey planner has for model ambiguity. To do this, we extend and apply the multiplier preferences of Hansen and Sargent (2001).

For each of our three types of ambiguity, we compute a robust Ramsey plan and an associated worst-case probability model. A worst-case distribution is sometimes called an *ex post* distribution, meaning *after* the robust decision maker's minimization over probabilities. *Ex post*, ambiguity of type 1 delivers a model of endogenously distorted *homogeneous* beliefs, while ambiguities of types 2 and 3 give distinct models of endogenously *heterogeneous* beliefs.

A Ramsey problem can be solved by having the planner choose a path for the private sector's decisions subject to restrictions on the private sector's co-state variable $\lambda_t$ at dates $t \geq 0$ that are implied by the private sector's optimization.[4] The private sector's Euler equation for $\lambda_t$ involves conditional expectations of future values of $\lambda_t$, which makes it differ from a standard 'backward-looking' state evolution equation in ways that we must take into account when we pose Ramsey problems that confront alternative types of ambiguity. A Ramsey plan can be represented recursively by using the "co-state on the private sector costate," $\lambda_t$, as a state variable $\psi_t$ for the Ramsey planner. The planner chooses the initial value $\psi_0$ to maximize its time 0 value function. The evolution of $\psi_t$ encodes the planner's commitment to confirm the private sector's earlier expectations about the Ramsey planner's time $t$ actions. It is particularly important for us to characterize the probability distribution with respect to which the private sector's expectations are formed and how $\psi_t$ responds to shocks.

For linear-quadratic problems without robustness, a certainty equivalence principle implies that shock exposures have no impact on decision rules.[5] But even in linear-quadratic problems, concerns about robustness make shock exposures affect decision rules by affecting the scope of concerns about statistical misspecification.

Along with others, in earlier work we have analyzed the effects of shock exposures on robust decisions too casually. In this paper, we proceed systematically by starting with fundamentals and distinguishing among conditional expectations associated with alternative probability models. We exploit the finding that, without concerns about robustness, the

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[4]Marcet and Marimon (2011) and the references cited there formulate a class of problems like ours under rational expectations. Marcet and Marimon (2011) discuss measurability restrictions on multipliers that are closely related to ones that we impose.

[5]Shock exposures do affect constant terms in value functions.

planner's commitment multiplier $\psi_t$ is "locally predictable" and hence has zero exposure to shocks in the current period. We then describe ways that a Ramsey planner seeks to be robust for each of our three types of statistical ambiguity and produce a Hamilton-Jacobi-Bellman equation for each.

Technically, this paper (1) uses martingales to clarify distinctions among the three types of ambiguity; (2) finds, to our initial surprise, that even in continuous time limits and even in our very simple linear New Keynesian model, ambiguity of types II and III lead to zero-sum games that are not linear-quadratic; (3) uses recursive formulations of Ramsey problems to impose local predictability of commitment multipliers in a direct way; and (4) finds, as a consequence of (3), that to reduce the dimension of the state in the recursive formulation, it is convenient to transform the commitment multiplier in a way to accommodate heterogeneous beliefs with ambiguity of types II and III.[6]

The *ex post* belief distortion that emerges from ambiguity of type I is reminiscent of some outcomes for a robust social planning problem appearing in some of our earlier research, but there are important differences. Hansen and Sargent (2008, chs. 12-13)) used a robust social planning problem to compute allocations as well as worst-case beliefs that we imputed to a representative agent in a model of competitive equilibrium without economic distortions. In effect, we appealed to welfare theorems and restrictions on preferences to justify a robust planner. We priced risky assets by taking the representative agent's first-order conditions for making trades in a decentralized economy, then evaluating them at the allocation chosen by a robust social planner under the imputed worst-case beliefs (e.g. Hansen and Sargent (2008, chs. 14)). In this paper, we can't appeal to the welfare theorems.[7]

Section 2 describes a simple New Keynesian model that we use as a laboratory in which to study our three types of ambiguity. Section 3 sets the stage by solving a Ramsey problem without robustness in two ways, one in the space of sequences, another recursively. Section 4 describes how to represent alternative probability models as distortions of a baseline approximating model. Section 5 solves a robust Ramsey problem under the first type

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[6]We do not analyze the type 0 ambiguity studied by Karantounias (2012) mainly for the technical reason that the trick we use to reduce the dimension of the state in the planner's Bellman equations for ambiguity of types II and III in sections 7 and 8 does not apply. The Bellman equation analyzed by Karantounias (2012) contains an additional state variable relative to ours.

[7]Even in heterogeneous-agent economies without economic distortions, where the welfare theorems do apply, formulating Pareto problems with agents who are concerned about robustness requires an additional endogenous state variable to characterize efficient allocations recursively. See Anderson (2005), who studies risk-sensitive preferences that also have an interpretation as expressing aversion to model ambiguity with what have come to be called multiplier preferences.

of ambiguity. Section 6 studies a Ramsey problem with exogenous belief heterogeneity between the private sector and the Ramsey planner. The model with arbitrary belief heterogeneity is of interest in its own right and is also useful in preparing for the analysis of the robust Ramsey problem under the second type of ambiguity to be presented in section 7. Section 8 then studies the robust Ramsey problem under the third type of ambiguity. Section 9 proposes new local approximations to compare outcomes under robust Ramsey plans constructed under the three types of ambiguity. We illustrate our analysis with a numerical example in section 10. After section 11 offers concluding remarks, appendices B and C describe calculations that illustrate how sequence formulations and recursive formulations of Ramsey plans agree.

# 2    Illustrative model

For concreteness, we use a simple version of a New Keynesian model of Woodford (2010). We begin by describing the model and Ramsey problems without ambiguity in discrete time and in continuous time.

Let time be discrete with $t = \epsilon j$ for $\epsilon > 0$ and integer $j \geq 0$. A cost-push shock $c_t$ is a function $f(x_t)$ of a Markov state vector $x_t$ described by

$$x_{t+\epsilon} = g(x_t, w_{t+\epsilon} - w_t, \epsilon), \tag{1}$$

where $\{w_t\}$ is a standard Brownian motion so that the increment $w_{t+\epsilon} - w_t$ is normally distributed with mean zero and variance $\epsilon$ and is independent of $w_s$ for $0 \leq s \leq t$. The private sector treats $c$ as exogenous to its decisions.

The private sector's first-order necessary conditions are

$$p_t - p_{t-\epsilon} = \epsilon \lambda_t \tag{2}$$
$$\lambda_t = \epsilon(\kappa y_t + c_t + c^*) + \exp(-\delta\epsilon)E\left[\lambda_{t+\epsilon}|\mathcal{F}_t\right] \tag{3}$$
$$\epsilon i_{\epsilon,t} - \epsilon \lambda_t = \rho E\left[y_{t+\epsilon}|\mathcal{F}_t\right] - \rho y_t + \epsilon d^*, \tag{4}$$

where $i_{\epsilon,t}$ is the one-period (of length $\epsilon$) nominal interest rate set at date $t$. Equation (3) is a New Keynesian Phillips curve and equation (4) is a consumption Euler equation.

To obtain a continuous-time model that is mathematically easier to analyze, we shrink the discrete-time increment $\epsilon$. Index the time increment by $\epsilon = \frac{1}{2^j}$ for some positive integer

$j$. Define the local mean $\mu_{\lambda_t}$ to be

$$\mu_{\lambda,t} = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} E\left[\lambda_{t+\epsilon} - \lambda_t | \mathcal{F}_t\right],$$

and drive $\epsilon$ to zero in (3) to get a continuous time version of a new Keynesian Phillips curve:

$$\mu_{\lambda,t} = \delta\lambda_t - \kappa y_t - c_t - c^*. \tag{5}$$

Applying a similar limiting argument to (4) produces a continuous-time consumption Euler equation:

$$\mu_{y,t} = \frac{1}{\rho}\left(i_t - \lambda_t - d^*\right) \tag{6}$$

where here $\lambda_t$ is the instantaneous inflation rate and $i_t$ is the instantaneous nominal interest rate. We depict the continuous-time counterpart to the exogenous state evolution equation (1) as

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dw_t.$$

These equations, or modifications of them that appropriately allow for alternative specifications of private sector beliefs, constrain our Ramsey planners.

# 3   No concern about robustness

In this section, we first pose a Ramsey problem as a Lagrangian and deduce a set of first-order conditions that restrict the dynamic evolution of the state variables and associated Lagrange multipliers. We can compute a Ramsey plan by solving these equations subject to the appropriate initial and terminal conditions. When these equations are linear, we could solve them using invariant subspace methods. We take a different route by developing and solving a recursive version of the Ramsey problem using the multiplier on the private sector Euler equation as a state variable. The idea of constructing a recursive representation of a Ramsey plan in this way has a long history. See (Ljungqvist and Sargent 2004, chs. 18,19) for an extensive discussion and references. In later sections, we will extend that literature by constructing robust counterparts to recursive formulation of the Ramsey problem in discrete and continuous time.

## 3.1 Planner's objective function

In discrete time and without concerns about robustness the Ramsey planner maximizes

$$-\frac{1}{2}E\left(\epsilon\sum_{j=0}^{\infty}\exp(-\epsilon\delta j)\left[(\lambda_{\epsilon j})^2+\zeta(y_{\epsilon j}-y^*)^2\right]|\mathcal{F}_0\right). \tag{7}$$

In a continuous-time limit, the planner's objective becomes

$$-\frac{1}{2}E\left(\int_0^{\infty}\exp(-\delta t)\left[(\lambda_t)^2+\zeta(y_t-y^*)^2\right]dt|\mathcal{F}_0\right).$$

In posing our Ramsey problem, we follow Woodford (2010) in specifying the Ramsey planner's objective function in a way that induces the Ramsey planner to trade off output and inflation dynamics. The Ramsey planner takes the firm's Euler equation (5) as an implementability constraint and chooses welfare-maximizing processes for $\{\lambda_t\}$ and $\{y_t\}$. The consumer's Euler equation (6) will then determine an implied interest rate rule $i_t = \lambda_t - \rho\mu_{y,t} + d^*$ that implements the Ramsey plan.

## 3.2 A discrete-time sequence formulation

A Ramsey planner chooses sequences $\{\lambda_{\epsilon j}, y_{\epsilon j}\}_{j=0}^{\infty}$ to maximize (7) subject to (3) and $c_t = f(x_t)$ with $x_t$ governed by (1). Form the Lagrangian

$$-\frac{1}{2}E\left[\epsilon\sum_{j=0}^{\infty}\exp(-\epsilon\delta j)\left[(\lambda_{\epsilon j})^2+\zeta(y_{\epsilon j}-y^*)^2\right]|\mathcal{F}_0\right]$$

$$+E\left[\sum_{j=0}^{\infty}\exp(-\epsilon\delta j)\psi_{\epsilon(j+1)}\left[\lambda_{\epsilon j}-\epsilon\left(\kappa y_{\epsilon j}+c_{\epsilon j}+c^*\right)-\exp(-\epsilon\delta)\lambda_{(j+1)\epsilon}\right]|\mathcal{F}_0\right]. \tag{8}$$

**Remark 3.1.** *The private sector Euler equation (3) is cast in terms of mathematical expectations conditioned on time t information. This makes it appropriate to restrict the Lagrange multiplier $\psi_{t+\epsilon}$ to depend on date t information. We shall exploit this measurability condition extensively when we drive $\epsilon$ to zero to obtain continuous-time limits. This measurability condition is the source of local predictability of $\psi_t$.*

First-order conditions for maximizing (8) with respect to $\lambda_t$, $y_t$, respectively, are

$$\psi_{t+\epsilon} - \psi_t - \epsilon \lambda_t = 0 \tag{9}$$

$$-\zeta(y_t - y^*) - \kappa \psi_{t+\epsilon} = 0.$$

Combine (9) with the equation system (1) that describes the evolution of $\{x_t\}$ and also the private-sector Euler equation (3). When the $x$ dynamics (1) are linear, a Ramsey plan without robustness is a stabilizing solution of the resulting system of equations, which can be computed using a stabilizing subspace method described by Hansen and Sargent (2008, chs. 4,16).

## 3.3   A recursive formulation

We now propose an alternative approach to the Ramsey problem without robustness that builds on recursive formulations of Stackelberg or Ramsey problems that were summarized by Ljungqvist and Sargent (2004, chs. 18,19) and extended by Marcet and Marimon (2011). To encode history, view $\psi$ as an endogenous state variable that evolves as indicated by (9), namely,

$$\psi_{t+\epsilon} = \epsilon \lambda_t + \psi_t.$$

Because the Brownian increment $w_{t+\epsilon} - w_t$ does not affect the evolution of $\psi_{t+\epsilon}$, $\psi_{t+\epsilon}$ is said to be "locally predictable".

In the spirit of dynamic programming, we transform a multi-period problem to a sequence of two-period problems. Recall that the cost-push shock $c$ is a function $f(x)$ of a Markov state vector $x$ that obeys (1). Guess that an appropriate state vector for next period is $(x^+, \psi^+)$. Soon we will argue that we can interpret $\psi^+$ as a commitment multiplier. Let $\lambda^+ = F^+(x^+, \psi^+)$ be a policy function for $\lambda^+$. Let $V^+(x^+, \psi^+)$ denote a planner's next-period value function inclusive of a term that encodes commitment. To be more precise $V(x,\psi) + \psi F(x,\psi)$ will be the discounted expected value of the single period contributions given by

$$-\frac{\epsilon}{2} \left[ (\lambda_t)^2 + \zeta(y_t - y^*)^2 \right]$$

to the Ramsey planner's objective. In our first recursive formulation, we will take to be the next period function $V^+(x^+, \psi^+) + \psi^+ F^+(x^+, \psi^+)$ and then compute the current-period functions $F$ and $V$. To ensure that commitments are honored we will subtract a term $\psi\lambda$ from the current-period objective when we optimize with respect $\lambda$ required for computing

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$F$. Notice that $V$ includes this term evaluated at $\lambda F(x, \psi)$.

It turns out that by virtue of optimization, we can restrict the two functions $V^+$ and $F^+$ to satisfy

$$V_2^+(x^+, \psi^+) = -F^+(x^+, \psi^+) \tag{10}$$

where $V_2^+$ is the derivative of $V^+$ with respect to its second argument $\psi^+$. We will show that property (10) is replicated under iteration on the Bellman equation for the Ramsey planner. The relations between $V^+$ and $F^+$ and between $V$ and $F$ will lead us to construct an alternative Bellman equation mapping $V^+$ to $V$. Our specific tasks in this section are to i) provide an evolution equation for $\psi^+$ and interpret $\psi$ and $\psi^+$ formally as commitment multipliers; ii) show that the counterpart to restriction (10) applies to $F$; and iii) construct a Bellman equation that applies to $V$ and $V^+$ with no specific reference to $F$ or $F^+$.

**Problem 3.2.** *Our first Bellman equation for the Ramsey planner is*

$$
\begin{aligned}
V(x, \psi) = \max_{y, \lambda} \; & -\psi\lambda - \frac{\epsilon}{2}\left[\lambda^2 + \zeta(y - y^*)^2\right] + \\
& + \exp(-\delta\epsilon)E\left[V^+(x^+, \psi^+) + \psi^+ F^+(x^+, \psi^+)|x, \psi\right]
\end{aligned} \tag{11}
$$

*where the maximization is subject to*

$$\lambda - \exp(-\delta\epsilon)E\left[F^+(x^+, \psi^+)|x, \psi\right] - \epsilon\left[\kappa y + f(x) + c^*\right] = 0 \tag{12}$$

$$\epsilon\lambda + \psi - \psi^+ = 0 \tag{13}$$

$$g(x, w^+ - w) - x^+ = 0.$$

Notice the term $-\psi\lambda$ on the right side of (11). This term remembers and confirms commitments and plays a vital role when it comes to optimizing with respect to $\lambda$. In the special case in which $\psi = 0$, which happens to be the initial value set at by the Ramsey planner at date zero, the only date at which the planner is free to set $\psi$, this commitment term vanishes. Soon we will display an alternative Bellman equation (17) that involves only the function $V$ but that nevertheless encodes the private sector Euler equation.

To justify our interpretation of $\psi^+$ and $\psi$ as commitment multipliers, we solve the Bellman equation (11) by first introducing multipliers $\ell_1$ and $\ell_2$ on the first two constraints (12) and (13) for Problem 3.2. First-order conditions for maximizing the resulting Lagrangian

with respect to $\lambda$ and $y$ are

$$-\epsilon\lambda + \ell_1 + \epsilon\ell_2 - \psi = 0,$$
$$-\zeta(y - y^*) - \kappa\ell_1 = 0. \tag{14}$$

Combining the first equation of (14) with the second constraint (13) for Problem 3.2 gives

$$\psi^+ = \ell_1 + \epsilon\ell_2.$$

Our next result justifies our interpretation of $\psi^+$ and the evolution that we posited for $\psi^+$ in the constraint (13). We link the multiplier $\ell_1$ to $\psi^+$ and verify that this constraint is slack.

**Lemma 3.3.** *In problem 3.2, the multiplier $\ell_1$ on constraint (12) equals $\psi^+$ and the multiplier $\ell_2$ on constraint (13) equals zero. Furthermore,*

$$y = y^* - \left(\frac{\kappa}{\zeta}\right)(\psi + \epsilon\lambda), \tag{15}$$

*where $\lambda = F(x, \psi)$ satisfies the private firm's Euler equation (12). Finally, $V_2(x, \psi) = -F(x, \psi)$.*

See Appendix A for a proof.

Finally, we construct a Bellman equation for the Ramsey planner that incorporates the private sector Euler equation by using our characterization of $\psi^+$ as a Lagrange multiplier. Express the contribution of the private sector Euler equation to a Lagrangian formed from the optimization on the right side of (11):

$$\psi^+ \left[\lambda - \exp(-\delta\epsilon)E\left[F^+(x^+, \xi^+)|x, \psi\right] - \epsilon\left(\kappa y + c + c^*\right)\right]$$
$$= -\exp(-\delta\epsilon)E\left[\psi^+ F^+(x^+, \psi^+)|x, \psi\right] + \psi^+\left[\lambda - \epsilon\left(\kappa y + c + c^*\right)\right],$$

where we have used the fact that $\psi^+$ is locally predictable. Adding this Lagrangian term to the Ramsey planner's objective results in:

$$-\psi\lambda - \frac{\epsilon}{2}\left[\lambda^2 + \zeta(y - y^*)^2\right] + \exp(-\delta\epsilon)E\left[V^+(x^+, \psi^+)|x, \psi\right]$$
$$+ \psi^+\left[\lambda - \epsilon\left(\kappa y + c + c^*\right)\right]. \tag{16}$$

Not surprisingly, by differentiating with respect to $y$, $\lambda$ and $\psi^+$, we reproduce consequence (15) of the first-order conditions reported in Lemma 3.3. This optimization has us maximize with respect to $\lambda$ and $y$. By maximizing with respect to $\lambda$ we obtain state evolution (13), and by minimizing with respect to $\psi^+$, we obtain the private sector Euler equation (12).

In what follows we consider $\psi^+$ as an endogenous state variable and $\lambda$ as a control. After substituting for $\psi^+$ into the Lagrangian (16), we are led to study the following recursive, zero-sum game.

**Problem 3.4.** *An alternative Bellman equation for a discrete-time Ramsey planner without robustness is*

$$V(x, \psi) = \min_{\lambda} \max_{y} \frac{\epsilon}{2} \left[\lambda^2 - \zeta(y - y^*)^2\right] + \exp(-\delta\epsilon)E\left[V^+(x^+, \psi^+)|x, \psi\right]$$
$$- \epsilon(\psi + \epsilon\lambda)\left[\kappa y + f(x) + c^*\right], \tag{17}$$

*where the extremization is subject to*

$$\psi + \epsilon\lambda - \psi^+ = 0 \tag{18}$$
$$g(x, w^+ - w, \epsilon) - x^+ = 0.$$

**Claim 3.5.** *Discrete-time problems 3.2 and 3.4 share a common value function $V$ and common solutions for $y$, $\lambda$ as functions of the state vector $(x, \psi)$.*

*Proof.* The first-order condition for $y$ implies the same formula given in Lemma 3.3. To verify the private sector Euler equation, introduce a multiplier $\ell$ on constraint (18). Differentiate with respect to $\lambda$ and divide by $\epsilon$:

$$\lambda + \ell - \epsilon\left[\kappa y + f(x) + c^*\right] = 0. \tag{19}$$

Differentiate with respect to $\psi^+$ and substitute $-F^+$ for $V_2^+$ to get

$$-\ell - \exp(-\delta\epsilon)E\left[F^+(x^+, \psi^+)|x, \psi\right] = 0.$$

Solving this equation for $\ell$ and substituting into (19) allows us to express the private sector Euler equation as constraint (12) in Problem 3.2. $\qquad\square$

**Remark 3.6.** *In Problem 3.4, the Ramsey planner* <u>minimizes</u> *with respect to $\lambda$, taking into account its contribution to the evolution of the multiplier $\psi^+$. That we minimize with respect*

to $\lambda$ is the outcome of our having substituted for $\psi^+$ into (16). In contrast to Problem 3.2, the constraint (13) ceases to be slack. Instead of being included as a separate constraint, Problem 3.4 embeds the private-sector Euler equation (i.e., equation (12)), in the criterion to be optimized.

**Remark 3.7.** *At time* $0$, $\psi$ *is a choice variable for the Ramsey planner. The optimal choice of* $\psi$ *solves*

$$\min_{\psi} V(x, \psi) + \psi F(x, \psi).$$

*First-order conditions are*

$$V_2(x, \psi) + F(x, \psi) + \psi F_2(x, \psi) = 0.$$

*Since* $V_2 = -F$, *a solution to the above equation is* $\psi = 0$, *which is consistent with our initial condition* $\psi_0 = 0$.

## 3.4 Continuous-time recursive formulation

In a continuous-time formulation of the Ramsey problem without concerns about robustness, the exogenous state vector evolves according to:

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dw_t$$
$$d\psi_t = \lambda_t dt.$$

Using Ito calculus, we characterize the effects of the evolution of $x, \psi$ on the value function $V$ by differentiating the value function. Subtract $V$ from both sides of (17) and divide by $\epsilon$ to obtain

**Problem 3.8.**

$$
\begin{aligned}
0 = \min_{\lambda} \max_{y} \; & \frac{1}{2}\lambda^2 - \frac{\zeta}{2}(y - y^*)^2 - \kappa\psi y - \psi f(x) - \psi c^* \\
& - \delta V + V_1 \cdot \mu_x + V_2\lambda \\
& + \frac{1}{2} trace\left(\sigma_x{}' V_{11}\sigma_x\right).
\end{aligned}
\tag{20}
$$

From the first-order conditions,

$$y = y^* - \frac{\kappa}{\zeta}\psi$$

$$\lambda = -V_2.$$

As in our discrete-time formulation, we used a Lagrangian to impose the private sector Euler equation under the approximating model. In Appendix A, we verify that satisfaction of the Hamilton-Jacobi-Bellman equation (20) implies that the Euler equation is also satisfied.

We end the section with a caveat. We have assumed attainment and differentiability without providing formal justification. We have not established the existence of smooth solutions to our Bellman equations. While we could presumably appeal to more general viscosity solutions to the Bellman equation, this would require a different approach to verifying that the private sector's Euler equation is satisfied than what we have done in Appendix A. In the numerical example of section 10, there is a quadratic solution to the Hamilton-Jacobi-Bellman (HJB) equation (20), so there the required smoothness prevails.

## 4    Representing probability distortions

To represent an alternative probability model, we use a positive martingale $z$ with a mathematical expectation with respect to the approximating model equal to unity. By setting $z_0 = 1$, we indicate that we are conditioning on time 0 information. A martingale $z$ is a likelihood ratio process for a probability model perturbed *vis a vis* an approximating model. It follows from the martingale property that the perturbed probability measure obeys a Law of Iterated Expectations. Associated with a martingale $z$ are the perturbed mathematical expectations

$$\widehat{E}\left(\rho_{t+\tau}|\mathcal{F}_t\right) = E\left(\frac{z_{t+\tau}}{z_t}\rho_{t+\tau}|\mathcal{F}_t\right),$$

where the random variable $\rho_{t+\tau}$ is in the date $t + \tau$ information set. By the martingale property

$$E\left(\frac{z_{t+\tau}}{z_t}|\mathcal{F}_t\right) = 1.$$

## 4.1 Measuring probability distortions

To measure probability distortions, we use relative entropy, an expected log-likelihood ratio, where the expectation is computed using a perturbed probability distribution. Following Hansen and Sargent (2007), the term

$$\sum_{j=0}^{\infty} \epsilon \exp[-\epsilon\delta(j+1)] E\left(z_{\epsilon(j+1)}\left[\log\ z_{\epsilon(j+1)} - \log z_{\epsilon j}\right] | \mathcal{F}_0\right)$$

$$= [1 - \exp(-\epsilon\delta)] \sum_{j=0}^{\infty} \epsilon \exp[-\epsilon\delta(j+1)] E\left[z_{\epsilon(j+1)} \log\ z_{\epsilon(j+1)} | \mathcal{F}_0\right] \qquad (21)$$

measures discounted relative entropy between a perturbed (by $z$) probability model and a baseline approximating model. The component

$$E\left[z_{\epsilon(j+1)} \log\ z_{\epsilon(j+1)} | \mathcal{F}_0\right]$$

measures conditional relative entropy of perturbed probabilities of date $\epsilon(j+1)$ events conditioned on date zero information, while

$$E\left(z_{\epsilon(j+1)}\left[\log\ z_{\epsilon(j+1)} - \log z_{\epsilon j}\right] | \mathcal{F}_{\epsilon j}\right)$$

measures conditional relative entropy of perturbed probabilities of date $\epsilon(j+1)$ events conditioned on date $\epsilon j$ information.

## 4.2 Representing continuous-time martingales

We acquire simplifications by working with a continuous time model that emerges from forming a sequence of discrete time models with time increment $\epsilon$ and driving $\epsilon$ to zero. For continuous Brownian motion information structures, altering the probability model changes the drift of the Brownian motion in a way conveniently described in terms of a multiplicative representation of the martingale $\{z_t\}$:

$$dz_t = z_t h_t \cdot dw_t.$$

Under the perturbed model associated with the martingale $z$, the drift of $dw_t$ is $h_t dt$. We use Ito's lemma to characterize the evolution of $\log z$ and $z \log z$:

$$d \log z_t = -\frac{1}{2}|h_t|^2 dt + h_t \cdot dw_t,$$

$$dz_t \log z_t = \frac{1}{2} z_t (h_t)^2 dt + z_t(1 + \log z_t) h_t \cdot dw_t.$$

The drift or local mean of $\left(\frac{z_{t+\epsilon}}{z_t}\right)(\log z_{t+\epsilon} - \log z_t)$ at $t$ for small positive $\epsilon$ is $\frac{1}{2}(h_t)^2$. Hansen et al. (2006) used this local measure of relative entropy. Discounted relative entropy in continuous time is

$$\frac{1}{2} E\left[\int_0^\infty \exp(-\delta t) z_t (h_t)^2 dt | \mathcal{F}_0 \right] = \delta E\left[\int_0^\infty \exp(-\delta t) z_t \log z_t dt | \mathcal{F}_0 \right].$$

In our continuous-time formulation, the robust Ramsey planner chooses $h$.

# 5    The first type of ambiguity

In the first type of ambiguity, the planner thinks that the private sector knows a model that is distorted relative to the planner's approximating model.

## 5.1    Managing the planner's ambiguity

To respond to its ambiguity about the private sector's statistical model, the Ramsey planner chooses $z$ to minimize and $y$ and $\lambda$ to maximize a *multiplier criterion*[8]

$$-\frac{1}{2} E\left(\epsilon \sum_{j=0}^\infty \exp(-\epsilon \delta j) z_{\epsilon j} \left[(\lambda_{\epsilon j})^2 + \zeta(y_{\epsilon j} - y^*)^2\right] | \mathcal{F}_0 \right)$$

$$+ \theta E\left(\sum_{j=0}^\infty \epsilon \exp[-\epsilon \delta(j+1)] z_{\epsilon(j+1)} \left[\log z_{\epsilon(j+1)} - \log z_{\epsilon j}\right] | \mathcal{F}_0 \right) \tag{22}$$

subject to the implementability constraint

$$\lambda_t = \epsilon(\kappa y_t + c_t + c^*) + \exp(-\delta \epsilon) E\left(\frac{z_{t+\epsilon}}{z_t}\lambda_{t+\epsilon} | \mathcal{F}_t\right) \tag{23}$$

---

[8]See Hansen and Sargent (2001).

and the exogenously specified cost-push process. Here the parameter $\theta$ penalizes martingales $z$ with large relative entropies. Setting $\theta$ arbitrarily large makes this problem approximate a Ramsey problem without robustness. In (22), the Ramsey planner evaluates its objective under the perturbed probability model associated with the martingale $z$. Also, in the private sector's Euler equation (23), the Ramsey planner evaluates the expectation under the perturbed model. These choices capture the planner's belief that the private sector knows a correct probability specification linked to the planner's approximating model by a probability distortion $z$ that is unknown to the Ramsey planner but known by the private sector.

Evidently

$$E\left[\frac{z_{t+\epsilon}}{z_t}(c_{t+\epsilon} - c_t)|\mathcal{F}_t\right] = \epsilon\nu_c c_t + E\left[\frac{z_{t+\epsilon}}{z_t}(w_{t+\epsilon} - w_t)|\mathcal{F}_t\right]$$

where $E\left[\frac{z_{t+\epsilon}}{z_t}(w_{t+\epsilon} - w_t)|\mathcal{F}_t\right]$ is typically not zero, so that the martingale $\{z_t\}$ alters the conditional mean of the cost-push process.

Form the Lagrangian

$$
\begin{aligned}
- \ & \frac{1}{2}E\left[\epsilon\sum_{j=0}^{\infty}\exp(-\epsilon\delta j)z_{\epsilon j}\left[(\lambda_{\epsilon j})^2 + \zeta(y_{\epsilon j} - y^*)^2\right]|\mathcal{F}_0\right]\\
& +\theta E\left(\sum_{j=0}^{\infty}\epsilon\exp[-\epsilon\delta(j+1)]z_{\epsilon(j+1)}\left[\log z_{\epsilon(j+1)} - \log z_{\epsilon j}\right]|\mathcal{F}_0\right)\\
& +E\left[\sum_{j=0}^{\infty}\exp(-\epsilon\delta j)z_{\epsilon(j+1)}\psi_{\epsilon(j+1)}\left[\lambda_{\epsilon j} - \epsilon\left(\kappa y_{\epsilon j} + c_{\epsilon j} + c^*\right) - \exp(-\epsilon\delta)\lambda_{(j+1)\epsilon}\right]|\mathcal{F}_0\right].
\end{aligned}
$$

(24)

First-order conditions for maximizing (24) with respect to $\lambda_t$ and $y_t$, respectively, are

$$z_t\psi_{t+\epsilon} - z_t\psi_t - \epsilon z_t\lambda_t = 0$$
$$-\zeta z_t(y_t - y^*) - \kappa z_t\psi_{t+\epsilon} = 0,$$

where we have used the martingale property $E(z_{t+\epsilon}|\mathcal{F}_t) = z_t$. Because $z_t$ is a common factor in both first-order conditions, we can divide both by $z_t$ and thereby eliminate $z_t$.

## 5.2 Recursive formulation with arbitrarily distorted beliefs

For our recursive formulation in discrete time, initially we posit that the cost-push process $c$ is a function $f(x)$ of a Markov state vector $x$ and that the martingale $z$ itself has a recursive representation, so that

$$x^+ = g(x, w^+ - w, \epsilon)$$
$$z^+ = zk(x, w^+ - w, \epsilon), \tag{25}$$

where we impose the restriction $E\left[k(x, w^+ - w, \epsilon)|x\right] = 1$ that lets us interpret $\frac{z^+}{z} = k(x, w^+ - w, \epsilon)$ as a likelihood ratio that alters the one-step transition probability for $x$. For instance, since $w^+ - w$ is a normally distributed random vector with mean zero and covariance $\epsilon I$, suppose that

$$k(x, w^+) = \exp\left[q(x)'(w^+ - w) - \frac{\epsilon}{2}q(x)'q(x)\right].$$

Then the multiplicative martingale increment $\frac{z^+}{z} = k(x, w^+ - w, \epsilon)$ transforms the distribution of the increment $(w^+ - w)$ from a normal distribution with conditional mean zero to a normal distribution with conditional mean $q(x)$.

Using this recursive specification, we can adapt the analysis in section 3.3 to justify solving

$$V(x, \psi) = \min_\lambda \max_y \frac{\epsilon}{2}\left[\lambda^2 - \zeta(y - y^*)^2\right] + \exp(-\delta\epsilon)E\left[k(x, w^+ - w, \epsilon)V^+(x^+, \psi^+)|x, \psi\right]$$
$$- \epsilon(\psi + \epsilon\lambda)\left[\kappa y + f(x) + c^*\right] + \theta E\left[k(x, w^+ - w, \epsilon)\log k(x, w^+ - w, \epsilon)|x, \psi\right],$$

where the extremization is again subject to (18). We minimize with respect to $\lambda$, taking into account the contribution of $\lambda$ to the evolution of $\psi$. This takes the specification of the martingale as given. To manage ambiguity of the first type, we must contemplate the consequences of alternative $z$'s.

## 5.3   A Ramsey planner's HJB equation for the first type of ambiguity

In a continuous-time formulation of the Ramsey problem with concerns about the first type of ambiguity, we confront the Ramsey planner with the state vector evolution

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dw_t$$
$$dz_t = z_t h_t \cdot dw_t$$
$$d\psi_t = \lambda_t dt.$$

We characterize the impact of the state evolution on continuation values by applying the rules of Ito calculus under the change of measure. We add a penalty term $\frac{\theta}{2}|h|^2$ to the continuous-time objective to limit the magnitude of the drift distortions for the Brownian motion and then by imitating the derivation of HJB equation (20) deduce

$$
\begin{aligned}
0 = \min_{\lambda,h} \max_{y} \; & \frac{1}{2}\lambda^2 - \frac{\zeta}{2}(y - y^*)^2 + \frac{\theta}{2}|h|^2 - \kappa\psi y - \psi f(x) - \psi c^* \\
& - \delta V + V_1 \cdot (\mu_x + \sigma_x h) + V_2 \lambda \\
& + \frac{1}{2}\text{trace}\left(\sigma_x' V_{11} \sigma_x\right).
\end{aligned} \tag{26}
$$

Notice how (26) minimizes over $h$.

The separable form of the objective implies that the order of minimization and maximization can be exchanged. First-order conditions imply

$$
\begin{aligned}
y &= y^* - \frac{\kappa}{\zeta}\psi \\
h &= -\frac{1}{\theta}(\sigma_x)'V_1 \\
\lambda &= -V_2.
\end{aligned} \tag{27}
$$

As in the Ramsey problem without robustness (see Appendix A), to verify that the private sector Euler condition is satisfied, differentiate the HJB equation (26) for $V$ with respect to $\psi$ and apply the envelope condition.

## 5.4 Interpretation of worst-case dynamics

The worst-case $h_t = -\frac{1}{\theta}(\sigma_x)'V_1(x_t, \psi_t)$ from (27) feeds back on the endogenous state variable $\psi_t$. As a consequence, the implied worst-case model makes this *endogenous* state influence the dynamics of the *exogenous* state vector $x_t$. The peculiar feature that $\{\psi_t\}$ Granger-causes $\{x_t\}$ can make the worst-case model difficult to interpret. What does it mean for the Ramsey planner to believe that its decisions influence the motion of exogenous state variables? To approach this question, Hansen et al. (2006) develop an alternative representation. As shown by Fleming and Souganidis (1989), in a two-player zero-sum HJB equation, if a Bellman-Isaacs condition makes it legitimate to exchange orders of maximization and minimization for the recursive problem, then orders of maximization and minimization can also be exchanged for a corresponding zero-sum game that constitutes a date zero, formulation of a robust Ramsey problem in the space of sequences. That allows us to construct an alternative representation of the worst-case model without dependence of the dynamics of the exogenous state vector $x_t$ on $\psi_t$. We accomplish this by augmenting the exogenous state vector as described in detail by Hansen et al. (2006) and Hansen and Sargent (2008, ch. 7) in what amounts to an application of the "Big $K$, little $k$" trick common in macroeconomics. In particular, we construct a worst-case exogenous state-vector process

$$d \begin{bmatrix} x_t \\ \Psi_t \end{bmatrix} = \begin{bmatrix} \mu_x(x_t) \\ F(c_t, \Psi_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_x(x_t) \\ 0 \end{bmatrix} \left[ -\frac{1}{\theta}\sigma_x(x_t)'V_1(x_t, \Psi_t)dt + d\tilde{w}_t \right] \tag{28}$$

for a multivariate standard Brownian increment $d\tilde{w}_t$. We then construct a Ramsey problem *without* robustness but with this expanded state vector. This yields an HJB equation for a value function $\tilde{V}(x, \Psi, \psi)$ that depends on both big $\Psi$ and little $\psi$. After solving it, we can construct $\tilde{F}$ via

$$\tilde{F} = -\tilde{V}_3.$$

Then

$$F(c, \psi) = \tilde{F}(c, \psi, \psi).$$

Provided that we set $\psi_0 = \Psi_0 = 0$, it will follow that $\psi_t = \Psi_t$ and that the resulting $\{\lambda_t\}$ and $\{y_t\}$ processes from our robust Ramsey plan with the first type of ambiguity will coincide with the Ramsey processes under specification (28) for the cost-push process.

## 5.5 Relation to previous literature

The form of HJB equation (26) occurs in the literature on continuous time robust control. For instance, see James (1992) and Hansen et al. (2006). It is also a continuous-time version of a discrete-time Ramsey problem studied by researchers including Walsh (2004), Giordani and Soderlind (2004), Leitemo and Soderstrom (2008), Dennis (2008), and Olalla and Gomez (2011). We have adapted and extended this literature by suggesting an alternative recursive formulation together with appropriate HJB equations. In the next subsection, we correct misinterpretations in some of the earlier literature.

### 5.5.1 Not sharing worst-case beliefs

Walsh (2004) and Giordani and Soderlind (2004) argue that private agents share the government's concern about robustness so that when the government chooses beliefs in a robust fashion, agents act on these same beliefs. We think that interpretation is incorrect and prefer the one we have described as the first type of ambiguity. In selecting a worst-case model, the private sector would look at its *own* objective functions and constraints, not the government's, so robust private agents' worst-case models would differ from the government's. Even if the government and the private agents were to share the same value of $\theta$, they would compute different worst-case models.[9] Dennis (2008) argues that "the Stackelberg leader believes the followers will use the approximating model for forming expectations and formulates policy accordingly." Our Ramsey problem for the second type of ambiguity has this feature, but not our Ramsey problem for the first type, as was mistakenly claimed by Dennis.

As emphasized above, we favor an interpretation of the robust Ramsey plans of Walsh and others as one in which the Ramsey planner believes that private agents know the correct probability model. Because the associated inference problem is so immense, the Ramsey planner cannot infer private agents' model by observing their decisions (see section 5.5.2). The Ramsey planner's worst-case $z$ is not intended to "solve" this impossible inference problem. It is just a device to construct a robust Ramsey policy. It is a cautious inference

---

[9]Giordani and Soderlind (2004), in particular, argue that "we follow Hansen and Sargent in taking the middle ground, and assume that the private sector and government share the same loss function, reference model and degree of robustness." But even if the government and private sector share the same loss function, the same reference model, and the same robustness parameter, they still might very well be led to different worst-case models because they face different constraints. We do not intend to criticize Walsh (2004) and Giordani and Soderlind (2004) unfairly. To the contrary, it is a strength that on this issue their work is more transparent and criticizable than many other papers.

about private agents' beliefs that helps the Ramsey planner design that robust policy. Since private firms know the correct model, they would actually make decisions by using a model that generally *differs* from the one associated with the Ramsey planner's minimizing $\{z_t\}$. Therefore, the Ramsey planner's *ex post* subjective decision rule for the firm as a function of the aggregate states, obtained by solving its Euler equation with the minimizing $\{z\}$, will not usually produce the observed value of $p_{t+\epsilon} - p_t$. This discrepancy will not surprise the Ramsey planner, who knows that discrepancy is insufficient to reveal the process $\{z_t\}$ actually believed by the private sector.

### 5.5.2   An intractable model inference problem

The martingale $\{z_t\}$ defining the private sector's model has insufficient structure to allow the Ramsey planner to infer the private sector's model from observed outcomes $\{p_{t+\epsilon} - p_t, x_t, y_t\}$. The Ramsey planner knows that the probability perturbation $\{z_t\}$ gives the private sector a model that has constrained discounted entropy relative to the approximating model. This leaves the immense set of unknown models so unstructured that it is impossible to infer the private sector's model from histories of outcomes for $y_t$, $x_t$, and $\lambda_t$. The Ramsey planner does not attempt to reverse engineer $\{z_t\}$ from observed outcomes because it cannot.

To indicate the magnitude of the inference problem, consider a discrete time specification and suppose that after observing inflation, the Ramsey planner solves an Euler equation forward to infer a discounted expected linear combination of output and a cost-push shock. If the Ramsey planner were to compare this to the outcome of an analogous calculation based on the approximating model, it would reveal a distorted expectation. But there are many consistent ways to distort dynamics that rationalize this distorted forecast. One would be to distort only the next period transition density and leave transitions for subsequent time periods undistorted. Many other possibilities are also consistent with the same observed inflation. The computed worst-case model is one among many perturbed models consistent with observed data.

## 6   Heterogeneous beliefs without robustness

In section 7, we shall study a robust Ramsey planner who faces our second type of ambiguity. The section 7 planner distrusts an approximating model but believes that private agents trust it. Because *ex post* the Ramsey planner and the private sector have disparate beliefs,

many of the same technical issues for coping with the second type of ambiguity arise in a class of Ramsey problems with exogenous heterogeneous beliefs. So we begin by studying situations in which both the Ramsey planner and the private agents completely trust different models.

To make a Ramsey problem with heterogeneous beliefs manageable, it helps to use the perturbed probability model associated with $\{z_t\}$ when computing the mathematical expectations that appear in the system of equations whose solution determines an equilibrium. To prepare a recursive version of the Ramsey problem, it also helps to transform the $\psi_t$ variable that measures the Ramsey planner's commitments in a way that reduces the number of state variables. We extend the analysis in section 3.3 to characterize the precise link between our proposed state variable and the multiplier on the private sector Euler equation.

With exogenous belief heterogeneity, it is analytically convenient to formulate the Lagrangian for a discrete time version of the Ramsey planner's problem as

$$
-\frac{1}{2}E\left[\epsilon\sum_{j=0}^{\infty}\exp(-\epsilon\delta j)z_{\epsilon j}\left[(\lambda_{\epsilon j})^2 + \zeta(y_{\epsilon j}-y^*)^2\right]|\mathcal{F}_0\right]
$$
$$
+E\left[\sum_{j=0}^{\infty}\exp(-\epsilon\delta j)z_{\epsilon j}\psi_{\epsilon(j+1)}\left[\lambda_{\epsilon j}-\epsilon\left(\kappa y_{\epsilon j}+c_{\epsilon j}+c^*\right)-\exp(-\epsilon\delta)\lambda_{(j+1)\epsilon}\right]|\mathcal{F}_0\right] \quad (29)
$$

## 6.1 Explanation for treatment of $\psi_{t+\epsilon}$

Compare (29) with the corresponding Lagrangian (24) for the robust Ramsey problem for the first type of ambiguity from section 5. There we used $z_{t+\epsilon}\psi_{t+\epsilon}$ as the Lagrange multiplier on the private firm's Euler equation at the date $t$ information set. What motivated that choice was that in the section 5 model with the first type of ambiguity, private agents use the $z$-perturbed model, so their expectations can be represented as

$$
E\left(\frac{z_{t+\epsilon}}{z_t}\lambda_{t+\epsilon}|\mathcal{F}_t\right),
$$

where $z_t$ is in the date $t$ information set. Evidently

$$
\frac{z_{t+\epsilon}}{z_t}z_t\psi_{t+\epsilon} = z_{t+\epsilon}\psi_{t+\epsilon},
$$

which in section 5 allowed us to adjust for the probability perturbation by multiplying $\psi_{t+\epsilon}$ by $z_{t+\epsilon}$ and then appropriately withholding $z_{t+\epsilon}$ as a factor multiplying $\lambda_{t+\epsilon}$ in the Euler equation that $\psi_{t+\epsilon} z_{t+\epsilon}$ multiplies. In contrast to the situation in section 5, here the private sector embraces the original benchmark model, so the private firm's Euler equation now involves the conditional expectation $E\left(\lambda_{t+\epsilon}|\mathcal{F}_t\right)$ taken with respect to the approximating model. The form of this conditional expectation leads us to attach Lagrange multiplier $z_t \psi_{t+\epsilon}$ to the private firm's Euler equation at the information set at date $t$, a choice that implies that the ratio $\frac{z_{t+\epsilon}}{z_t}$ does not multiply $\lambda_{t+\epsilon}$ in the Lagrangian (29).

## 6.2   Analysis

First-order conditions associated with $\lambda_t$ for $t \geq 0$ are

$$z_t \psi_{t+\epsilon} - \epsilon z_t \lambda_t - z_{t-\epsilon}\psi_t = 0, \tag{30}$$

and first-order conditions for $y_t$ for $t \geq 0$ are

$$-\epsilon \zeta z_t (y_t - y^*) - \epsilon \kappa \psi_{t+\epsilon} z_t = 0.$$

To facilitate a recursive formulation, define

$$\xi_{t+\epsilon} = \frac{z_t}{z_{t+\epsilon}} \psi_{t+\epsilon}, \tag{31}$$

which by virtue of (30) implies

$$\xi_{t+\epsilon} = \epsilon \frac{z_t}{z_{t+\epsilon}}\lambda_t + \frac{z_t}{z_{t+\epsilon}}\xi_t.$$

While the process $\{\xi_t\}$ is not locally predictable, the exposure of $\xi_{t+\epsilon}$ to shocks comes entirely through $z_{t+\epsilon}$. The conditional mean of $\xi_{t+\epsilon}$ under the perturbed measure associated with $\{z_t\}$ satisfies

$$E\left(\frac{z_{t+\epsilon}}{z_t}\xi_{t+\epsilon}|\mathcal{F}_t\right) = \epsilon \lambda_t + \xi_t.$$

First-order conditions for $y_t$ imply

$$(y_t - y^*) = -\left(\frac{\kappa}{\zeta}\right)\frac{z_{t+\epsilon}}{z_t}\xi_{t+\epsilon}.$$

25

Evidently,

$$E\left[\left(\frac{z_{t+\epsilon}}{z_t}\right)\xi_{t+\epsilon}\lambda_{t+\epsilon}|\mathcal{F}_t\right] = \psi_{t+\epsilon}E\left(\lambda_{t+\epsilon}|\mathcal{F}_t\right),$$

a prediction formula that suggests a convenient way to pose the Ramsey planner's optimization under the $z$ model.

## 6.3   Recursive formulation with exogenous heterogeneous beliefs

We continue to view the cost-push shock $c$ is a function $f(x)$ of a Markov state vector $x$ and use evolution equation (25) for $x^+$ and $z^+$. As a prolegomenon to studying robustness, we extend the analysis of section 3.3 to describe a recursive way to accommodate exogenous heterogeneity in beliefs described by the likelihood ratio $k(x, w^+ - w, \epsilon)$. We again work backwards from a continuation-policy function $F^+(x^+, \xi^+)$ for the private-sector co-state variable $\lambda^+$ and a continuation-value function $V^+(x^+, \xi^+)$. To start our backwards recursions, we assume that

$$V_2^+(x^+, \xi^+) = -F^+(x^+, \xi^+). \tag{32}$$

**Problem 6.1.** *The Ramsey planner's Bellman equation is*

$$V(x, \xi) = \max_{y, \lambda} -\xi\lambda - \frac{\epsilon}{2}\left[\lambda^2 + \zeta(y - y^*)^2\right]$$
$$+ \exp(-\delta\epsilon)E\left[\left(\frac{z^+}{z}\right)\left[V^+(x^+, \xi^+) + \xi^+ F^+(x^+, \xi^+)\right]|x, \xi\right],$$

*where the maximization is subject to*

$$\lambda - \exp(-\delta\epsilon)E\left[F^+(x^+, \xi^+)|x, \xi\right] - \epsilon\left[\kappa y + f(x) + c^+\right] = 0 \tag{33}$$

$$\left(\frac{z}{z^+}\right)(\epsilon\lambda + \xi) - \xi^+ = 0 \tag{34}$$

$$g(x, w^+ - w, \epsilon) - x^+ = 0$$

$$zk(x, w^+ - w, \epsilon) - z^+ = 0.$$

We now construct an alternative Bellman equation for the Ramsey planner. It absorbs the forward-looking private sector Euler equation into the planner's objective function. We still carry along a state transition equation for $\xi^+$.

Introduce multipliers $\ell_1$ and $\left(\frac{z^+}{z}\right)\ell_2$ on the constraints (33) and (34). Maximizing the

resulting Lagrangian with respect to $\lambda$ and $y$ gives

$$-\epsilon\lambda + \ell_1 + \epsilon\ell_2 - \xi = 0,$$
$$-\zeta(y - y^*) - \kappa\ell_1 = 0.$$

Thus,

$$\left(\frac{z^+}{z}\right)\xi^+ - \ell_1 = \epsilon\ell_2.$$

Therefore, from what we have imposed so far, it seems that $\psi^+$ can differ from $\ell_1$, so we cannot yet claim that $\psi^+$ is "the multiplier on the multiplier". Fortunately, there is more structure to exploit.

**Lemma 6.2.** *The multiplier $\ell_1$ on constraint (33) equals $\left(\frac{z^+}{z}\right)\xi^+$ and the multiplier $\ell_2$ on constraint (34) equals zero. Furthermore,*

$$y = y^* - \left(\frac{\kappa}{\zeta}\right)(\xi + \epsilon\lambda),$$

*where $\lambda = F(x, \xi)$ solves the private firm's Euler equation (33). Finally, $V_2(x, \xi) = -F(x, \xi)$.*

See Appendix A for a proof. Lemma 6.2 extends Lemma 3.3 to an environment with heterogeneous beliefs.

Finally, we deduce an alternative Bellman equation that accommodates heterogeneous beliefs. From Lemma 6.2, the Ramsey planner's value function $V(x, \xi)$ satisfies

$$V(x, \xi) = \max_{y, \lambda} -\xi\lambda - \frac{\epsilon}{2}\left[\lambda^2 + \zeta(y - y^*)^2\right] +$$
$$+ \exp(-\delta\epsilon)E\left[\left(\frac{z^+}{z}\right)[V^+(x^+, \xi^+) + \xi^+ F^+(x^+, \xi^+)] \,|x, \xi\right],$$

where the maximization is subject to constraints (33) and (34) and where $\lambda = F(x, \xi)$. Express the contribution of the private sector Euler equation to a Lagrangian as

$$\left(\frac{z^+}{z}\right)\xi^+\left[\lambda - \exp(-\delta\epsilon)E\left[F^+(x^+, \xi^+)|x, \xi\right] - \epsilon\left(\kappa y + c + c^+\right)\right]$$
$$= -\exp(-\delta\epsilon)E\left[\left(\frac{z^+}{z}\right)[\xi^+ F^+(x^+, \xi^+)] \,|x, \xi\right] + \left(\frac{z^+}{z}\right)\xi^+\left[\lambda - \epsilon\left(\kappa y + c + c^*\right)\right],$$

where we have used the fact that $\left(\frac{z^+}{z}\right)\xi^+$ is locally predictable. Adding this term to the Ramsey planner's objective results in the Lagrangian

$$- \xi\lambda - \frac{\epsilon}{2}\left[\lambda^2 + \zeta(y - y^*)^2\right] + \exp(-\delta\epsilon)E\left[\left(\frac{z^+}{z}\right)\left[V^+(x^+, \xi^+)\right]\vert x, \xi\right]$$

$$+ \left(\frac{z^+}{z}\right)\xi^+\left[\lambda - \epsilon\left(\kappa y + c + c^+\right)\right].$$

Next we substitute from

$$\left(\frac{z^+}{z}\right)\xi^+ = \xi + \epsilon\lambda$$

to arrive at

**Problem 6.3.** *An alternative Bellman equation for a discrete-time Ramsey planner with belief heterogeneity is*

$$V(x, \psi) = \min_\lambda \max_y \frac{\epsilon}{2}\left[\lambda^2 - \zeta(y - y^*)^2\right] + \exp(-\delta\epsilon)E\left[k(x, w^+ - w, \epsilon)\left[V^+(x^+, \xi^+)\right]\vert x, \xi\right]$$

$$- \epsilon(\xi + \epsilon\lambda)\left[\kappa y + f(x) + c^*\right],$$

$$(35)$$

*where the extremization is subject to*

$$\left(\frac{z}{z^+}\right)(\epsilon\lambda + \xi) - \xi^+ = 0$$

$$g(x, w^+ - w, \epsilon) - x^+ = 0,$$

*where we have used $z^+ = zk(x, w^+ - w, \epsilon)$ to eliminate the ratio $\frac{z^+}{z}$.*

**Claim 6.4.** *Discrete-time problems 6.1 and 6.3 share a common value function $V$ and common solutions for $y$, $\lambda$ as functions of the state vector $(x, \xi)$.*

In problem 6.3, we *minimize* with respect to $\lambda$, taking into account its contribution to the evolution of the transformed multiplier $\xi^+$.

In the next subsection, we study the continuous-time counterpart to Problem 6.3. Taking a continuous-time limit adds structure and tractability to the probability distortions in ways that we can exploit in formulating a robust Ramsey problem.

## 6.4 Heterogeneous beliefs in continuous time

Our first step in producing a continuous-time formulation is to characterize the state evolution. For a Brownian motion information structure, a positive martingale $\{z_t\}$ evolves as

$$dz_t = z_t h_t \cdot dw_t$$

for some process $\{h_t\}$. In this section where we assume exogenous belief heterogeneity, we suppose that $h$ is a given function of the state, but in section 7 we will study how a robust planner chooses $h_t$. When used to alter probabilities, the martingale $z_t$ changes the distribution of the Brownian motion $w$ by appending a drift $h_t dt$ to a Brownian increment. Recall from (31) that $\xi_{t+\epsilon} = \frac{z_t}{z_{t+\epsilon}} \psi_{t+\epsilon}$. The "exposure" of $dz_t$ to the Brownian increment $dw_t$ determines the exposure of $d\xi_t$ to the Brownian increment and induces a drift correction implied by Ito's Lemma. By differentiating the function $\frac{1}{z}$ of the real variable $z$ with respect to $z$ and adjusting for the scaling by $z_t = z$, it follows that the exposure is $-\xi_t h_t dw_t$. By computing the second derivative of $\frac{1}{z}$ and applying Ito's Lemma, we obtain the drift correction $\xi_t |h_t|^2$. Thus,

$$d\xi_t = \lambda_t dt + \xi_t |h_t|^2 dt - \xi_t h_t' dw_t.$$

Also suppose that

$$dx_t = \mu_x(x_t) dt + \sigma_x(x_t) dw_t.$$

While we can avoid using $z_t$ as an additional state variable, the $\{\xi_t\}$ process has a local exposure to the Brownian motion described by $-h_t \cdot dw_t$. It also has a drift that depends on $h_t$ under the approximating model.

Write the law of motion in terms of $dw_t$ as

$$d \begin{bmatrix} x_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} \mu_x(x_t) \\ \lambda_t + \xi_t |h_t|^2 \end{bmatrix} dt + \begin{bmatrix} \sigma_x(x_t) \\ -\xi_t h_t' \end{bmatrix} dw_t,$$

where $\{w_t\}$ is standard Brownian motion under the approximating model. Under the distorted model,

$$d \begin{bmatrix} x_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} \mu(x_t) + \sigma_x(x_t) h_t \\ \lambda_t \end{bmatrix} dt + \begin{bmatrix} \sigma_x(x_t) \\ -\xi h_t' \end{bmatrix} d\hat{w}_t,$$

where $\{\hat{w}_t\}$ is a Brownian motion.

In continuous time, we characterize the impact of the state evolution using Ito calculus to differentiate the value function. We subtract $V$ from both sides of (35) and divide by $\epsilon$ to obtain

$$
\begin{aligned}
0 = \min_{\lambda} \max_{y} \ & \frac{1}{2}\lambda^2 - \frac{\zeta}{2}(y - y^*)^2 - \kappa\xi y - \xi c - \xi c^* \\
& - \delta V + V_1 \cdot \mu_x + V_2\lambda \\
& + (V_1)'\sigma_x h - \xi V_{21}\sigma_x h + \frac{1}{2}\xi^2 V_{22}|h|^2 \\
& + \frac{1}{2}\text{trace}\left(\sigma_x' V_{11}\sigma_x\right),
\end{aligned}
\tag{36}
$$

where we use the distorted evolution equation. From the first-order conditions

$$
y = y^* - \frac{\kappa}{\zeta}\xi
$$

$$
\lambda = -V_2.
$$

As hoped, the private sector Euler equation under the approximating model imposed by the Lagrangian is satisfied as we verify in Appendix A.

**Remark 6.5.** *To accommodate belief heterogeneity, we have transformed the predetermined commitment multiplier. Via the martingale used to capture belief heterogeneity, the transformed version of this state variable acquires a nondegenerate exposure to the Brownian increment. This structure is reminiscent of the impact of belief heterogeneity in continuoustime recursive utility specifications. Dumas et al. (2000) show that conveniently chosen Pareto weights are locally predictable when beliefs are homogeneous, but with heterogeneous beliefs Borovička (2012) shows that the Pareto weights inherit an exposure to a Brownian increment from the martingale that alters beliefs of some economic agents.*

# 7  The second type of ambiguity

By exploiting the structure of the exogenous heterogeneous beliefs Ramsey problem of section 6, we now analyze a concern about robustness for a Ramsey planner who faces our second type of ambiguity. In continuous time, we add a penalty term $\theta\frac{|h|^2}{2}$ to the planner's

objective and minimize with respect to $h$:

$$0 = \min_{\lambda, h} \max_{y} \frac{1}{2}\lambda^2 - \frac{\zeta}{2}(y - y^*)^2 + \frac{\theta}{2}|h|^2 - \kappa\xi y - \xi c - \xi c^*$$
$$- \delta V + V_1 \cdot \mu_x + V_2\lambda$$
$$+ (V_1)'\sigma_x h - \xi V_{12}\sigma_x h - \frac{1}{2}\xi^2 V_{22}|h|^2$$
$$+ \frac{1}{2}\text{trace}\left(\sigma_x{}'V_{11}\sigma_x\right).$$

Recursive formulas for $y$ and $\lambda$ remain

$$y = y^* - \frac{\kappa}{\zeta}\xi$$
$$\lambda = -V_2,$$

but now we add minimization over $h$ to the section 6 statement of the Ramsey problem. First-order conditions for $h$ are

$$\theta h + (\sigma_x)'V_1 - \xi(\sigma_x)'V_{12} + \xi^2 V_{22}h = 0,$$

so the minimizing $h$ is

$$h = -\left(\frac{1}{\theta + \xi^2 V_{22}}\right)\left[(V_1)'\sigma_x - \xi V_{12}\sigma_x\right]'. \tag{37}$$

As was the case for the Ramsey plan under the first type of ambiguity, separability of the recursive problem implies that a Bellman-Isaacs condition is satisfied. Again in the spirit of Hansen and Sargent (2008, ch. 7), we can use a date zero sequence formulation of the worst-case model to avoid having the exogenous state vector feed back onto the endogenous state variable $\xi_t$. For a Ramsey plan under the second type of ambiguity, we use this construction to describe the beliefs of a Ramsey planner while the private sector continues to embrace the approximating model. This makes heterogeneous beliefs endogenous.

## 8  The third type of ambiguity

We now turn to our third type of ambiguity. Here, following Woodford (2010), a Ramsey planner trusts an approximating model but does not know the beliefs of private agents.

31

We use $\{z_t\}$ to represent the private sector's unknown beliefs.

## 8.1 Discrete time

Here the Lagrangian associated with designing a robust Ramsey plan is

$$
\begin{aligned}
- \;& \frac{1}{2} E \left[ \epsilon \sum_{j=0}^{\infty} \exp(-\epsilon\delta j) \left[ (\lambda_{\epsilon j})^2 + \zeta(y_{\epsilon j} - y^*)^2 \right] |\mathcal{F}_0 \right] \\
+ \;& \theta \left[ \sum_{j=0}^{\infty} \epsilon \exp[-\epsilon\delta(j+1)] \left( \frac{z_{\epsilon(j+1)}}{z_{\epsilon j}} \right) \left[ \log z_{\epsilon(j+1)} - \log z_{\epsilon j} \right] |\mathcal{F}_0 \right] \\
+ \;& E \left[ \sum_{j=0}^{\infty} \exp(-\epsilon\delta j)\psi_{\epsilon(j+1)} \left[ \lambda_{\epsilon j} - \epsilon \left( \kappa y_{\epsilon j} + c_{\epsilon j} + c^* \right) - \exp(-\epsilon\delta) \left( \frac{z_{\epsilon(j+1)}}{z_{\epsilon j}} \right) \lambda_{(j+1)\epsilon} \right] |\mathcal{F}_0 \right].
\end{aligned}
$$

First-order conditions for $\lambda_t$ are

$$
\psi_{t+\epsilon} - \epsilon\lambda_t - \left( \frac{z_t}{z_{t-\epsilon}} \right) \psi_t = 0.
$$

Let

$$
\xi_{t+\epsilon} = \left( \frac{z_{t+\epsilon}}{z_t} \right) \psi_{t+\epsilon}
$$

so that

$$
\xi_{t+\epsilon} = \epsilon \left( \frac{z_{t+\epsilon}}{z_t} \right) \lambda_t + \left( \frac{z_{t+\epsilon}}{z_t} \right) \xi_t. \tag{38}
$$

We can imitate the argument underlying Claim 6.4 to construct a Bellman equation

$$
\begin{aligned}
V(x, \xi) = \min_{\lambda} \max_{y} \; & \frac{\epsilon}{2} \left[ \lambda^2 - \zeta(y - y^*)^2 \right] + \exp(-\delta\epsilon) E \left[ V^+(x^+, \xi^+) | x, \xi \right] \\
& - \epsilon(\xi + \epsilon\lambda) \left( \kappa y + c + c^* \right), \tag{39}
\end{aligned}
$$

where the extremization is subject to

$$
\begin{aligned}
x^+ &= g(x, w^+ - w, \epsilon) \\
\xi^+ &= k(x, w^+ - w, \epsilon)\xi + \epsilon k(x, w^+ - w, \epsilon)\lambda,
\end{aligned}
$$

where we have used $z^+ = zk(x, w^+ - w, \epsilon)$ to rewrite the evolution equation for $\xi^+$.

## 8.2 Woodford's way of restraining perturbations of beliefs

His assumption that the Ramsey planner embraces the approximating model prompted Woodford (2010) to measure belief distortions in his own special way. Thus, while we have measured model discrepancy by discounted relative entropy (21), Woodford (2010) instead uses

$$\sum_{j=0}^{\infty} \epsilon \exp[-\epsilon\delta(j+1)] E\left(\left[\frac{z_{\epsilon(j+1)}}{z_{\epsilon j}}\right] \left[\log z_{\epsilon(j+1)} - \log z_{\epsilon j}\right] | \mathcal{F}_0\right). \tag{40}$$

Whereas at date zero we weight $(\log z_{t+\epsilon} - \log z_t)$ by $z_{t+\epsilon}$, Woodford weights it by $\frac{z_{t+\epsilon}}{z_t}$.

**Remark 8.1.** *In discrete time, Woodford's measure* (40) *is not relative entropy, but a continuous-time counterpart* $\frac{1}{2} E\left[\int_0^{\infty} \exp(-\delta t)(h_t)^2 dt | \mathcal{F}_0\right]$ *is relative entropy with a reversal of models. To see this, consider the martingale evolution*

$$dz_t = z_t h_t \cdot dw_t \tag{41}$$

*for some process* $\{h_t\}$. *By applying Ito's Lemma,*

$$\lim_{\epsilon \downarrow 0} E\left[\frac{z_{t+\epsilon}}{z_t} \left(\log z_{t+\epsilon} - \log z_t\right) | \mathcal{F}_t\right] = \frac{1}{2}|h_t|^2.$$

*Thus, the continuous-time counterpart to Woodford's discrepancy measure is*

$$\frac{1}{2} E\left[\int_0^{\infty} \exp(-\delta t)(h_t)^2 dt | \mathcal{F}_0\right] = -\delta E\left[\int_0^{\infty} \exp(-\delta t) \log z_t dt | \mathcal{F}_0\right],$$

*where the right side is a measure of relative entropy that switches roles of the* $\{z_t\}$*-perturbed model and the approximating model.*

## 8.3 Third type of ambiguity in continuous time

We use equation (41) for $dz_t$ to depict the small $\epsilon$ limit of (38) as

$$d\xi_t = \lambda_t dt + \xi_t h_t \cdot dw_t.$$

For a Ramsey planner confronting our third type of ambiguity, we compute a robust Ramsey plan under the approximating model. Stack the evolution equation for $\xi_t$ together with the

evolution equation for $x_t$:

$$d \begin{bmatrix} x_t \\ \xi_t \end{bmatrix} = \begin{bmatrix} \mu(x_t) \\ \lambda_t \end{bmatrix} dt + \begin{bmatrix} \sigma_x(x_t) \\ \xi_t h_t' \end{bmatrix} dw_t.$$

The continuous-time counterpart to the Hamilton-Jacobi-Bellman equation (39) adjusted for a robust choice of $h$ is

$$0 = \min_{\lambda, h} \max_{y} \frac{1}{2} \left[ \lambda^2 - \zeta(y - y^*)^2 \right] - \kappa \xi y - \xi c - \xi c^*$$
$$+ V_1 \mu_x + V_2 \lambda - \delta V(x)$$
$$+ \frac{\theta}{2} |h|^2 + \frac{1}{2} \text{trace} \left[ \sigma_x' V_{11} \sigma_x \right] + \xi h' \sigma_x' V_{12} + \frac{1}{2} (\xi)^2 |h|^2 V_{22}.$$

First-order conditions for extremization are

$$y = y^* - \frac{\kappa}{\zeta} \xi$$
$$\lambda = -V_2$$
$$h = -\frac{1}{\theta + \xi^2 V_{22}} \xi \sigma_x' V_{12}. \tag{42}$$

We can verify the private sector Euler equation as we did earlier, except that now we have to make sure that the private sector expectations are computed with respect to a distorted model that assumes that $dw_t$ has drift $h_t dt$, where $h_t$ is described by equation (42).

As with the robust Ramsey planner under the first and second types of ambiguity, we can verify the corresponding Bellman-Isaacs condition. Under the third type of ambiguity, the worst-case model is attributed to the private sector while the Ramsey planner embraces the approximating model.

# 9    Comparisons

In this section, we use new types of local approximations to compare models. We modify earlier local approximations in light of the special structures of our three types of robust Ramsey problems, especially the second and third types, which appear to be unprecedented in the robust control literature. It is convenient to refer to robust Ramsey plans under our three types of ambiguity as Types I, II, and III, respectively.

James (1992) constructs expansions that simultaneously explore two dimensions unleashed by increased conditional volatility, namely, increased noise and increased concern about robustness.[10] In particular, within the context of our model, he would set $\sigma_x = \sqrt{\tau}\varsigma_x$, $\theta = \frac{1}{\vartheta\tau}$, and then compute first derivatives with respect to $\tau$ and $\vartheta$. James's approach is enlightening for Type I, but not for Type II or Type III. To provide insights about Type II and Type III, we compute two first-order expansions, one that holds $\theta < \infty$ fixed when we differentiate with respect to $\tau$; and another that holds fixed $\tau$ when we differentiate with respect $\gamma = \frac{1}{\theta}$. For both computations, our New Keynesian economic example is simple enough to allow us to attain quasi-analytical solutions for the parameter configurations around which we approximate. We use these first-order approximations to facilitate comparisons.[11]

Suppose that

$$dx_t = A_{11}x_t dt + \sigma_x dw_t$$
$$c_t = H \cdot x_t,$$

where $\sigma_x$ is a vector of constants.

Recall the adjustments (27), (37), and (42) in the drift of the Brownian motion that emerge from our three types of robustness:

$$\text{Type I: } h^* = -\frac{1}{\theta}\left[\sigma_x{}' V_1(x,\xi)\right]$$
$$\text{Type II: } h^* = -\frac{1}{\theta + \xi^2 V_{22}(x,\xi)}\left[\sigma_x{}' V_1(x,\xi) - \xi\sigma_x{}' V_{12}(x,\xi)\right]$$
$$\text{Type III: } h^* = -\frac{1}{\theta + \xi^2 V_{22}(x,\xi)}\left[\xi\sigma_x{}' V_{12}(x,\xi)\right],$$

where the value functions $V(x,\xi)$ and the scaling of the commitment multiplier $\xi_t$ differs across our three types of ambiguity. In particular, for Type I we used the commitment multiplier $\psi_t$ and did not rescale it as we did for the Type II and III models.. To facilitate comparisons, for the Type I Ramsey problem we take $\xi_t = \psi_t$. For Type I, the drift adjustment includes only a contribution from the first derivative of the value function as is typical for problems studied in the robust control literature. For our Type II and III

---

[10]See Anderson et al. (2012) and Borovička and Hansen (2011) for related approaches.

[11]James (1992) provides formal justification for his bi-variate expansion. Our presentation is informal in some respects. Modifications of our calculations will be required before they can be applied to a broader class of models.

problems, the second derivative also makes contributions. The associated adjustments to the planner's value function in our three types of Ramsey problems are:

$$\text{Type I:} \quad -\frac{1}{2\theta}|\sigma_x{}'V_1(x,\xi)|^2 + \frac{1}{2}\text{trace}\left[\sigma_x{}'V_{11}(x,\xi)\sigma_x\right]$$

$$\text{Type II:} \quad -\frac{1}{2[\theta + \xi^2 V_{22}(x,\xi)]}|\sigma_x{}'V_1(x,\xi) - \xi\sigma_x{}'V_{12}(x,\xi)|^2 + \frac{1}{2}\text{trace}\left[\sigma_x{}'V_{11}(x,\xi)\sigma_x\right]$$

$$\text{Type III:} \quad -\frac{1}{2[\theta + \xi^2 V_{22}(x,\xi)]}|\xi\sigma_x{}'V_{12}(x,\xi)|^2 + \frac{1}{2}\text{trace}\left[\sigma_x{}'V_{11}(x,\xi)\sigma_x\right], \tag{43}$$

where we have included terms involving $\sigma_x$. For each Ramsey plan, let $\Phi(V, \sigma_x, \theta)$ denote the adjustment described in (43).

These adjustment formulas are suggestive but also potentially misleading as a basis for comparison because the Ramsey planner's value functions themselves differ across our three types of ambiguity. In the following section, we propose more even-footed comparisons by taking small noise and small robustness approximations around otherwise linear-quadratic economies.

## 9.1  Common baseline value function

The baseline value function is the same as that given in Appendix B except the constant term differs because we now set $\sigma_x = 0$ when computing $W$. The minimization problem

$$0 = \min_{\lambda} \frac{1}{2}\lambda^2 + \frac{\kappa^2}{2\zeta}(\xi)^2 - \kappa\xi y^* - \xi c - \xi c^*$$
$$- \delta W(x,\xi) + [W_1(x,\xi)] \cdot A_{11}x + W_2(x,\xi)\lambda$$

yields a quadratic value function $W(x,\xi)$ that we propose to use as a baseline with respect to which we compute adjustments for our three types of robust Ramsey problems. The Riccati equation is the same one given in Appendix B for the stochastic problem without a concern for robustness except that initially we ignore a constant term contributed by the shock exposure $\sigma_x$, allowing us to solve a deterministic problem.

## 9.2  A small-noise approximation

To facilitate comparisons, we study effects of variations in $\tau$ for small $\tau$ under the "small noise" parameterization $\sigma_x = \sqrt{\tau}\varsigma_x$, where $\varsigma_x$ is a vector with the same number of columns

as $x$.

We deduce the first-order value function expansion

$$V(x, \xi) \approx W(x, \xi) + \tau N(x, \xi).$$

We approximate the optimal $\lambda$ by

$$\lambda \approx -W_2(x, \xi) - \tau N_2(x, \xi),$$

where $N_2$ differs across our three types of robust Ramsey problems.

We study a parameterized HJB equation of the form

$$0 = -\frac{1}{2}V_2(x, \xi)^2 + \frac{\kappa^2}{2\varsigma}(\xi)^2 - \kappa\xi y^* - \xi c - \xi c^*$$
$$- \delta V(x, \xi) + [V_1(x, \xi)] \cdot A_{11}x + \Phi(V, \tau\varsigma_x, \theta)(x, \xi). \tag{44}$$

We can ignore the impact of minimization with respect to $\lambda$ and $h$ because of the usual "Envelope Theorem" that exploits first-order conditions to eliminate terms involving derivatives of $\lambda$ and $h$.

We start by computing derivatives with respect to $\tau$ of the terms included in (43). Thus, we differentiate $\Phi(V, \tau\varsigma_x, \theta)$ with respect to $\tau$ for all three plans. These derivatives are

$$\text{Type I: } S(x, \xi) = -\frac{1}{2\theta}|\varsigma_x' W_1(x, \xi)|^2 + \frac{1}{2}\text{trace}[\varsigma_x' W_{11}\varsigma_x]$$

$$\text{Type II: } S(x, \xi) = -\frac{1}{2[\theta + \xi^2 W_{22}]}|\varsigma_x' W_1(x, \xi) - \xi\varsigma' W_{12}|^2 + \frac{1}{2}\text{trace}[\varsigma_x' W_{11}\varsigma]$$

$$\text{Type III: } S(x, \xi) = -\frac{1}{2[\theta + \xi^2 W_{22}]}|\xi\varsigma_x' W_{12}|^2 + \frac{1}{2}\text{trace}[\varsigma_x' W_{11}\varsigma_x].$$

The function $S$ is then used to compute $N$. To obtain the equation of interest, differentiate the (parameterized by $\tau$) HJB equation (44) with respect to $\tau$ to obtain:

$$0 = -W_2(x, \xi) \cdot N_2(x, \xi) - \delta N(x, \xi) + N_1(x, \xi)' A_{11}x + S(x, \xi), \tag{45}$$

where we have used the first-order conditions for $\lambda$ to inform us that

$$\lambda\frac{\partial\lambda}{\partial\tau} + V_2\frac{\partial\lambda}{\partial\tau} = 0.$$

Then $N$ solves the Lyapunov equation (45). Notice that $S$ is a quadratic function of the states for Type I, but not for Types II and III. For Type II and III, this equation must be solved numerically, but it has a quasi-analytic, quadratic solution for Type I.

## 9.3   A small robustness approximation

So far we have kept $\theta$ fixed. Instead, we now let $\theta = \frac{1}{\gamma}$ and let $\gamma$ become small and hence $\theta$ large. The relevant parameterized HJB equation becomes

$$0 = -\frac{1}{2}V_2(x, \xi)^2 + \frac{\kappa^2}{2\zeta}(\xi)^2 - \kappa \xi y^* - \xi c - \xi c^*$$
$$- \delta V(x, \xi) + [V_1(x, \xi)] \cdot A_{11}x + \Phi\left(V, \sigma_x, \frac{1}{\gamma}\right)(x, \xi), \qquad (46)$$

where $\Phi(V, \sigma_x, \theta)$ is given by (43). Write the three respective adjustment terms $\Phi(V, \tau\varsigma_x, \frac{1}{\gamma})$ defined in (43) as

Type I:   $-\dfrac{\gamma}{2}|\sigma_x{'}V_1(x, \xi)|^2 + \dfrac{1}{2}\text{trace}\left[\sigma_x{'}V_{11}(x, \xi)\sigma_x\right]$

Type II:   $-\dfrac{\gamma}{2[1 + \gamma\xi^2 V_{22}(x, \xi)]}|\sigma_x{'}V_1(x, \xi) - \xi\sigma_x{'}V_{12}(x, \xi)|^2 + \dfrac{1}{2}\text{trace}\left[\sigma_x{'}V_{11}(x, \xi)\sigma_x\right]$

Type III:   $-\dfrac{\gamma}{2[1 + \gamma\xi^2 V_{22}(x, \xi)]}|\xi\sigma_x{'}V_{12}(x, \xi)|^2 + \dfrac{1}{2}\text{trace}\left[\sigma_x{'}V_{11}(x, \xi)\sigma_x\right]. \qquad (47)$

Since $\sigma_x$ is no longer made small in this calculation, we compute the limiting value function as $\gamma$ becomes small to be

$$W(x, \xi) + \frac{1}{2\delta}\text{trace}\left[\sigma_x{'}W_{11}\sigma_x\right],$$

where the additional term is constant and identical for all three robust Ramsey plans. This outcome reflects a standard certainty equivalent property for linear-quadratic optimization problems.

We now construct a first-order robustness adjustment

$$V \approx W + \frac{1}{2\delta}\text{trace}\left[\sigma_x{'}W_{11}\sigma_x\right] + \gamma G$$
$$\lambda \approx -W_2 - \gamma G_2.$$

As an intermediate step on the way to constructing $G$, first differentiate (47) with respect

to $\gamma$:

$$\text{Type I: } H(x,\xi) = -\frac{1}{2}|\sigma_x{}'W_1(x,\xi)|^2$$

$$\text{Type II: } H(x,\xi) = -\frac{1}{2}|\sigma_x{}'W_1(x,\xi) - \xi\sigma_x{}'W_{12}|^2$$

$$\text{Type III: } H(x,\xi) = -\frac{1}{2}|\xi\sigma_x{}'W_{12}|^2.$$

To obtain the equation of interest, differentiate the parameterized HJB equation (46) with respect to $\gamma$ to obtain

$$0 = -W_2(x,\xi) \cdot G_2(x,\xi) - \delta G(x,\xi) + G_1(x,\xi)'A_{11}x + H(x,\xi). \tag{48}$$

Given $H$, we compute the function $G$ by solving Lyapunov equation (48). See Appendix D for more detail.

## 9.4 Relation to previous work

To relate our expansions to an approach taken in Hansen and Sargent (2008, ch. 16), we revisit Type II. Using the same section 9.3 parameterization that we used to explore small concerns about robustness, we express the HJB equation as

$$\begin{aligned}
0 = \min_{\lambda,h} \max_y \; & \frac{1}{2}\lambda^2 - \frac{\zeta}{2}(y - y^*)^2 + \frac{1}{2\gamma}|h|^2 - \kappa\xi y - \xi c - \xi c^* \\
& - \delta V + V_1 \cdot \mu_x + V_2\lambda \\
& + (V_1)'\sigma_x h - \xi V_{21}\sigma_x h - \frac{1}{2}\xi^2 V_{22}|h|^2 \\
& + \frac{1}{2}\text{trace}\left(\sigma_x{}'V_{11}\sigma_x\right). 
\end{aligned} \tag{49}$$

In Hansen and Sargent (2008, ch. 16), we arbitrarily modified this HJB equation to become

$$\begin{aligned}
0 = \min_{\lambda,h} \max_y \; & \frac{1}{2}\lambda^2 - \frac{\zeta}{2}(y - y^*)^2 + \frac{1}{2\gamma}|h|^2 - \kappa\xi y - \xi c - \xi c^* \\
& - \delta U + U_1 \cdot \mu_x + U_2\lambda \\
& + (U_1)'\sigma_x h - \xi U_{21}\sigma_x h \\
& + \frac{\tau}{2}\text{trace}\left(\sigma_x{}'U_{11}\sigma_x\right), 
\end{aligned} \tag{50}$$

which omits the term $-\frac{1}{2}\xi^2 V_{22}|h|^2$ that is present in (49). A quadratic value function solves the modified HJB equation (50) provided that $\gamma$ is not too large. Furthermore, it shares the same first-order robustness expansions that we derived for Type II. The worst-case $h$ distortion associated with the modified HJB equation (50) is

$$h = -\gamma\sigma_x{}' \left[U_1(x,\xi) - \xi U_{12}\right].$$

Hansen and Sargent (2008) solve a version of the modified HJB equation (50) iteratively. They guess $\sigma_x{}'U_{12}$, solve the resulting Riccati equation, compute a new guess for $\sigma_x{}'U_{12}$, and then iterate to a fixed point. Thus, the Hansen and Sargent (2008, ch. 16) approach yields a correct first-order robustness expansion for a value function that itself is actually incorrect because of the missing term that appears in the HJB equation (49) but not in (50).[12]

Consider the first-order robustness expansion for Type II. Since $W$ is quadratic, $W_1(x,\xi)-\xi W_{12}$ depends only on $x$ and not on $\xi$. Also, $H$ and $G$ both depend only on $x$ and not on $\xi$, so $G_2$ is zero and there is no first-order adjustment for $\lambda$. The approximating continuation value function is altered, but only those terms that involve $x$ alone. Given the private sector's trust in the approximating model, even though the Ramsey planner thinks that the approximating model might misspecify the evolution of $\{x_t\}$, there is no impact on the outcome for $\lambda$. That same statement applies to $U(x,\xi) - \xi U_{12}$, which illustrates an observation made by Dennis (2008) in the context of the approach suggested in Hansen and Sargent (2008, ch. 16). When we use that original HJB equation to compute the value function, this insensitivity of $\lambda$ to $\gamma$ may not hold.

## 10  Numerical example

Using parameter values given in Appendix C and a robustness parameter $\theta = .014$, we illustrate the impact of a concern for robustness. Most of these parameter values are borrowed from Woodford (2010). Woodford takes the cost-push shock to be independent and identically distributed. In our continuous-time specification, we assume an AR process with the same unconditional standard deviation .02 assumed by Woodford. Since $\theta$ acts as a penalty parameter, we find it revealing to think about the consequences of $\theta$ for the

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[12]Hansen and Sargent (2008) take the shock exposure of $d\xi_t$ to be zero, as is the case for $d\psi_t$. The correct shock exposure for $d\xi_t$ scales with $\gamma$ and is zero only in the limiting case. Hansen and Sargent (2008) interpret $\sigma_x{}'U_{12}$ as the shock exposure for $\lambda_t$, which is only an approximation.

worst-case model when setting $\theta$. Under the worst-case model, the average drift distortion for the standardized Brownian increment is about .34. We defer to later work a serious quantitative investigation including the calibration of $\theta$.[13] What follows is for illustrative purposes only. Appendix C contains more numerical details.

## 10.1  Type I

For Type I ambiguity, we have quasi-analytical solutions. Under the approximating model, the cost-push shock evolves as

$$dc_t = -.15c_t dt + .011 dw_t, \tag{51}$$

while under the worst-case model it evolves as

$$d \begin{bmatrix} c_t \\ \Psi_t \end{bmatrix} = \begin{bmatrix} -.0983 & .0107 \\ 1.2485 & -.6926 \end{bmatrix} \begin{bmatrix} c_t \\ \Psi_t \end{bmatrix} dt + \begin{bmatrix} .0017 \\ .0173 \end{bmatrix} dt + \begin{bmatrix} .011 \\ 0 \end{bmatrix} dw_t, \tag{52}$$

a system in which $\{\Psi_t\}$ Granger causes $\{c_t\}$. In what follows we construct ordinary (non-robust) Ramsey plans for both cost-push shock specifications (51) and (52). If we set $\Psi_0 = 0$ in (52), the time series trajectories under the ordinary Ramsey plan constructed assuming that the planner completely trusts the above worst-case cost-push shock model will coincide with time series trajectories chosen by the robust Ramsey planner who distrusts the approximating model (51).

To depict dynamic implications, we report impulse response functions for the output gap and inflation using the two specifications (51) and (52) for the cost-push process. Figure 2 reports impulse responses under the approximating model (51) and these same responses under the worst-case model (52). Outcomes for the different cost-push shock models are depicted in the two columns of this figure. We also compute optimal plans for both cost-push shock specifications and consider the impact of misspecification. Thus, we plot two impulse response functions depending on which cost-push shock model, (51) or (52), is imputed to the planner who computes an ordinary non-robust Ramsey plan. The impulse response functions plotted in each of the panels line up almost on top of each other even though the actual cost processes are quite different. The implication is that the important differences in outcomes do not come from misspecification in the mind of the

---

[13]See Anderson et al. (2003) for a discussion of an approach to calibration based on measures of statistical discrimination.

Ramsey planner but from what we can regard as different models of the cost-push process imputed to an ordinary non-robust Ramsey planner.

The worst-case drift distortion includes a constant term that has no impact on the impulse response functions. To shed light on the implications of the constant term, we computed trajectories for the output gap and inflation under the approximating model, setting the initial value of the cost-push variable to zero. Again we compare outcomes under a robust Ramsey plan with those under a Ramsey planner who faces type I ambiguity. The left panel of Figure 3 reports differences in logarithms scaled by one-hundred. By construction, the optimal Ramsey plan computed under the approximating model gives a higher value of the objective function when the computations are done under the approximating model. The optimal plan begins at $y^*$ and diminishes to zero. Under the robust Ramsey plan (equivalently the plan that is optimal under the worst-case cost model), output starts higher than the target $y^*$ and then diminishes to zero. Inflation is also higher under the robust Ramsey plan. The right panel of Figure 3 reports these differences under the worst-case model for the cost process. We initialize the calculation at

$$
\begin{bmatrix} c_0 \\ \Psi_0 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} .0249 \\ 0 \\ 0 \end{bmatrix},
$$

where .0249 is the mean of the cost-push shock under the worst-case model. Again the output gap and inflation are higher under this robust Ramsey plan. If the worst-case model for the cost-push shock were to be completely trusted by a Ramsey planner, he would choose the same plan as the robust Ramsey planner. As a consequence, the output gap starts at $y^*$ and diminishes to zero. The optimal plan under the approximating model starts lower and diminishes to zero. The percentage differences depicted in the right panel of Figure 3 are substantially larger than those depicted in the left panel.

To summarize our results for type I ambiguity, while the impulse response function depend very little on whether or not the robustness adjustment is made, shifts in constant terms do have a nontrivial impact on the equilibrium dynamics that are reflected in transient responses from different initial conditions.
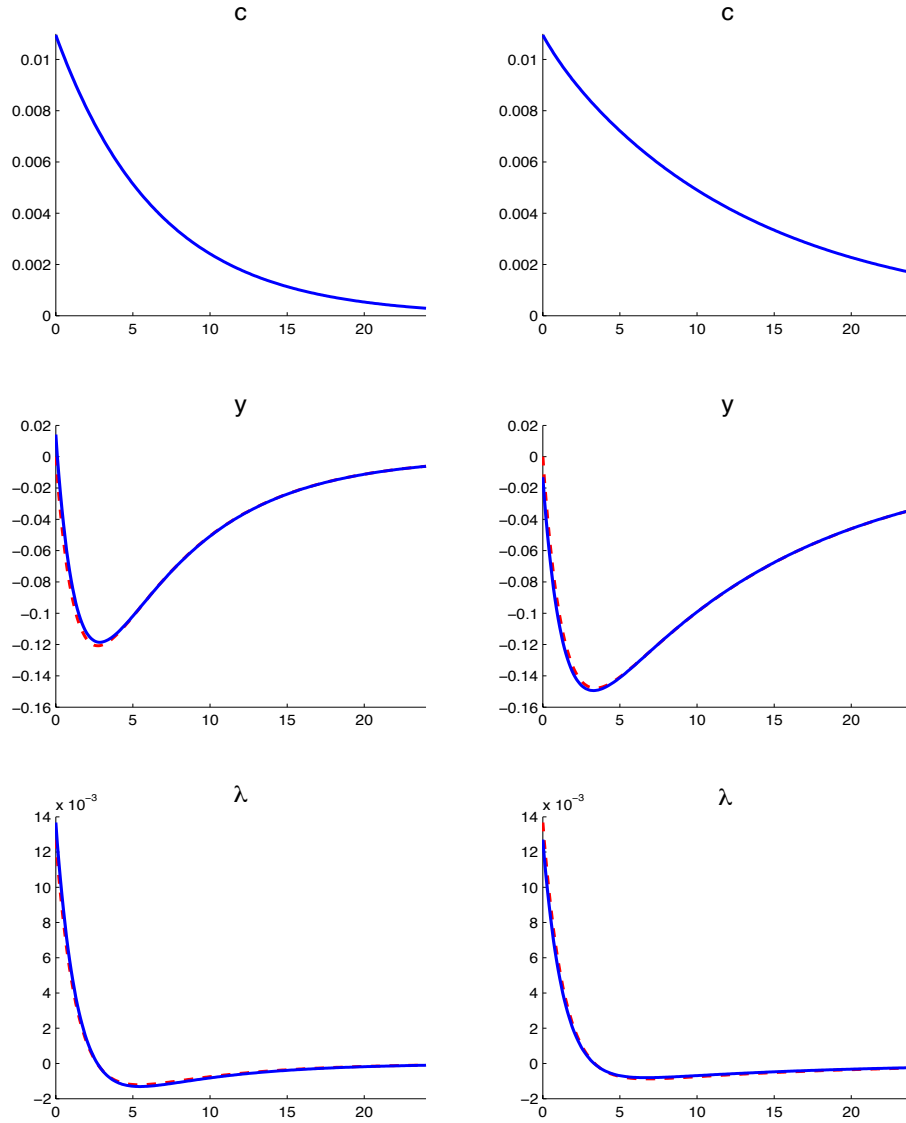
Impulse-response functions



Figure 2: The left panels assume the approximating model for the cost process, and the right panels assume the worst-case models for the cost process. The top panels give the impulse response functions for the cost process, the middle panels for the logarithm of the output gap, the bottom panels for inflation. The dashed line uses the approximating model solution and the solid line uses the worst-case model solution. The time units on the horizontal axis are quarters.

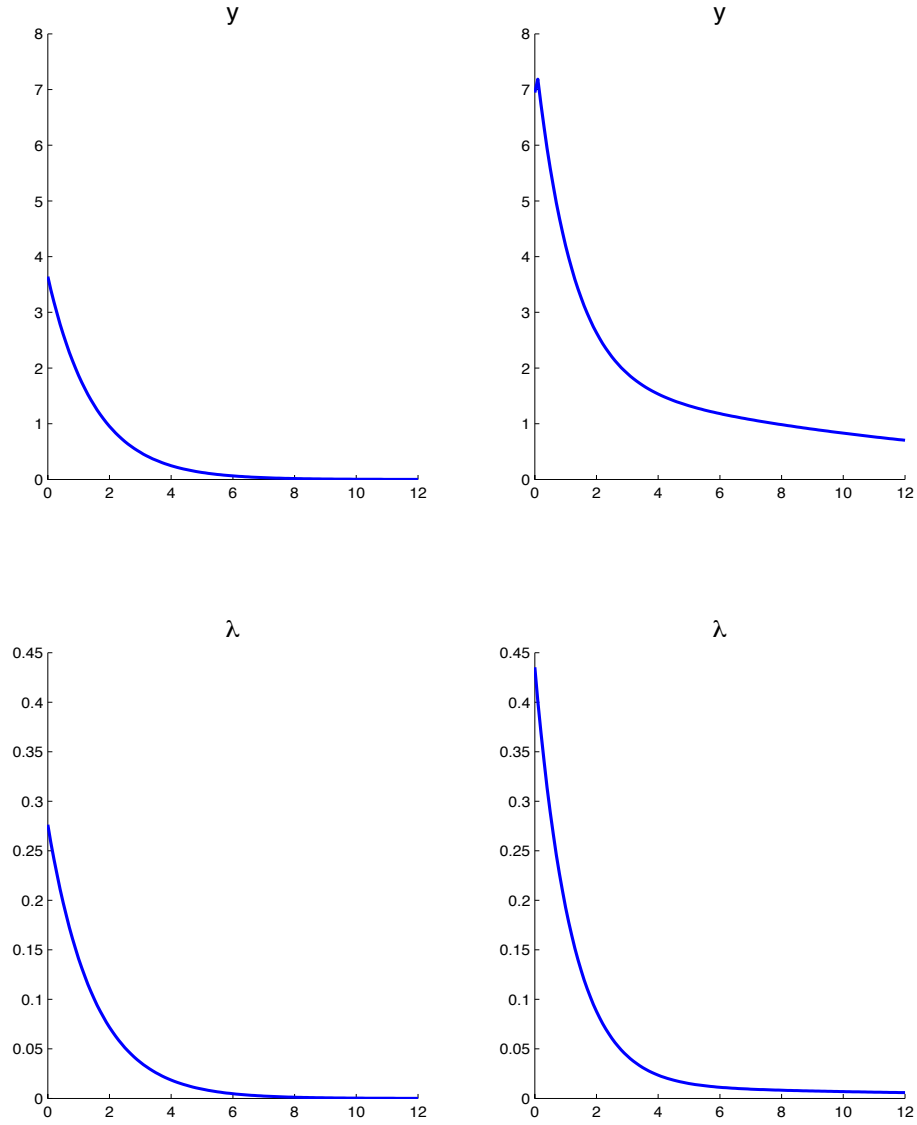Extrapolation from alternative initial conditions



Figure 3: The left panels assume the approximating model for the cost process initialized at its unconditional mean, 0. The right panels assume the worst-case models for the cost process initialized at its unconditional mean, .0249. The top panels give trajectory differences without shocks for the logarithm of the output gap (times one hundred), and the bottom panels give trajectory differences (times one hundred) for inflation without shocks. The time units on the horizontal axis are quarters.

## 10.2 Comparing Types II and III to Type I

To compare Type I with Types II and III, we compute derivatives for the worst-case drift distortion and for the decision rule for $\lambda$. The worst-case drift coefficients are shown in table 1. Notice the structure in these coefficients. Recall that the Type II problem has the

| Ambiguity Type | c | $\xi$ | 1 |
|---|---|---|---|
| I | .4752 | .1271 | .0111 |
| II | .4752 | 0 | .0111 |
| III | 0 | - .1271 | 0 |

Table 1: Coefficients for the derivatives of the drift distortion with respect to $\gamma$ times 10.

private sector embracing the approximating model, and that this substantially limits the impact of robustness. The coefficient on the (transformed) commitment multiplier is zero, but the other two coefficients remain the same as in Type I. In contrast, for Type III only the coefficient on $\xi$ is different from zero. The coefficient is the negative of that for Type I because the Ramsey planner now embraces the approximating model in contrast to Type I. Since the constant term is zero for Type III, the impact of robustness for a given value of $\theta$, say $\theta = .014$ as in our previous calculations, will be small. A calibration of $\theta$ using statistical criteria in the style of Anderson et al. (2003) would push us to much lower values of $\theta$.

Robustness also alters the decision rule for $\lambda$ as reflected in the derivatives with respect to $\gamma$, as shown in table 2. The Type II adjustments are evidently zero because the private

| Ambiguity Type | c | $\xi$ | 1 |
|---|---|---|---|
| I | 0.0854 | 0.0114 | 0.0022 |
| II | 0.0000 | 0.0000 | 0.0000 |
| III | 0.0154 | 0.0114 | 0.0002 |

Table 2: Coefficients for the derivatives for inflation with respect to $\gamma$ times 100.

sector embraces the approximating model. Type III derivatives are relatively small for the coefficients on $c_t$ and $\xi_t$.

While we find these derivatives to be educational, the numerical calculations for Type I reported in section 10 are apparently outside the range to which a linear approximation in $\gamma$ is accurate. This suggests that better numerical approximations to the HJB equations

for Type II and III ambiguity will be enlightening. We defer such computations to future research.

# 11    Concluding remarks

This paper has made precise statements about the seemingly vague topic of model ambiguity within a setting with a timing protocol that allows a Ramsey planner who is concerned about model misspecification to commit to history-contingent plans to which a private sector adjusts. There are different things that decision makers can be ambiguous about, which means that there are different ways to formulate what it means for either the planner or the private agents to experience ambiguity. We have focused on three types of ambiguity. We chose these three partly because we think they are intrinsically interesting and have potential in macroeconomic applications, and partly because they are susceptible to closely related mathematical formulations. We have used a very simple New Keynesian model as a laboratory to sharpen ideas that we aspire to apply to more realistic models.

We are particularly interested in type II ambiguity because here there is endogenous belief heterogeneity. Since our example precluded endogenous state variables other than a commitment multiplier, robustness influenced the Ramsey planner's value function but not Ramsey policy rules. In future research, we hope to study settings with other endogenous state variables and with pecuniary externalities that concern a Ramsey planner and whose magnitudes depend partly on private-sector beliefs.

In this paper, we started with a model that might be best be interpreted as the outcome of a log-linear approximation, but then ignored the associated approximation errors when we explored robustness. Interestingly, even this seemingly log-linear specification ceased to be log-linear in the presence of the Type II and Type III forms of ambiguity. In the future, we intend to analyze more fully the interaction between robustness and approximation. The small noise and small robustness expansions and related work in Adam and Woodford (2011) are steps in this direction, but we are skeptical about the sizes of the ranges of parameters to which these local approximations apply and intend to explore global numerical analytic approaches. Our exercises in the laboratory provided by the New Keynesian model of this paper should pave the way for attacks on more quantitatively ambitious Ramsey problems.

# A    Some basic proofs

Lemma 3.3 is a special case of Lemma 6.2 with $z^+ = z > 0$, $\psi^+ = \xi^+$ and $\psi = \xi$. We restate and prove Lemma 6.2.

**Lemma A.1.** *The multiplier $\ell_1$ on constraint (33) equals $\left(\frac{z^+}{z}\right)\xi^+$ and the multiplier $\ell_2$ on constraint (34) equals zero. Furthermore,*

$$y = y^* - \left(\frac{\kappa}{\zeta}\right)(\xi + \epsilon\lambda),$$

*where $\lambda = F(x, \xi)$ solves the private firm's Euler equation (33). Finally, $V_2(x, \xi) = -F(x, \xi)$.*

*Proof.* From relation (32)

$$\frac{\partial}{\partial\xi^+}\left[V^+(x^+, \xi^+) + \xi^+ F^+(x^+, \xi^+)\right] = \xi^+ F_2^+(x^+, \xi^+).$$

Differentiate the Lagrangian with respect to $\xi^+$ to obtain

$$-\left(\frac{z^+}{z}\right)\ell_2 - \ell_1\exp(-\delta\epsilon)F_2^+(x^+, \xi^+) + \exp(-\delta\epsilon)\left(\frac{z^+}{z}\right)\xi^+ F_2^+(x^+, \xi^+) = 0.$$

Taking conditional expectations gives

$$-\ell_2 + \left[\left(\frac{z^+}{z}\right)\xi^+ - \ell_1\right]\exp(-\delta\epsilon)E\left[F_2^+(x^+, \xi^+)|x, \xi\right] = 0$$

so that

$$\ell_2\left(1 - \epsilon\exp(-\delta\epsilon)E\left[F_2^+(x^+, \xi^+)|x, \xi\right]\right) = 0.$$

We conclude that $\ell_1 = \left(\frac{z^+}{z}\right)\xi^+$. The relation $V_2(x, \psi) = -F(x, \psi)$ follows from an envelope condition. $\qquad\square$

Next we verify that HJB equation (20) or (36) assures that the firm's Euler equation is satisfied. We carry out this verification for HJB equation (36), but the same argument applies for HJB equation (20) after we set $h = 0$ and $\xi = \psi$. Differentiating the objective

of the planner with respect to $\xi$ and using $V_2 = -F$ gives

$$
\begin{aligned}
0 = & -\kappa y - c - c^* + \delta F - F_1 \cdot \mu_x - F_2 \lambda \\
& - (F_1)' \sigma_x h + \xi F_{12} \sigma_x h - \frac{1}{2} \xi^2 F_{22} |h|^2 \\
& - \frac{1}{2} \text{trace} \left( \sigma_x' F_{11} \sigma_x \right) + (F_1)' \sigma_x h - \xi F_2 |h|^2,
\end{aligned}
$$

where we have used the envelope condition to adjust for optimization. Multiplying by minus one and simplifying gives

$$
\begin{aligned}
0 = & \kappa y + c + c^* - \delta F + F_1 \cdot \mu_x + F_2 \lambda + \xi F_2 |h|^2 \\
& + \frac{1}{2} \text{trace} \left( \sigma_x' F_{11} \sigma_x \right) - \xi F_{12} \sigma_x h + \frac{1}{2} \xi^2 F_{22} |h|^2.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\mu_{\lambda,t} = & F_1(x_t, \psi_t) \cdot \mu_x(x_t) + F_2(x_t, \psi_t) \lambda_t + \xi_t F_2(x_t, \psi_t) |h_t|^2 \\
& + \frac{1}{2} \text{trace} \left[ \sigma_x(x_t)' F_{11}(x_t, \psi_t) \sigma_x(x_t) \right] - \xi_t F_{12}(x_t, \xi_t) \sigma_x(x_t) h_t + \frac{1}{2} (\xi_t)^2 F_{22}(x_t, \xi_t) |h_t|^2.
\end{aligned}
$$

Thus, the Euler equation $\mu_{\lambda,t} = -\kappa y_t - c_t - c^* + \delta F(x_t, \psi_t)$ is satisfied.

# B    Example without robustness

If we suppose the exogenous linear dynamics

$$
\begin{aligned}
dx_t &= A_{11} x_t dt + \sigma_x dw_t \\
c_t &= H \cdot x_t,
\end{aligned}
$$

where $\sigma_x$ is a vector of constants, it is natural to guess that the Ramsey planner's value function is quadratic:

$$
V(x, \psi) = \frac{1}{2} \begin{bmatrix} x & \psi & 1 \end{bmatrix} \Lambda \begin{bmatrix} x \\ \psi \\ 1 \end{bmatrix} + \mathsf{v}.
$$

Then

$$F(x, \psi) = - \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Lambda \begin{bmatrix} c \\ \psi \\ 1 \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} - \frac{\delta}{2} & 0 & 0 \\ 0 & -\frac{\delta}{2} & 0 \\ 0 & 0 & -\frac{\delta}{2} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & -H & 0 \\ -H' & \frac{\kappa^2}{\zeta} & -\kappa y^* - c^* \\ 0 & -\kappa y^* - c^* & 0 \end{bmatrix}.$$

The matrix $\Lambda$ solves what is not quite a standard Riccati equation because the matrix $Q$ is indefinite:

$$-\Lambda B B' \Lambda + A' \Lambda + \Lambda A + Q. \tag{53}$$

The last thing to compute is the constant

$$\mathsf{v} = \frac{(\sigma_c)^2}{\delta} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We have confirmed numerically that we can compute the same Ramsey plan by using either the sequential formulation of section 3.2 that leads us to solve for the stabilizing solution of a linear equation system or the recursive method of section 3.3 that leads us to solve the Riccati equation (53). We assume the parameter values:

$$\begin{aligned} \delta &= .0101 & A_{11} &= -.15 \\ \kappa &= .05 & H &= 1 \\ \zeta &= .005 & \sigma_x &= \sqrt{.3} \times .02 \\ y^* &= .2 & c^* &= 0 \end{aligned}$$

Most of these parameter values are borrowed from Woodford (2010). Woodford takes the cost shock to be independent and identically distributed. In our continuous-time specification, we assume an AR process with the same unconditional standard deviation .02 assumed by Woodford.

The Matlab Riccati equation solver `care.m` applied to (53) gives[14]

$$F(c, \psi) = \begin{bmatrix} 1.1599 & -0.7021 & 0.0140 \end{bmatrix} \begin{bmatrix} c \\ \psi \\ 1 \end{bmatrix}$$

$$d \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} = \begin{bmatrix} -0.15 & 0 \\ 1.1599 & -0.7021 \end{bmatrix} \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.014 \end{bmatrix} dt + \begin{bmatrix} .011 \\ 0 \end{bmatrix} dw_t$$

$$V = \begin{bmatrix} -4.3382 & -1.1599 & -0.1017 \\ -1.1599 & 0.7021 & -0.0140 \\ -0.1017 & -0.0140 & -0.0195 \end{bmatrix}$$

# C  Example with first type of ambiguity

For our linear-quadratic problem, it is reasonable to guess that the value function is quadratic:

$$V(c, \psi) = \frac{1}{2} \begin{bmatrix} c & \psi & 1 \end{bmatrix} \Lambda \begin{bmatrix} c \\ \psi \\ 1 \end{bmatrix} + \mathsf{v}.$$

Then

$$F(x, \psi) = - \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Lambda \begin{bmatrix} c \\ \psi \\ 1 \end{bmatrix}.$$

---

[14]As expected, the invariant subspace method for solving (9), (1), and (3) gives identical answers.

Let

$$B = \begin{bmatrix} 0 & \sigma_c \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} - \frac{\delta}{2} & 0 & 0 \\ 0 & -\frac{\delta}{2} & 0 \\ 0 & 0 & -\frac{\delta}{2} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & -H & 0 \\ -H' & \frac{\kappa^2}{\zeta} & -\kappa y^* - c^* \\ 0 & -\kappa y^* - c^* & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix}.$$

The matrix $\Lambda$ solves

$$-\Lambda B R^{-1} B' \Lambda + A' \Lambda + \Lambda A + Q.$$

Again, this Riccati equation is not quite standard because the matrix $Q$ is indefinite. Finally,

$$v = \frac{(\sigma_c)^2}{\delta} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

### C.0.1  Example

Parameter values are the same as those in Appendix B except that now $\theta = .014$.

Using the Matlab program `care`,

$$\lambda = F(c, \psi) = \begin{bmatrix} 1.2485 & -0.6926 & 0.0173 \end{bmatrix} \begin{bmatrix} c \\ \psi \\ 1 \end{bmatrix} \tag{54}$$

$$h = \begin{bmatrix} 4.7203 & 0.9769 & 0.1556 \end{bmatrix} \begin{bmatrix} c \\ \psi \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} -6.0326 & -1.2485 & -0.1988 \\ -1.2485 & 0.6926 & -0.0173 \\ -0.1988 & -0.0173 & -0.0630 \end{bmatrix}.$$

The function $\tilde{F}$ that emerges by solving the Ramsey problem without robustness is

$$\tilde{F}(c, \Psi, \psi) = \begin{bmatrix} 1.2485 & 0.0095 & -0.7021 & 0.0173 \end{bmatrix} \begin{bmatrix} c \\ \Psi \\ \psi \\ 1 \end{bmatrix}.$$

Notice that the first coefficient and last coefficients equal the corresponding ones on the right side of (54) and that the sum of the second two coefficients equals the second coefficient in (54).

# D   Sensitivity to robustness

To compute the first-order adjustments for robustness, form

$$-H(x, \psi) = \frac{1}{2} \begin{bmatrix} x' & \xi & 1 \end{bmatrix} \Upsilon \begin{bmatrix} x \\ \xi \\ 1 \end{bmatrix}.$$

Guess a solution of the form

$$-G(x, \psi) = \frac{1}{2} \begin{bmatrix} x' & \xi & 1 \end{bmatrix} \Gamma \begin{bmatrix} x \\ \xi \\ 1 \end{bmatrix}.$$

The Lyapunov equation

$$(A^*)'\Gamma + \Gamma A^* + \Upsilon = 0$$

can be solved using the Matlab routine `lyap`. We used this approach to compute the derivatives reported in section 9.

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