

Syllabus

Wednesday, September 07, 2011
11:40 AM

Thomas Sargent
New York University
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Econ UA 367 - Econ UB 234

Syllabus

Math and advanced macro and finance

1. Math tools

Laplace transform

Z-transform

state space representation of linear difference equation

stochastic linear difference equation

Forecasting a geometric sum of future random variables

"Solving stable roots backwards and unstable roots forward"

or

Effects of foresight and transient decay

Kuhn-Tucker and the min-max theorem

Markov chains - joint distributions over histories that they induce

Population least squares projection

2. "A little knowledge of geometric series goes a long way"

- A linear asset pricing formula
- "dynamic programming solution" of asset pricing formula
- "bubbles"
- Friedman's permanent income model with rational expectations .
- Borrowing limits .
- Friedman meets Math
- consumption growth as forecaster of income
- Theory of interest rates - term structure

3. Complete markets

- The event tree & the joint distribution
- Arrow-Debreu complete markets
- Generalization of Lucas model —
asset pricing in an endowment economy .

4. Financial Crises and deposit insurance

- The Diamond-Dybvig model
- The Kaseken-Wallace model
- Problems with "too big to fail"

5. Growth theory

- A Malthusian model
- The Solow model
- The Cass-Koopmans optimal growth model

6. Fiscal policy in the growth model

- Lump sum and distorting taxes on capital, labor, consumption
- The government budget constraint
- The shooting method
- Foreseen and unforeseen policy changes
- Linear approximations
- Feed back and feed forward contributions

7. Incomplete markets

8. Asset pricing in incomplete markets

- Stochastic discount factors
- Implications of $E_t m_{t+1} R_{t+1}^i = 1$
- The Hansen-Jagannathan bounds

- Equity premium and risk-free rate puzzles
- Tallarini's model
- affine models of the term structure of interest rates

Lecture 1

Wednesday, September 07, 2011
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Use full math:

$$t \geq 0, t \in \mathbb{R}^+$$

$$f(t) : T \rightarrow \mathbb{R}$$

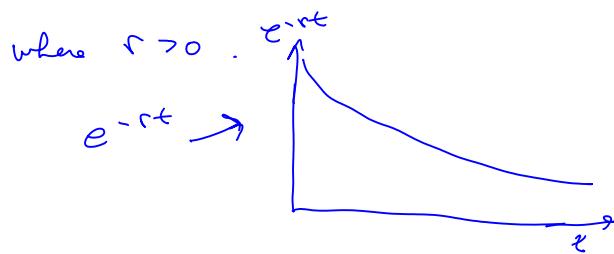
example ill $f(t) = e^{+\delta t}, \delta > 0$

(2) $f(t) = t e^{\delta t}, \delta > 0$

(3) $f(t) = t^n e^{\delta t}, \delta > 0, n$ a positive integer

Want to compute present value

$$F(r) = \int_0^\infty e^{-rt} f(t) dt$$



$r > 0$ is the "instantaneous interest rate"

$F(r)$ is the (one sided) Laplace transform
of $f(t)$.

$$F(r) \leftrightarrow f(t)$$

Remark: There are tables of Laplace transforms. —
see Wikipedia.

Example: $f(t) = B e^{+\delta t}, 0 < \delta < r$.

$$F(r) = B \int_0^\infty e^{-rt} e^{\delta t} dt$$

$$= B \int_0^\infty e^{-(r-\delta)t} dt$$

$$= -\frac{B}{(r-\delta)} e^{-(r-\delta)t} \Big|_0^\infty$$

$$F(r) = \frac{B}{r-\delta}$$

This is called the "Gordon formula".

Special case: $\delta = 0 \Rightarrow$

$$F(r) = \frac{B}{r}$$

— Ricardo formula
for price of a constant
income stream.

Richard Feynman secret weapon:

differentiate under integral sign.

$$F(r) = \int_0^\infty e^{-rt} f(t) dt$$

$$F'(r) = \int_0^\infty -t e^{-rt} f(t) dt$$

$$\Rightarrow -F'(r) \longleftrightarrow t f(t)$$

Application:

$$g(t) = B t e^{\delta t}, t \geq 0$$

$\not\propto f(t)$

$$F(r) = \frac{B}{r-\delta}$$

$$F'(r) = \frac{-B}{(r-\delta)^2}$$

$$\Rightarrow n, -\delta t \dots B - R \int_0^\infty -e^{-rt} t e^{\delta t} dt$$

$$\Rightarrow B + e^{-\delta t} \leftrightarrow \frac{B}{(s-\delta)^2} = B \int_0^\infty e^{-st} t e^{\delta t} dt$$

Exercise: show that

$$F''(s) \leftrightarrow s^2 f(t)$$

Digression: another trick with Laplace transforms

$$G(s) = \int_0^\infty e^{-st} f'(t) dt = s F(s) - f(0^-), \text{ where again } F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$\int u \cdot dv = uv - \int v \cdot du$$

$$dv = f'(t) dt \quad \cdot \quad v = f(t)$$

$$u = e^{-st} \quad du = -s e^{-st} dt$$

$$\int e^{-st} f'(t) dt = e^{-st} f(t) \Big|_0^\infty + s \int e^{-st} f(t) dt$$

$$f'(t) \leftrightarrow -f(0^-) + s F(s)$$

$$\text{e.g. } f(t) = e^{+\delta t}, \quad f'(t) = \delta e^{+\delta t}$$

$$\text{Then } F(s) = \frac{1}{s-\delta}$$

$$G(s) = \frac{s}{s-\delta} - 1 = \frac{s}{s-\delta} - \frac{s-\delta}{s-\delta} = \frac{\delta}{s-\delta} \quad \checkmark$$

Discrete time formulas:

$$x_t, \quad t \geq 0, \quad t = 0, 1, 2, \dots$$

$$\text{e.g. } x_t = \lambda^t, \quad ,$$

To compute:

$$\sum_{t=0}^{\infty} \beta^t x_t \quad \text{where} \quad \beta = \frac{1}{1+r}$$

$\downarrow \rightarrow$

$r = \text{one-period interest rate}$

$$= \sum_{t=0}^{\infty} \beta^t r^t = \sum_{t=0}^{\infty} (\beta r)^t = \frac{1}{1-\beta r}$$

provided $|\beta r| < 1$.

Some matrix algebra

vector $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ column vector

$$x' = (x_1 \ x_2 \ \dots \ x_n)$$
 row vector

matrix $A = \begin{pmatrix} A_{11} & \dots & A_{1m} \\ A_{21} & \dots & A_{2m} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{nm} \end{pmatrix}$

$$\left[A_{ij} \right] \quad i = 1, \dots, n; j = 1, \dots, m$$

matrix multiplication:

$$\begin{matrix} C \\ \uparrow \\ n \times p \end{matrix} \quad \begin{matrix} A + B \\ \uparrow \\ m \times m \end{matrix} \quad \begin{matrix} \\ \uparrow \\ m \times p \end{matrix}$$

$$C_{ik} = \sum_j A_{ij} B_{jk}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \downarrow$$

$$C_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21}$$

Inverse of a matrix: A^{-1}

$$A^{-1} A = I = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & & & 0 & 0 & \ddots & 0 & 1 \end{pmatrix}}$$

identity matrix.

Matrix equation:

$$A x = b$$

$m \times n \quad n \times 1 \quad m \times 1$

$$A^{-1} A x = A^{-1} b \quad n$$

$$x = A^{-1} b$$

Application of matrix algebra to model dynamics.

First order linear difference equation:

$$x_{t+1} = A x_t \quad , A \text{ is } n \times n$$

$$y_t = G x_t \quad , G \text{ is } 1 \times n$$

x_0 given

Example:

$$y_{t+1} = f_1 y_t + f_2 y_{t-1}$$

$$1 + \dots - /y_0 \backslash$$

let $x_t = \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}$

Then

$$\begin{pmatrix} y_{t+1} \\ y_t \\ \vdots \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \\ 1 & 0 \\ \vdots & \vdots \\ A & \end{pmatrix} \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ x_t \end{pmatrix}$$

$$y_t = \begin{bmatrix} 1 & 0 \\ \vdots & \\ G & \end{bmatrix} \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ x_t \end{pmatrix}$$

Want:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} \quad \beta = \frac{1}{1+r}$$

↑ ↑ ↑
 asset price discount factor dividends

note: $x_{t+1} = A x_t$

$$x_{t+2} = A^2 x_t$$

$$\vdots$$

$$x_{t+\infty} = A^\infty x_t$$

$$\Rightarrow P_t = \sum_{j=0}^{\infty} \beta^j G A^j x_t$$

$$= G \sum_{j=0}^{\infty} (\beta A)^j x_t$$

remark: $\sum_{j=0}^{\infty} \beta^j A^j = (I - \beta A)^{-1}$

this converges if the maximum eigenvalue of A
is less than $\frac{1}{\beta}$ in modulus.

Eigenvalue of A : a scalar (possibly complex)
such that

$$Ax = \lambda x \quad \text{for some } x \neq 0. \text{ such an } x \text{ is called an } \underline{\text{eigenvector}}.$$

evidently

$$(*) \quad p_t = G(I - \beta A)^{-1} x_t$$

Other way to solve for p_t - "guess and verify"
or "undetermined coefficients" method

recursive argument

$$\begin{aligned} p_t &= \sum_{j=0}^{\infty} \beta^j y_{t+j} \\ &= y_t + \beta \sum_{j=0}^{\infty} \beta^j y_{t+j+1} \end{aligned}$$

$$p_t = y_t + \beta p_{t+1}$$

Guess $p_t = H x_t$ x_t is state

$$\begin{aligned} p_t &= y_t + \beta H x_{t+1}, \quad x_{t+1} = A x_t \\ y_t &= G x_t \end{aligned}$$

$$H x_t = G x_t + \beta H A x_t$$

$$H = G + \beta H A$$

$$H(I - \beta A) = G$$

$$H(I - \beta A) = G$$

$$(*) \quad H = G(I - \beta A)^{-1}.$$

$$\text{thus } p_t = G(I - \beta A)^{-1} x_t$$

Example:

$$y_{t+1} = .9 y_t + .04 y_{t-3}$$

Compute p_t for $\beta = .95$.

Probability theory:

$i = 1, \dots, I$ there are I possible "states of the world"

$$\pi_i = \text{Prob}\{ \text{state of world is } i \}$$

$$\pi_i \geq 0, \sum_{i=1}^I \pi_i = 1$$

A random variable - a vector

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_I \end{pmatrix}$$

where y_i is value in state of the world i .

mathematical expectation (mean)

$$\underline{\mathbb{E} y = \sum_{i=1}^I \pi_i y_i = \pi \cdot y}$$

\nearrow inner product

Here are some Laplace transform tables from Wikipedia:

Laplace transform - Wikipedia, the free encyclopedia - Mozilla Firefox

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W http://en.wikipedia.org/wiki/Laplace_transform

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	Time domain	's' domain	Comment
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$	Can be proved using basic rules of integration.
Frequency differentiation	$tf(t)$	$-F'(s)$	F' is the first derivative of F .
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, (n)th derivative of $F(s)$.
Differentiation	$f'(t)$	$sF(s) - f(0)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second Differentiation	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General Differentiation	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$	f is assumed to be n -times differentiable, with n^{th} derivative of exponential type. Follow by mathematical induction.
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma) d\sigma$	
Integration	$\int_0^t f(\tau) d\tau = (u * f)(t)$	$\frac{1}{s} F(s)$	$u(t)$ is the Heaviside step function. Note $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Scaling	$f(at)$	$\frac{1}{ a } F\left(\frac{s}{a}\right)$	

Done



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ID	Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence
1	ideal delay	$\delta(t - \tau)$	$e^{-\tau s}$	
1a	unit impulse	$\delta(t)$	1	all s
2	delayed n th power with frequency shift	$\frac{(t - \tau)^n}{n!} e^{-\alpha(t-\tau)} \cdot u(t - \tau)$	$\frac{e^{-\tau s}}{(s + \alpha)^{n+1}}$	$\text{Re}\{s\} > -\alpha$
2a	n th power (for integer n)	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	$\text{Re}\{s\} > 0$ ($n > -1$)
2a.1	q th power (for complex q)	$\frac{t^q}{\Gamma(q + 1)} \cdot u(t)$	$\frac{1}{s^{q+1}}$	$\text{Re}\{s\} > 0$ ($\text{Re}\{q\} > -1$)
2a.2	unit step	$u(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
2b	delayed unit step	$u(t - \tau)$	$\frac{e^{-\tau s}}{s}$	$\text{Re}\{s\} > 0$
2c	ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} > 0$
2d	n th power with frequency shift	$\frac{t^n}{n!} e^{-\alpha t} \cdot u(t)$	$\frac{1}{(s + \alpha)^{n+1}}$	$\text{Re}\{s\} > -\alpha$
2d.1	exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s + \alpha}$	$\text{Re}\{s\} > -\alpha$
3	exponential approach	$(1 - e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s + \alpha)}$	$\text{Re}\{s\} > 0$
4	sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2 + \omega^2}$	$\text{Re}\{s\} > 0$
5	cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2 + \omega^2}$	$\text{Re}\{s\} > 0$

Done



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	coseine	$\cos(\omega t) \cdot u(t)$	$\frac{s^2 + \omega^2}{\alpha}$	$\text{Re}\{s\} > 0$
6	hyperbolic sine	$\sinh(\alpha t) \cdot u(t)$	$\frac{\alpha}{s^2 - \alpha^2}$	$\text{Re}\{s\} > \alpha $
7	hyperbolic cosine	$\cosh(\alpha t) \cdot u(t)$	$\frac{s}{s^2 - \alpha^2}$	$\text{Re}\{s\} > \alpha $
8	Exponentially-decaying sine wave	$e^{\alpha t} \sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s - \alpha)^2 + \omega^2}$	$\text{Re}\{s\} > \alpha$
9	Exponentially-decaying cosine wave	$e^{\alpha t} \cos(\omega t) \cdot u(t)$	$\frac{s - \alpha}{(s - \alpha)^2 + \omega^2}$	$\text{Re}\{s\} > \alpha$
10	n th root	$\sqrt[n]{t} \cdot u(t)$	$s^{-(n+1)/n} \cdot \Gamma\left(1 + \frac{1}{n}\right)$	$\text{Re}\{s\} > 0$
11	natural logarithm	$\ln\left(\frac{t}{t_0}\right) \cdot u(t)$	$-\frac{t_0}{s} [\ln(t_0 s) + \gamma]$	$\text{Re}\{s\} > 0$
12	Bessel function of the first kind, of order n	$J_n(\omega t) \cdot u(t)$	$\frac{\omega^n (s + \sqrt{s^2 + \omega^2})^{-n}}{\sqrt{s^2 + \omega^2}}$	$\text{Re}\{s\} > 0$ ($n > -1$)
13	Modified Bessel function of the first kind, of order n	$I_n(\omega t) \cdot u(t)$	$\frac{\omega^n (s + \sqrt{s^2 - \omega^2})^{-n}}{\sqrt{s^2 - \omega^2}}$	$\text{Re}\{s\} > \omega $
14	Bessel function of the second kind, of order 0	$Y_0(\alpha t) \cdot u(t)$	$-\frac{2 \sinh^{-1}(s/\alpha)}{\pi \sqrt{s^2 + \alpha^2}}$	$\text{Re}\{s\} > 0$
15	Modified Bessel function of the second kind, of order 0	$K_0(\alpha t) \cdot u(t)$		
16	Error function	$\text{erf}(t) \cdot u(t)$	$\frac{e^{s^2/4} (1 - \text{erf}(s/2))}{s}$	$\text{Re}\{s\} > 0$

Done



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Formal definition [edit]

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

The parameter s is a complex number:

$$s = \sigma + i\omega,$$

with real numbers σ and ω .

The meaning of the integral depends on types of functions of interest. For functions that decay at infinity or are of exponential type, it can be understood as a (proper) Lebesgue integral. However, for many applications it is necessary to regard it as a conditionally convergent improper integral at ∞ . Still more generally, the integral can be understood in a weak sense, and this is dealt with below.

One can define the Laplace transform of a finite Borel measure μ by the Lebesgue integral:

$$(\mathcal{L}\mu)(s) = \int_{[0,\infty)} e^{-st} d\mu(t).$$

An important special case is where μ is a probability measure or, even more specifically, the Dirac delta function. In operational calculus, the Laplace transform of a measure is often treated as though the measure came from a distribution function f . In that case, to avoid potential confusion, one often writes

$$(\mathcal{L}f)(s) = \int_{0^-}^\infty e^{-st} f(t) dt$$

where the lower limit of 0^- is short notation to mean

$$\lim_{\varepsilon \rightarrow 0^+} \int_{-\varepsilon}^\infty.$$

This limit emphasizes that any point mass located at 0 is entirely captured by the Laplace transform. Although with the Lebesgue integral, it is not necessary to take such a limit, it does appear more naturally in connection with the Laplace–Stieltjes transform.

Probability theory [edit]

In pure and applied probability, the Laplace transform is defined by means of an expectation value. If X is a random variable with probability density function $f_X(x)$, then the Laplace transform is given by

$$\mathbb{E}[e^{-stX}] = \int_{-\infty}^\infty e^{-stx} f_X(x) dx.$$


Kuhn-Tucker

Wednesday, September 07, 2011
2:01 PM

Notes on Kuhn-Tucker -

(taken from Wikipedia, but changed
"min" to "max")

Problem: $\max f(x)$ s.t. $g(x) \geq 0$
 $h(x) = 0$
 x a scalar

Form:

$$L = f(x) + \mu g(x) + \lambda h(x)$$

First order necessary conditions:

$$f'(x) + \mu g'(x) + \lambda h'(x) = 0$$

$$g(x) \geq 0, \mu \geq 0$$

$$\mu g(x) = 0 ; \because h(x) = 0$$

e.g. $g(x) = x \Leftrightarrow x \geq 0$ non negativity constraint on x .

$$f'(x) + \lambda h'(x) + \mu = 0, \mu \cdot x = 0 \Rightarrow$$

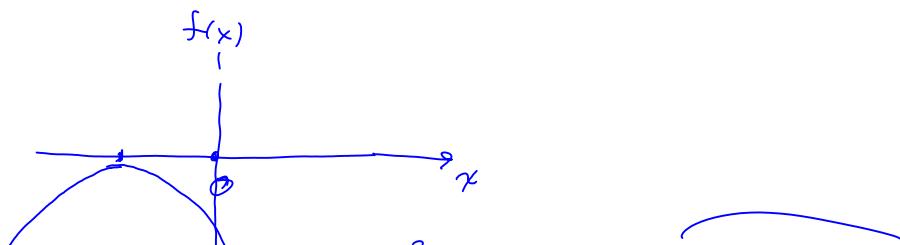
\Rightarrow

$$f'(x) + \lambda h'(x) \leq 0, = \text{if } x > 0$$

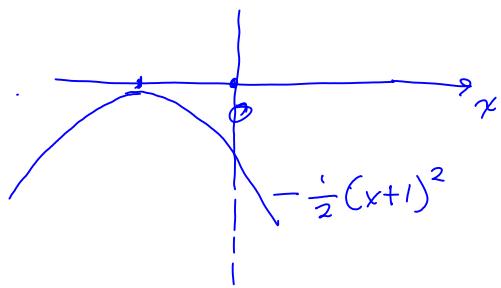
because $\mu x = 0$
and $\mu \geq 0$.

$$f(x)$$

Example:



E xample:



$$-\frac{1}{2}(x+1)^2$$

$$\max -\frac{1}{2}(x+1)^2 \quad \text{s.t. } x \geq 0$$

$$\begin{aligned} f'(x) &= -(x+1) = 0 \\ \Rightarrow x+1 &= 0 \\ x &= -1 \end{aligned}$$

$$L = -\frac{1}{2}(x+1)^2 + \mu x$$

$$\text{FONC: } x: - (x+1) + \mu = 0$$

$$-(1+x) + \mu = 0 \quad (\star)$$

$$\stackrel{\sigma}{=} (x+1) \leq 0, \quad = 0 \quad \text{if } x > 0$$

$$\mu \cdot x = 0,$$

$$\text{Solv is } x = 0 \text{ and}$$

$$-(1+x) + \mu = 0$$

$$-1 + \mu = 0 \Rightarrow \mu = 1.$$

$$\mu x = 0.$$

So far this problem, we can write the Kuhn-Tucker conditions in shorthand as

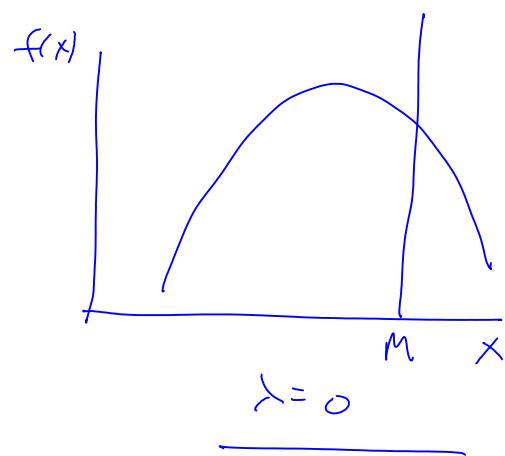
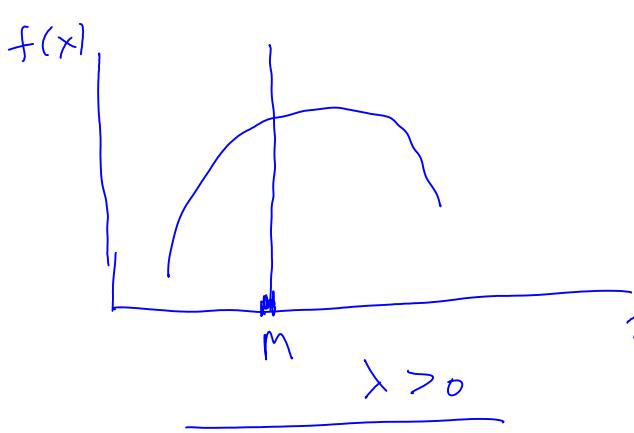
$$-(x+1) \leq 0, \quad = 0 \quad \text{if } x > 0$$

$$\Rightarrow x = 0.$$

Another example:

Another example:

$$\max f(x) \text{ s.t. } x \leq m$$



$$\text{Func} \quad L = f(x) + \lambda [m - x], \quad \lambda \geq 0$$

$$\text{FONC: } x: \quad f'(x) - \lambda = 0$$

$$\text{and } \lambda [m - x] = 0, \quad \lambda \geq 0$$

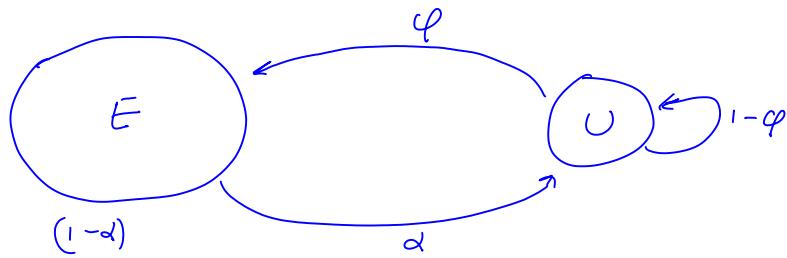
$$\text{if } \lambda > 0 \Rightarrow m - x = 0 \Rightarrow x = m$$

$$\text{if } \lambda = 0, \text{ can have } f'(x) = 0$$

Lake model

Saturday, September 10, 2011
9:54 PM

Lake model of gross labor market flows



Markov Chain: transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \varphi & 1-\varphi \end{pmatrix} \quad \begin{array}{l} \text{state 1} \leftrightarrow E \\ \text{state 2} \leftrightarrow U \end{array}$$

To compute invariant distribution (i.e. "long run"

distribution π = prob of E

$1-\pi$ = prob of U

method: $\begin{pmatrix} \pi \\ 1-\pi \end{pmatrix}$ is a left eigenvector of P
affiliated with a unit eigenvalue

thus it solves

$$\begin{bmatrix} \pi & 1-\pi \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \varphi & 1-\varphi \end{bmatrix} = \begin{bmatrix} \pi & 1-\pi \end{bmatrix}$$

\Rightarrow

$$\pi(1-\alpha) + \varphi(1-\pi) = \pi$$

$$-\alpha\pi + (1-\pi)\varphi = 0$$

$$-\varphi - \pi(\alpha + \varphi) = 0$$

$$\pi = \frac{\alpha}{\alpha + \varphi} = \frac{\text{probability of being employed}}{E}$$

$$1-\pi = \frac{\varphi}{\alpha + \varphi} = \frac{\text{probability of being unemployed}}{U}$$

$$1 - \pi = \frac{\frac{1}{\phi}}{\frac{1}{\phi} + \frac{1}{d}}$$

divide both numerator
& denominator by
 $d - \phi$

Some numbers for U.S. - 2007

(Source : President's Economic Report, 2009).

average duration of unemployment = 16.8 weeks

median " " = 8.5 weeks

Civilian Unemployment rate: 4.7%

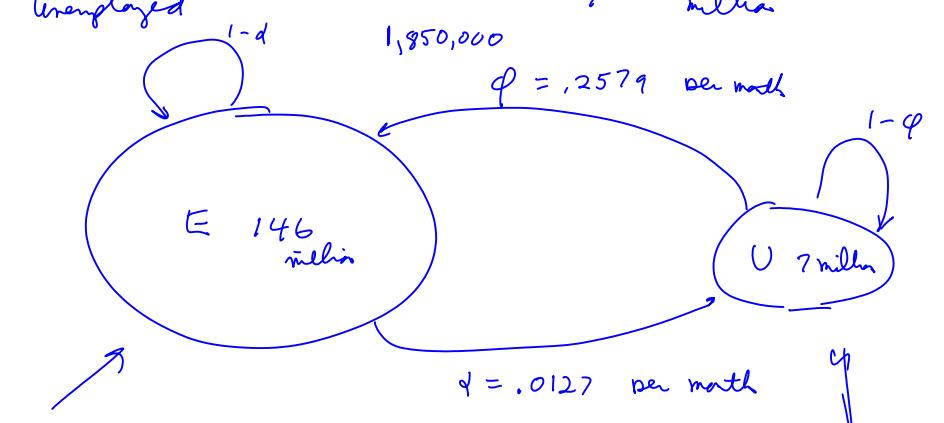
α employment / population 63.0%

labor force / population 66%

Civilian population (non institutional) 231 million

Civilian labor force 153 million

Unemployed ? million



average duration
78.62 months
= 6.55 years

average duration
3.87 months

$\approx 1,850,000$ jobs created

1,850,000 jobs destroyed.

$$\frac{U}{\text{labor force}} = \frac{\frac{1}{\phi}}{\frac{1}{\alpha} + \frac{1}{\phi}} = \frac{\alpha}{\alpha + \phi}$$

$$\text{Inflows} \quad \frac{1}{q} + \frac{1}{\alpha} \quad \alpha + q$$

$$\downarrow \text{rate inflow, } \quad \uparrow \text{1-outflow}$$

$$u_{bt+1} = \alpha(1-u_b) + (1-q)u_b$$

this eqn describes unemploy dynamics

$$u = \frac{\cup}{\text{labor force}} = \frac{\text{unemploy.}}{\text{rate}}$$

Note: to estimate α we took the 2007 data

$$\text{on } u = .047 \text{ and } q = .2579 \text{ per month}$$

$$\text{and then solved } u = \alpha(1-u) + (1-q)u$$

$$\text{for } \alpha, \Rightarrow$$

$$\alpha = q \frac{u}{1-u} = .0127 \text{ per month.}$$

Then we computed the gross flows from , eq.

$$\text{gross job creation} = \cup \cdot q$$

Z-transforms

Friday, September 09, 2011
12:33 PM

\mathbb{Z} -transform

Unilateral \mathbb{Z} -transform

$\{x_n\}_{n=0}^{\infty}$ a one-sided sequence

$\{x_n\} \leftrightarrow X(z)$, $z \in \mathbb{C}$ space of complex variables

$X(z) = \sum_{n=0}^{\infty} x_n z^n$, z^n is a "generating function"

differentiate w.r.t. $z \Rightarrow$

$$X'(z) = \sum_{n=0}^{\infty} n x_n z^{n-1} = z^{-1} \sum_{n=0}^{\infty} n x_n z^n$$

or
 $z X'(z) = \sum_{n=0}^{\infty} n x_n z^n$

so
 $z X'(z) \leftrightarrow \{n x_n\}$

Example:

let $x_n = \lambda^n$, $\lambda \in \mathbb{R}$

$$X(z) = \sum_{n=0}^{\infty} \lambda^n z^n = \frac{1}{1-\lambda z}$$

$$X'(z) = \frac{\lambda}{(1-\lambda z)^2}$$

$$z X'(z) = \frac{\lambda z}{(1-\lambda z)^2}$$

$$\{\dots, \lambda^n\} \leftrightarrow \frac{\lambda z}{(1-\lambda z)^2}$$

Application 1: Compute present value of $\{n\lambda^n\}_{n=0}^{\infty}$:

$$v(\beta) = \sum_{n=0}^{\infty} n \lambda^n \beta^n$$
$$= \frac{\lambda \beta}{(1 - \lambda \beta)^2} .$$

Application 2: let $\{p_j\}_{j=0}^{\infty}$ with $p_j \geq 0$, $\sum_{j=0}^{\infty} p_j = 1$

be a probability density for a nonnegative random variable N taking values

$j=0, 1, 2, 3, \dots$. Here

$$p_j = \text{Prob}(N=j) .$$

The probability generating function is

$$P(z) = \sum_{j=0}^{\infty} p_j z^j .$$

here $\text{Prob}\{N=j\} = \text{coefficient on } j^{\text{th}} \text{ power of } z$.

The mean of N is defined as

$$\mu = \sum_{j=0}^{\infty} j p_j .$$

Exercise: Use the probability generating function to compute μ .

... .

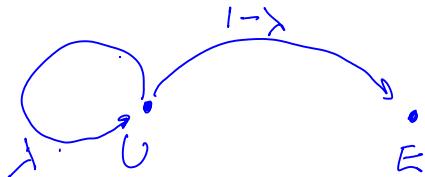
Note:

$$z^j P'(z) = \sum_{j=0}^{\infty} j p_j z^j$$

Evaluate at $z=1$ to get

$$P'(1) = \sum_{j=0}^{\infty} j \cdot p_j = M$$

Example: waiting time distribution.
Each period, an unemployed person finds a job with probability
 $0 < (1-\lambda) < 1$. With probability λ , each period the person
unemployed person remains unemployed. So the picture is



Let N be the random variable "length of time it takes to find a job", with $N=1$ meaning that it takes 1 period.

let $p_j = \text{probability } (N=j)$

Evidently,

$$p_1 = (1-\lambda) \quad \text{succeed}$$

$$p_2 = \lambda(1-\lambda) \quad \text{fail, succeed}$$

$$p_3 = \lambda^2(1-\lambda) \quad \text{fail, fail, succeed}$$

$$p_4 = \lambda^3(1-\lambda) \quad \underbrace{\text{fail, fail, fail}}_{\lambda^3}, \text{succeed}$$

$$\text{Thus } P(z) = (1-\lambda) \sum_{j=1}^{\infty} \lambda^{j-1} z^j \quad \begin{matrix} \leftarrow \text{ note the sum} \\ \text{starts at } j=1. \end{matrix}$$

j=1

starts at j=1.

$$P_0 = 0$$

The mean waiting time is $\geq P(z)$:

But $P(z) = \frac{(1-\lambda)z}{(1-\lambda z)}$

$$P'(z) = \frac{(1-\lambda)(1-\lambda z) + \lambda(-\lambda z)}{(1-\lambda z)^2}$$

$$= \frac{(1-\lambda) \cdot 1}{(1-\lambda z)^2}$$

$$\geq P'(z) = \frac{(1-\lambda)}{(1-\lambda z)^2}$$

$$P'(1) = \frac{1}{(1-\lambda)}$$

= mean waiting time for
a geometric distribution

Bubbles

Monday, September 12, 2011
11:16 AM

More asset pricing.

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j}, \quad \beta = \frac{1}{1+r}$$

$$\text{For example, } r = .05 \Rightarrow \beta = \frac{1}{1.05} \approx .95$$

$$P_t = y_t + \beta y_{t+1} + \beta^2 y_{t+2} + \dots$$

$$P_{t+1} = y_{t+1} + \beta y_{t+2} + \beta^2 y_{t+3} + \dots$$

$$\beta P_{t+1} = \beta y_{t+1} + \beta^2 y_{t+2} + \dots$$

$$\therefore P_t - \beta P_{t+1} = y_t$$

on

$$(*) \quad P_t = y_t + \beta P_{t+1}$$

(*) is a difference equation.

Take a model of y_t :

$$X_{t+1} = A X_t$$

$$y_t = G X_t$$

we have shown

$$(*) \quad P_t = G(I - \beta A)^{-1} X_t$$

is a solution of the difference equation (*)

Question : is (†) the unique solⁿ of (*) ?

Answer : No. To show this, consider the special case

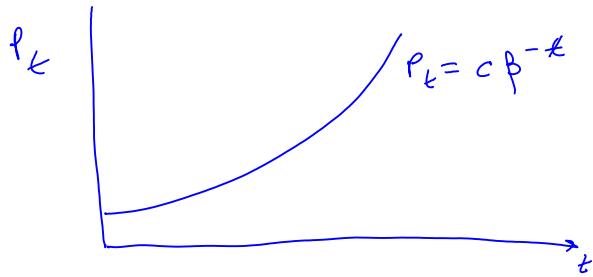
$$(\text{e}^{\text{to}}) \quad p_t = \beta p_{t+1}$$

here $y_t = 0$. Clearly $p_t = 0$ is a solⁿ of (*)[†]. But so is

$$p_t = c \left(\frac{1}{\beta} \right)^t = c \beta^{-t}$$

for any constant c .

This is a pure bubble, because $\beta < 1$.



The general solⁿ of (*) is

$$p_t = G(I - \beta A)^{-1} x_t + c \beta^{-t}$$

for any c . The solⁿ is not unique.

$c \beta^{-t}$ is the bubble.

"... = 1. + ... + 1, i.e."

"prices rise because they are expected to rise"

Homework 1

Wednesday, September 14, 2011
1:05 PM

Homework number 1.

1. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers.

Let $\beta \in (0, 1)$ be a constant

discount factor.

The present value of $\{x_n\}_{n=0}^{\infty}$ is

$$PV(\beta) = \sum_{n=0}^{\infty} \beta^n x_n .$$

Please compute the present values of the following sequences as functions of β :

a. $x_n = A \lambda^n , n \geq 0 , |\lambda| < \frac{1}{\beta}$

b. $x_n = A n \lambda^n , n \geq 0 , |\lambda| < \frac{1}{\beta}$

c. $x_n = A_0 n \lambda^n + A_1 n^2 \delta^n , |\lambda| < \frac{1}{\beta}$
 $n \geq 0 \quad |\delta| < \frac{1}{\beta}$

d. $x_n = A n^3 \lambda^n , n \geq 0 , |\lambda| < \frac{1}{\beta}$

2. A stock price P_t obeys the following formula:

$$(*) \quad p_t = y_t + \beta p_{t+1}$$

↑ ↑ ↑
 price today dividend today discount factor price tomorrow

The dividend obeys

$$n \times 1 \quad n \times n \quad n \times 1$$

$$x_{t+1} = A x_t, \text{ where maximum eigenvalue is less than } \frac{1}{\beta} \text{ in modulus}$$

$$y_t = G x_t$$

$$1 \times 1 \quad 1 \times n \quad n \times 1$$

A friend of yours guesses that (*) implies that the following equation for p_t must be true that

$$p_t = H x_t + c \lambda^t$$

for a matrix $H_{1 \times n}$ and scalars c and λ .

- Get as far as you can in finding formulas for H , λ , and c . (Hint: use the guess and verify method.)
- Is H unique? Is λ unique? Is c unique?
- "Finding the state is an art"

A dividend process obeys the law of motion

$$y_{t+1} = \lambda_0 + \lambda_1 y_t + \lambda_2 y_{t-1}$$

The price of a stock obeys

$$p_t = \sum_{j=0}^{\infty} \beta^j y_{t+j}, \quad \beta \in (0, 1)$$

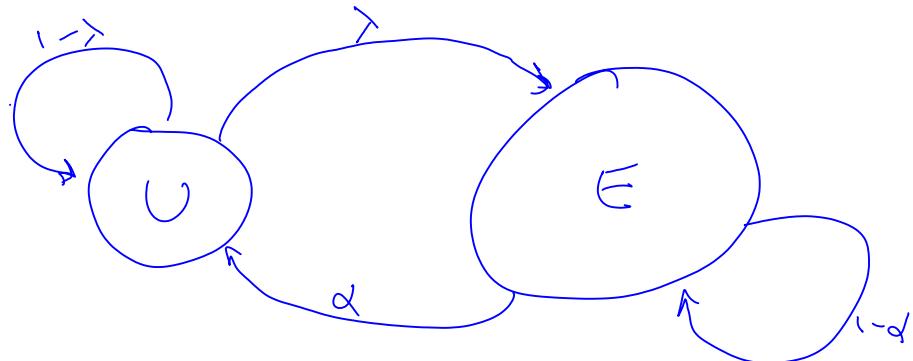
Please give another formula for p_t of the form

$$p_t = a_0 + a_1 y_t + a_2 y_{t-1}$$

Please tell how to compute a_0, a_1 , and a_2 .

4. There are two states U (for unemployment) and E (for employment). With probability $\lambda \in (0, 1)$ a person unemployed today becomes employed tomorrow. With probability $\alpha \in (0, 1)$ a person employed today becomes unemployed tomorrow.

So the situation is



a. Let $N \geq 1$ be the number of periods elapsed until a currently unemployed person becomes employed.

Let $p_n = \text{Prob}\{N = n\}$

Please compute p_n . Please compute

$$E N = \sum_{n=1}^{\infty} n p_n$$

b. Let $M \geq 1$ be the number of periods elapsed until a currently employed person loses his or her job. Please compute $q_m = \text{Prob}\{M > m\}$.

Please compute $E M = \sum_{m=0}^{\infty} q_m m$.

c. Please compute the fraction of time an infinitely lived person can expect to be unemployed, and the fraction that he or she can expect to be employed.

Permanent income model

Monday, September 12, 2011
12:08 PM

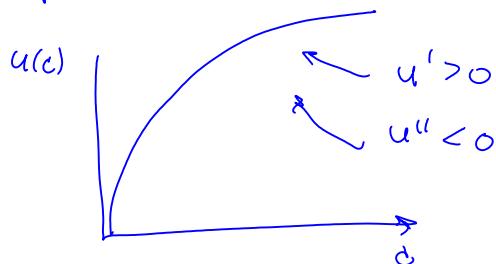
Permanent income model. (Friedman)

$$\max_{\{c_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) , \quad \beta \in (0, 1)$$

$$\text{s.t. } \sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + A_t = 0$$

↓
 labor income ↓
 consumption ↑ assets

$$u'(c) > 0, u''(c) < 0$$



Formulate as a Lagrangian:

$$\begin{aligned} L = & \sum_{j=0}^{\infty} \beta^j u(c_j) \\ & + \lambda \left[\sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + A_t \right] \end{aligned}$$

Use Feynman trick ~ "differentiate under integral"

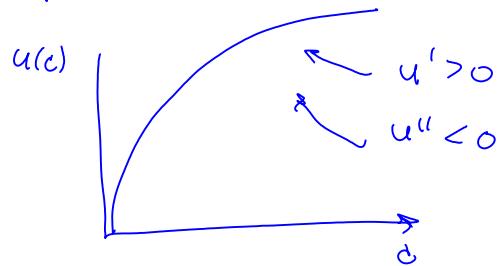
FONC:

$$c_{t+j}: \beta^j u'(c_{t+j}) - \lambda \beta^j = 0, \quad j=0, \dots, \infty$$

$$\sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + H_t = 0$$

↑
labor income ↓
consumption ↑ assets

$$u'(c) > 0, u''(c) < 0$$



Formulate as a Lagrangian:

$$L = \sum_{j=0}^{\infty} \beta^j u(c_j) + \lambda \left[\sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + H_t \right]$$

Use Feynman trick ~ "differentiate under integral"

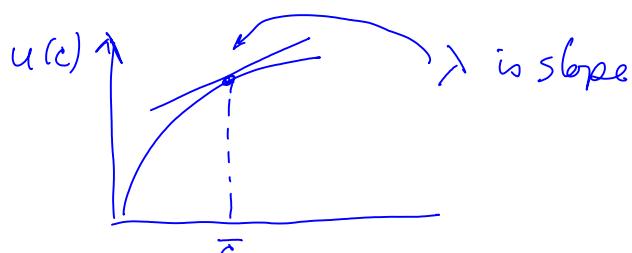
FONC:

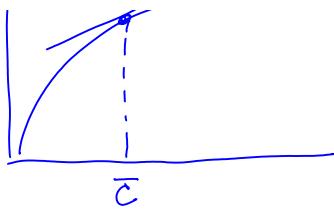
$$c_{t+j} : \beta^j u'(c_{t+j}) - \lambda \beta^j = 0, \quad j=0, \dots, \infty$$

$$\rightarrow u'(c_{t+j}) = \lambda \quad \forall j$$

a constant

$$\Rightarrow c_{t+j} = u'^{-1}(\lambda) - \text{a constant}.$$





Now find λ :

FUNC w.r.t. λ (minimizes wrt λ)

$$\sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + A_t = 0$$

$$\text{or } \sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t = \sum_{j=0}^{\infty} \beta^j c_{t+j}$$

But $c_{t+j} = \bar{c}$ (from above) \Rightarrow

$$\begin{aligned} \sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t &= \bar{c} \sum_{j=0}^{\infty} \beta^j \\ &= \frac{\bar{c}}{(1-\beta)} \end{aligned}$$

$$\text{or } \bar{c} = (1-\beta) \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t \right]$$

\nearrow \nwarrow \nearrow
 marginal propensity to consume out of wealth.

Note: $\beta = \frac{1}{1+r}$ ↗ net interest rate
 $\Rightarrow (1-\beta) = \frac{r}{1+r}$

so $\bar{c} = \frac{c}{(1+r)} \left(\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} + A_t \right)$

Remark: manipulate adjacent budget constraints:

$$+ \quad \sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t = \sum_{j=0}^{\infty} \beta^j c_{t+j}$$

$$t+1 \quad \sum_{j=0}^{\infty} \beta^j y_{t+j+1} + A_{t+1} = \sum_{j=0}^{\infty} \beta^j c_{t+j+1}$$

multiply second equation by β & subtract second from first

⇒

$$y_t + A_t - \beta A_{t+1} = c_t$$

or

$$A_{t+1} = \beta^{-1} (A_t + y_t - c_t)$$

or

$$A_{t+1} = (1+r) \underbrace{(A_t + y_t - c_t)}_{\text{savings}}$$

Permanent Income model, 2

Wednesday, September 14, 2011
8:59 PM

When asked "What is the most powerful force in nature?"
Albert Einstein answered "compound interest."

Finite horizon permanent income or life-cycle model.

A household has exogenous labor income process

$\{y_t\}_{t=0}^T$ and chooses $\{c_t\}_{t=0}^T$

to maximizing

$$\sum_{t=0}^T \beta^t u(c_t), \quad u' > 0, u'' < 0 \\ \beta \in (0, 1)$$

s.t.

$$A_{t+1} = R(A_t + y_t - c_t), \quad t = 0, 1, \dots, T$$

$$R > 1$$

R is gross rate of return on assets

$$R \approx (1+p)$$

\uparrow net rate of return

A_0 is given

— — — — —

Digression:

Write time t budget constraint as:

$$A_t = R^{-1} A_{t+1} + c_t - y_t$$

view as like asset pricing equation $p_t = \beta p_{t+1} + d_t -$
solve in same way \Rightarrow

$$A_t = \sum_{j=0}^{T-t} R^{-j} (c_{t+j} - y_{t+j})$$

on

$$\sum_{j=0}^{T-t} R^{-j} c_{t+j} = \sum_{j=0}^{T-t} R^{-j} y_{t+j} + A_t$$

PV of assets = PV of labor + financial wealth

end of discussion

Form Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t [R(A_t + y_t - c_t) - A_{t+1}] \right\}$$

↑
are for each $t = 0, \dots, T$

Note: we impose $A_{T+1} \geq 0$

FONC:

$$c_t: \beta^t u'(c_t) - \beta^t \lambda_t R = 0$$

$$A_t: \lambda_t R - \lambda_{t+1} \beta^{-1} = 0, t=1, \dots, T$$

$$A_{T+1}: -\beta^T \lambda_T \leq 0, = 0 \text{ if } A_{T+1} > 0$$

$$\text{or } \beta^T \lambda_T A_{T+1} = 0 \\ \Rightarrow A_{T+1} = 0 \text{ if }$$

$$\text{FONC} \Rightarrow u'(c_t) = \beta R u'(c_{t+1}), t=0, \dots, T-1$$

$$\begin{matrix} \text{budget} \\ \text{constraint} \end{matrix} \quad A_t = R^{-1} A_{t+1} + c_t - y_t$$

$$\Rightarrow A_t = \sum_{j=0}^{T-t} R^{-j} (c_{t+j} - y_{t+j})$$

$$A_{-} = \dots$$

$$A_{T+1} = 0$$

$$A_T = c_T - y_T$$

$$A_{T-1} = R^{-1}(c_T - y_T) + (c_{T-1} - y_{T-1})$$

⋮

\equiv

Special case : $T=1$

$$\sum_{t=0}^1 \beta^t u(c_t) = u(c_0) + \beta u(c_1) = U$$

$$dU = u'(c_0) dc_0 + \beta u'(c_1) dc_1 = 0$$

$$\Rightarrow \frac{dc_1}{dc_0} = -\frac{u'(c_0)}{\beta u'(c_1)}$$

$$\max_{c_0, c_1, A_2} u(c_0) + \beta u(c_1)$$

$$\text{s.t. } A_1 = R(A_0 + y_0 - c_0)$$

$$A_2 = R(A_1 + y_1 - c_1)$$

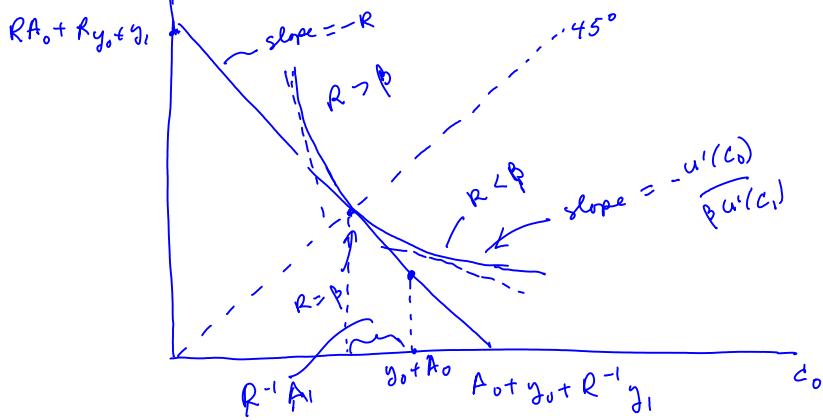
FONC:

$$u'(c_0) = \beta R u'(c_1)$$

$$A_0 = R^{-1}(c_1 - y_1) + (c_0 - y_0)$$

$$c_0 + R^{-1}c_1 = A_0 + y_0 + R^{-1}y_1$$

$$Rc_0 + c_1 = RA_0 + Ry_0 + y_1$$



Suppose $u(c_t) = \ln c_t \Rightarrow u'(c_t) = \frac{1}{c_t}$
 then $u'(c_t) = \beta R u'(c_{t+1})$

$$\Rightarrow \frac{1}{c_t} = \beta R \frac{1}{c_{t+1}}$$

or

$$c_{t+1} = \beta R c_t, \quad t=0, \dots, T-1$$

Examples:

$$\text{set } T = +\infty, \quad y_t = \delta^t y_0, \quad c_t = (\beta R)^t c_0$$

$A_0 = 0$. Then

$$0 = \sum_{j=0}^{\infty} R^{-j} ((\beta R)^j c_0 - \delta^j y_0)$$

$$c_0 \sum_{j=0}^{\infty} \beta^j = \sum_{j=0}^{\infty} \left(\frac{\delta}{R}\right)^j y_0 \quad \text{Assume}$$

or

$$\frac{1}{1-\beta} c_0 = \frac{1}{1-\delta/R} y_0 \quad \left| \left(\frac{\delta}{R} \right) \right| < 1$$

Example 2: $T < +\infty$.

$$c_t = (\beta R)^t c_0, \quad y_t = \delta^t y_0, \quad A_0 = 0$$

$$0 = \sum_{j=0}^{T-1} R^{-j} ((\beta R)^j c_0 - \delta^j y_0) \quad \left| \frac{\delta}{R} \right| < 1$$

$$\frac{(1-\beta^{T+1})}{1-\beta^T} c_0 = \left[\frac{(1-(\frac{\delta}{R})^{T+1})}{(1-(\frac{\delta}{R}))} \right] y_0$$

Example 3: $T = 70$

$$c_t = (\beta R)^t c_0; \quad y_t = \begin{cases} \delta^t y_0, & t=0, \dots, 50 \\ 0, & t=51, \dots, 70 \end{cases}$$

" " + + - .

"0" stands for age 20

"70" stands for age 90 , $A_0 = 0$, $A_{T+1} = 0$

$$\text{Recall: } \beta \frac{u'(c_{t+1})}{u'(c_t)} R = 1 , \quad u'(c) = \frac{1}{c} \Rightarrow$$

$$\beta \frac{c_t}{c_{t+1}} R = 1 \Rightarrow$$

$$c_{t+1} = \beta R c_t \Rightarrow c_t = (\beta R)^t c_0 .$$

$$\sum_{t=0}^{70} R^{-t} (\beta R)^t c_0 = \sum_{t=0}^{50} \delta^t R^{-t} y_0$$

$$\Rightarrow \sum_{t=0}^{70} \beta^t c_0 = \sum_{t=0}^{50} (\delta/R)^t y_0$$

$$c_0 \frac{1 - \beta^{71}}{1 - \beta} = \frac{1 - (\delta/R)^{51}}{1 - (\delta/R)} y_0$$

$$c_0 = \frac{(1-\beta)}{(1-\beta^{71})} \frac{(1 - (\delta/R)^{51})}{(1 - \delta/R)} \quad \begin{array}{l} \text{this OK even} \\ \text{if } |\frac{\delta}{R}| > 1 \end{array}$$

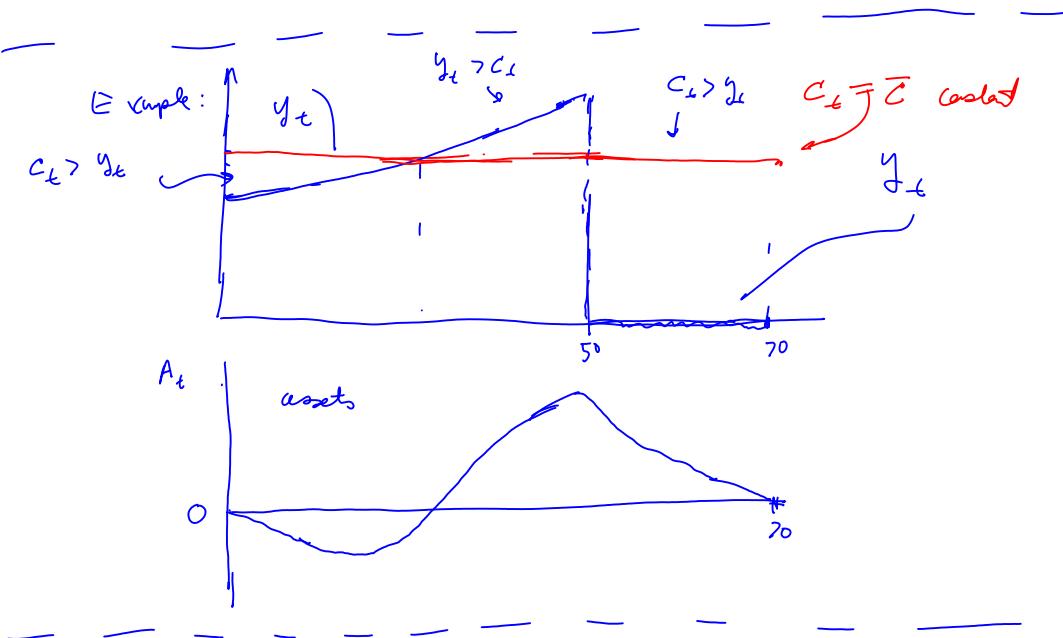
$$\text{e.g. } (\beta R) = 1 \Rightarrow c_t = c_0 \quad \forall t = 0, 1, \dots, 70$$

Interesting case: $\beta R = 1$, $R > 1$, $\frac{\delta}{R} < 1$, $\delta > 1$.

— — — — — — — — — — — — —

$$\text{Note: } \frac{1 + b + \dots + b^T}{1 - b} \quad \text{OK for } |b| > 1$$

$$\begin{aligned} & \frac{1-b}{b} \\ & \frac{b-b^2}{b^2} \\ & \vdots \\ & \frac{b^T}{b^T - b^{T+1}} \end{aligned}$$



More on permanent income model.

Consider infinite horizon problem with
no-borrowing allowed. Problem becomes

$$L = \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + \lambda_t [R(A_t + y_t - c_t) - A_{t+1}] \}$$

$\beta < 1$
 $R > 1$

max s.t. $c_t \geq 0, A_{t+1} \geq 0$ no borrowing

FONC:

$$c_t: \beta^t u'(c_t) - \beta^t \lambda_t R = 0$$

$$t=0, 1, 2, \dots$$

$$A_t: \beta^t \lambda_t R - \beta^{t-1} \lambda_{t-1} \leq 0, \quad \text{if } A_t > 0.$$

$$\Rightarrow \lambda_t = u'(c_t)/R$$

$t=1, 2, \dots$

and

$$u'(c_{t-1}) \geq \beta R u''(c_t)$$

$$= \text{if } c_{t-1} \leq A_{t-1} + y_t \quad \left. \right\} t=1, 2, \dots$$

$$\text{so that } A_t > 0$$

Dissussion: Explanation of first-order conditions. (Kuhn-Tucker)

Constraints:

multiplier

$$R(A_t + y_t - c_t) - A_{t+1} \leq 0 ; \quad \beta^t \lambda_t , \lambda_t \geq 0$$

$$c_t \geq 0 ; \quad \beta^t \alpha_t , \alpha_t \geq 0$$

$$A_{t+1} \geq 0 ; \quad \beta^t N_{t+1} , N_{t+1} \geq 0$$

$t \geq 0$

$$\boxed{\begin{aligned} \alpha_t c_t &= 0 \\ N_{t+1} A_{t+1} &= 0 \end{aligned}}$$

Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t [R(A_t + y_t - c_t) - A_{t+1}] + N_{t+1} A_{t+1} + \alpha_t c_t \right\}$$

FONC:

$$(1) \quad c_t: \beta^t u'(c_t) + \beta^t \alpha_t - \beta^t \lambda_t R = 0 \quad , t \geq 0$$

$$(2) \quad A_t: \beta^t \lambda_t R - \beta^{t+1} \lambda_{t+1} + \beta^{t+1} N_t = 0 \quad , t \geq 1$$

$$(1) \Rightarrow \lambda_t R = u'(c_t) + \alpha_t$$

where $\alpha_t \geq 0$ and $\alpha_t c_t = 0$

$$\alpha_t \geq 0 \Rightarrow \lambda_t R \geq u'(c_t)$$

but

Inada condition $u'(0) = +\infty \Rightarrow c_t > 0$

$$\Rightarrow \alpha_t = 0 \Rightarrow$$

$$\lambda_t R = u'(c_t) \Rightarrow \lambda_t = \frac{u'(c_t)}{R}$$

$$(2) \quad \text{and} \quad \lambda_t = \frac{u'(c_t)}{R} \Rightarrow$$

$$R\beta u'(c_t) + R N_t = u'(c_{t-1}) \quad , t \geq 1$$

where $N_t \geq 0$ and $N_t A_t = 0$

Thus $R\beta u'(c_t) \leq u'(c_{t-1}) \quad , t \geq 1$
 $=$ if $A_t > 0$, (because then $N_t = 0$)

End of digression on how to apply the Kuhn-Tucker conditions.

Thus, along an optimal path either

- * $\begin{cases} u'(c_{t-1}) = \beta R u'(c_t) ; \text{ or} \\ u'(c_{t-1}) > \beta R u'(c_t) \text{ and } c_{t-1} = A_{t-1} + y_{t-1} \end{cases}$

Examples: Assume $\beta \cdot R = 1$, $u(c) = \ln c$.

then (*) becomes:

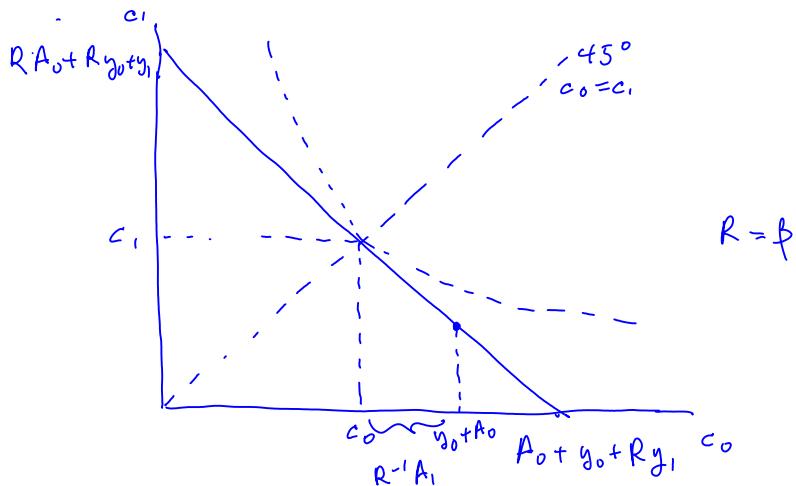
$$\frac{1}{c_{t-1}} = \frac{1}{c_t} \Rightarrow c_t = c_{t-1} \quad \text{or}$$

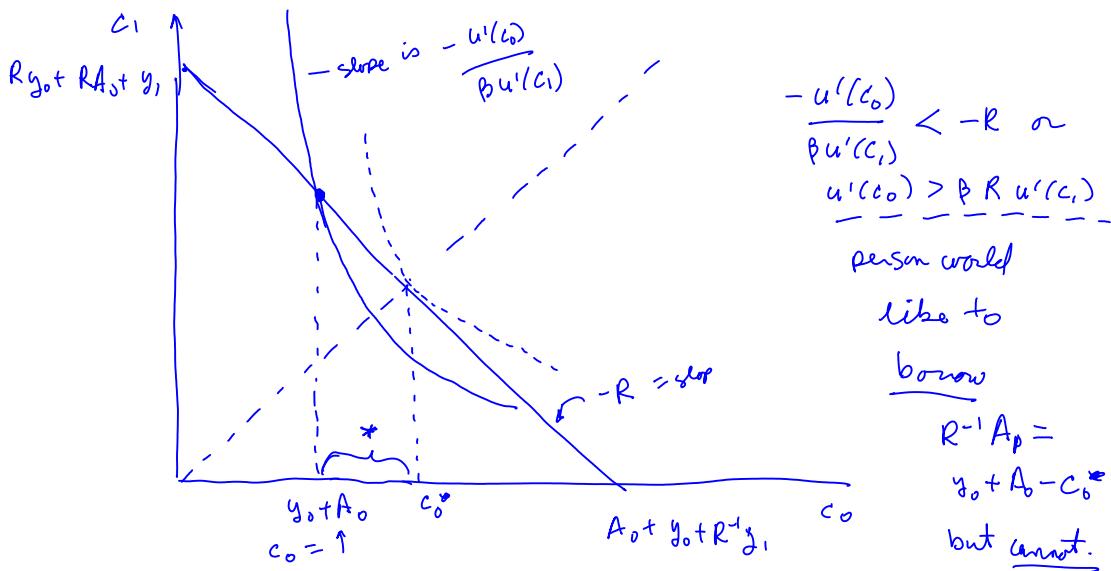
$$\frac{1}{c_{t-1}} > \frac{1}{c_t} \Rightarrow c_t > c_{t-1} \quad \text{and} \quad c_{t-1} = A_{t-1} + y_{t-1}$$

Example: $T=1$, no borrowing, $A_t \geq 0$.

$$u'(c_0) \geq \beta R u'(c_1)$$

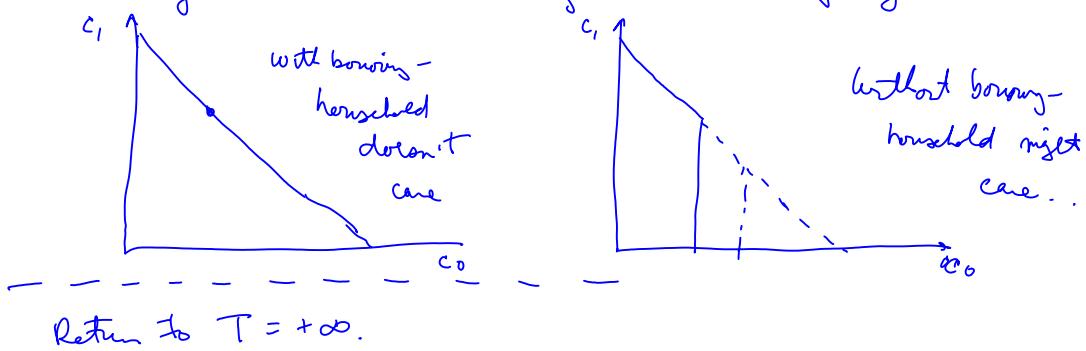
$$= \text{if } c_0 < A_0 + y_0 \text{ and } A_1 > 0$$





Do "Ricardian" experiment - more endowment along

budget line under borrowing & no-borrowing regimes.



Return to $T = +\infty$.

Example 1:

$$y_{t+1} = \delta y_t \Rightarrow y_t = \delta^t y_0 \quad \delta > 1, \\ A_0 = 0, \quad (\beta \delta t < 1)$$

Then solution is

$$c_t = y_t \quad \forall t, \\ A_t = 0 \quad \forall t.$$

The person is always borrowing constrained.

Example 2:

$$y_t = \delta^t y_0, \quad 0 < \delta < 1$$

The $\rightarrow -\bar{r}$ if $t \rightarrow \infty$ (when)

Then $c_t = \bar{c}$ & t where \bar{c} solves

$$\frac{1}{1-\beta} \bar{c} = y_0 \sum_{t=0}^{\infty} \delta^t \beta^t = \frac{y_0}{1-\beta\delta}$$

$$\bar{c} = (1-\beta) \frac{y_0}{1-\beta\delta} \quad \text{if } t \geq 0$$

$$A_{t+1} = \beta^{-1} (A_t + y_t - \bar{c})$$

$$\text{Note: } c_0 = \frac{(1-\beta)}{(1-\beta\delta)} y_0 < \frac{1-\beta}{1-\beta} y_0 = y_0$$

because $0 < \delta < 1$.

\therefore for $t = 0$, the individual saves

$$A_1 = R(y_0 - c_0) \quad \text{because } A_0 = 0$$

In the limit as $t \rightarrow \infty$ $y_t \rightarrow 0$

but $c_t = \bar{c}$. \therefore

$$A_{t+1} = R(A_t + 0 - \bar{c})$$

and $A_{t+1} = A_t$ — seek a steady state \bar{A}

we get

$$R \bar{c} = (R-1) \bar{A}$$

$$\bar{c} = \frac{R-1}{R} \bar{A} \quad , \quad \frac{R-1}{R} = \frac{r}{1+r}$$

$$\bar{c} = \frac{r}{1+r} \bar{A} \quad \text{line off of savings eventually}$$

Example 3: (Remember Einstein's answer)

$$y_t = \begin{cases} y_0 & \text{for } t = 0, \dots, T_w \\ 0 & \text{for } t > T_w \end{cases}$$

$A_t > 0$ constant never binds.

$c_t = \bar{c}$ & t (from our FONC's (\leftarrow above))

$c_t = \bar{c} + t$. (from our FONC's (\Leftarrow above))

$$\begin{aligned}\bar{c} &= (1-\beta) \sum_{t=0}^T \beta^t y_0 \\ &= (1-\beta) \cdot \frac{(1-\beta^{T_w+1})}{(1-\beta)} y_0 \\ \bar{c} &= (1 - \beta^{T_w+1}) y_0\end{aligned}$$

$$A_{t+1} = R [A_t + y_0 - \bar{c}] \quad \text{for } t=0, \dots, T_w$$

$$A_{t+1} = R [A_t - \bar{c}] \quad \text{for } t > T_w$$

$$\bar{c} = \left(\frac{R-1}{R}\right)^n \bar{A} \quad \text{for } t > T_w$$

Saving while early in life:

$$y_0 - \bar{c} = \beta^{T_w+1} \cdot y_0$$

Additional exercise:

could allow person to borrow but not lend,

$$1. \&: A_{t+1} \leq 0 . \quad -$$

then we would have

$$R \beta u'(c_{t+1}) \geq u'(c_t)$$

$$\Rightarrow \text{if } A_{t+1} < 0$$

Costs of no borrowing

Saturday, September 24, 2011
11:07 AM

Compute welfare cost of no borrowing constraint.

Compare two situations to a consumer with preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t \log c_t , \quad 0 < \beta < 1$$

$$y_t = y_0 \delta^t , \quad y_0 > 0 , \quad 1 < \delta < R$$

$$A_0 = 0$$

$$A_{t+1} = R(A_t + y_t - c_t) , \quad R > 1$$

$$\text{Assume } 1 \leq \beta R < \delta$$

Situation 1: Consumer is free to borrow

$$\text{subject to } \lim_{t \rightarrow \infty} A_{t+1} \beta^t = 0$$

(no Ponzi-scheme)

Solution:

$$\beta R \frac{u'(c_{t+1})}{u'(c_t)} = 1 \quad \forall t \geq 0 \Rightarrow$$

$$\beta R \frac{c_t}{c_{t+1}} = 1 \Rightarrow c_{t+1} = \beta R c_t$$

$$\Rightarrow c_t = (\beta R)^t c_0$$

$$c_t = (\beta R)^t c_0$$

Compute PV of consumption & income

$$\sum_{t=0}^{\infty} R^{-t} c_t = \sum_{t=0}^{\infty} R^{-t} (\beta R)^t c_0 = c_0 \sum_{t=0}^{\infty} \beta^t$$

$$= \frac{c_0}{1-\beta}$$

$$\sum_{t=0}^{\infty} R^{-t} y_t = y_0 \sum_{t=0}^{\infty} R^{-t} \delta^t = \frac{y_0}{1 - \delta/R}$$

∴

$$\frac{c_0}{1-\beta} = \frac{y_0}{1 - \delta/R}$$

↑

PV of consumption = PV of income

$$\text{note: } \sum_{t=0}^{\infty} y_t R^{-t} = \sum_{t=0}^{\infty} y_0 \delta^t R^{-t} \\ = y_0 / (1 - \delta R^{-1}) \quad \checkmark$$

⇒

$$\frac{c_0}{1-\beta} = \frac{y_0}{1 - \delta/R}$$

$$c_0 = (1-\beta) \frac{y_0}{1 - \delta/R}$$

Now evaluate the consumer's life-time utility:

$$\begin{aligned} U_l &= \sum_{t=0}^{\infty} \beta^t \log c_t \\ &= \sum_{t=0}^{\infty} \beta^t \log \left[(\beta R)^t \frac{(1-\beta)}{1 - \delta/R} y_0 \right] \\ &= \sum_{t=0}^{\infty} \beta^t \left[\log y_0 + \log \frac{(1-\beta)}{1 - \delta/R} + t \log(\beta R) \right] \\ &= \underbrace{\log y_0 + \log \frac{(1-\beta)}{1 - \delta/R}}_{1-\beta} + \sum_{t=0}^{\infty} \beta^t t \log(\beta R) \end{aligned}$$

$$U_1 = \frac{1}{1-\beta} \left[\log y_0 + \log(1-\beta) - \log(1-\delta/R) \right]$$

$$+ \log(\beta R) \left[\frac{\beta}{(1-\beta)^2} \right]$$

Situation 2:

Same preferences, Same income $y_t = y_0 \delta^t$

but now, no borrowing: $A_{t+1} \geq 0$ for $t \geq 0$.

Solution: $c_t = y_0 \delta^t$ (consume all income)

$$\begin{aligned} U_2 &= \sum_{t=0}^{\infty} \beta^t \log(y_0 \delta^t) \\ &= \sum_{t=0}^{\infty} \beta^t [\log y_0 + t \log \delta] \\ &= \log y_0 \sum_{t=0}^{\infty} \beta^t + \log \delta \sum_{t=0}^{\infty} t \beta^t \\ U_1 &= \frac{\log y_0}{1-\beta} + \log \delta \left[\frac{\beta}{(1-\beta)^2} \right] \end{aligned}$$

Note: from our \mathbb{E} -transform lecture

$$\sum_{t=0}^{\infty} t \beta^t = \frac{\beta}{(1-\beta)^2}$$

Thus,

$$U_1 = \left(\frac{1}{1-\beta} \right) \left(\log \left(\frac{y_0 (1-\beta)}{1-\delta/R} \right) \right) + \frac{\beta}{(1-\beta)^2} \log(\beta R)$$

$1 \leq \beta R < \delta$

and

$$U_2 = \frac{\log y_0}{1-\beta} + \log \delta \left[\frac{\beta}{(1-\beta)^2} \right]$$

Exercise 1: Suppose that $y_0 = 1$, $\beta = .95$,
 $R = 1.04$, $\delta = 1.02$. Compute U_1 and U_2 .

Exercise 2: Suppose that $\beta = .95$, $R = 1.04$, $\delta = 1.02$.

Consider two consumers. For the one who
can borrow, $y_0 = y_0^B$. For the one who

cannot borrow, $y_0 = y_0^{NB}$. Let $y_0^{NB} = 1$.

Find a value y_0^B that makes

$U_1 = U_2$. This is the amount of initial
 income the NB person would be willing to
 sacrifice in order to be free to borrow.

It is a measure of the welfare loss of
 the no-borrowing constraint.

=====

Dynamic programming way to evaluate
 "value function"

Suppose $c_t = c_0 \delta^t$, $t \geq 0$, δ

wants to evaluate ∞

$$V(c_0) = \sum_{t=0}^{\infty} \delta^t \log c_t$$

way to do this: note that

$$\begin{aligned} V(c_0) &= \log c_0 + \beta \sum_{j=0}^{\infty} \delta^j \log (c_{1+j}) \\ &= \log c_0 + \beta V(c_1) \end{aligned}$$

$$v(c_1) = \sum_{j=0}^{\infty} \beta^j \log(c_{1+j})$$

where $c_1 = \delta c_0$.

Solve the Bellman equation

$$v(c_0) = \log c_0 + \beta v(c_1)$$

where $c_1 = \delta c_0$

for a value function $v(\cdot)$.

Thus want to solve

$$(*) v(c_0) = \log c_0 + \beta v(\delta c_0)$$

For a function $v(c_0)$.

Guess $v(c_0) = k_0 + k_1 \log c_0$

where k_0 are coefficients to be determined.

(method of "undetermined coefficients")

Plug into (*)

$$v(c_0) = \log c_0 + \beta [k_0 + k_1 (\log \delta + \log c_0)]$$

$$\text{''} = \log c_0 (1 + \beta k_1) + \beta k_0 + \beta k_1 \log \delta$$

\Rightarrow

$$\begin{matrix} k_0 + k_1 \log c_0 & = \underbrace{(\beta k_0 + \beta k_1 \log \delta)}_{k_0} + \underbrace{(1 + \beta k_1) \log c_0}_{k_1} \end{matrix}$$

\therefore must solve

$$k_1 = (1 + \beta k_1) \Rightarrow k_1 = \frac{1}{1 - \beta}$$

$$k_0 = \beta k_0 + \beta k_1 \log \delta \Rightarrow$$

$$k_0 = \beta k_0 + \beta k_1 \log \delta \Rightarrow$$

$$k_0 = \frac{\beta}{(1-\beta)^2} \log \delta$$

$$\therefore N(c_0) = \frac{1}{1-\beta} \log c_0 + \frac{\beta}{(1-\beta)^2} \log \delta$$

Note: this agrees with our earlier answer.

Stochastic asset prices

Friday, September 16, 2011
2:35 PM

Asset pricing of random dividend stream
under risk-neutrality.

Assume

$$x_{t+1} = A x_t + C \varepsilon_{t+1}$$

$$y_t = G x_t$$

where $\{\varepsilon_{t+1}\}$ is an independent & identically distributed sequence of random variables that are Gaussian - i.e., multivariate normal with mean $\mathbf{0}_{m \times 1}$ and covariance matrix $I_{m \times m} = I_{m \times m}^{-1} \sim (m \times m \text{ identity matrix})$

$$f(\varepsilon_{t+1}) \propto \exp\left(-\frac{1}{2} \varepsilon_{t+1}' I^{-1} \varepsilon_{t+1}\right)$$

Gaussian density

Notice that

$$\begin{aligned} E_t x_{t+1} &= E_t(A x_t + C \varepsilon_{t+1}) \\ &= A x_t + C E_t \varepsilon_{t+1} \\ &= A x_t + C \cdot \mathbf{0} \end{aligned}$$

Because $E_t \varepsilon_{t+1} = \int \varepsilon_{t+1} f(\varepsilon_{t+1}) d \varepsilon_{t+1} = \mathbf{0}$

We will write $E_t z_{t+1} \equiv E z_{t+1} \mid x_t$
 ↑ conditional on
 knowing x_t

In summary $E x_t = A x_t$

In summary, $E_t x_{t+1} = Ax_t$

Problem: to solve

$$* p_t = \beta E_t p_{t+1} + y_t \quad \beta = \frac{1}{1+r}, r > 0$$

digression: (*) $\Rightarrow p_t = \sum_{j=0}^{\infty} \beta^j E_t y_{t+j} + \text{possible bubble (see below)}$

where $E_t y_{t+j}$ = best forecast of y_{t+j} given information at time t .

= end of digression

method: guess and verify

$$\text{Guess: } p_t = H X_t \quad , H \text{ to be determined}$$

Verify: - substitute guess into (*) and rearrange -

$$H X_t = \beta E_t (H X_{t+1}) + y_t$$

$$= \beta H E_t (A X_t + C \varepsilon_{t+1}) + y_t$$

$$= \beta H A X_t + \beta H C E_t \varepsilon_{t+1} + y_t$$

$$H X_t = \beta H A X_t + G X_t \quad \& X_t$$

$$\Rightarrow H = \beta H A + G$$

$$H(I - \beta A)^{-1} = G$$

$$H = G(I - \beta A)^{-1} X_t$$

$$\therefore p_t = G(I - \beta A)^{-1} X_t$$

Remark: this is the same formula we had earlier without the random term ε_{t+1} .

Stochastic bubbles:

return to (*) and consider the special case

$$y_t \equiv 0 \quad \forall t$$

want to solve

$$p_t = \beta E_t p_{t+1}, \quad \beta = \frac{1}{1+r}$$

to find a solution - use guess and
verify

Guess:

$$p_t = c_t \beta^{-t}, \quad c_t \sim 1 \times 1 - \text{scalar}$$

where c_t is any martingale.

A martingale $\{c_t\}$ is a random process
that satisfies

$$E_t c_{t+1} = c_t \quad -$$

best forecast of future value is today's
value. (random walk)

Verify:

$$\begin{aligned} p_t &= \beta E_t p_{t+1} \\ c_t \beta^{-t} &= \beta E_t \beta^{-(t+1)} c_{t+1} \\ &= \beta^{-t} E_t c_{t+1} \\ &= \beta^{-t} c_t \quad . \quad \therefore \text{ we have} \end{aligned}$$

verified that $p_t = c_t \beta^{-t}$

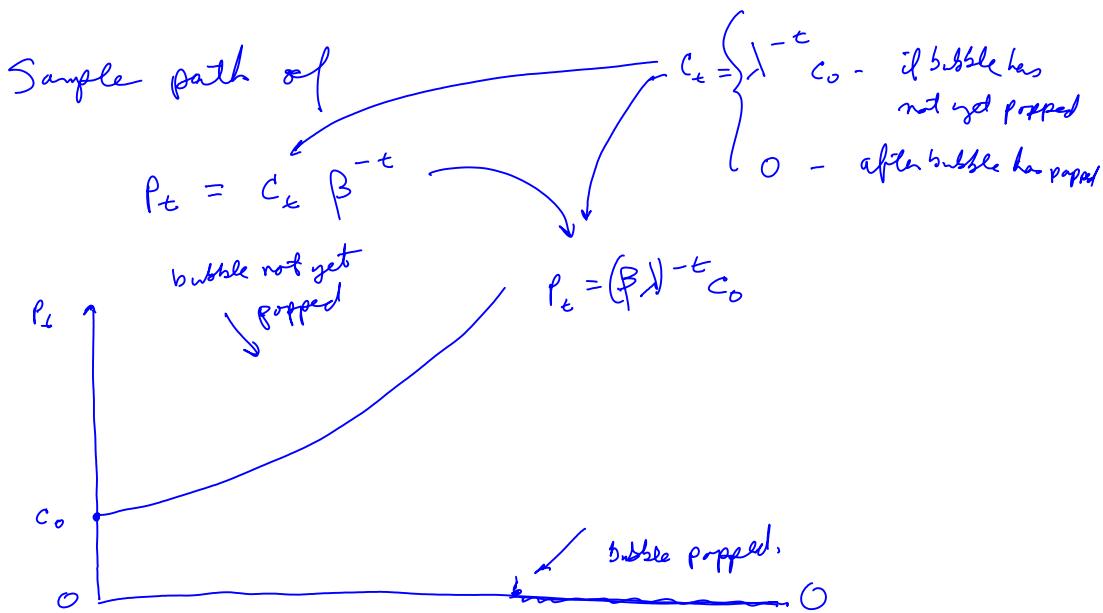
satisfies $p_t = \beta E_t p_{t+1}$.

satisfies $p_t = \beta E_t p_{t+1}$.

Example of a martingale:

$$c_{t+1} = \begin{cases} \lambda^{-1} c_t & \text{with probability } \lambda \in (0, 1) \\ 0 & \text{with probability } (1-\lambda) \end{cases}$$

$$\begin{aligned} E_t c_{t+1} &= \lambda (\lambda^{-1} c_t) + (1-\lambda) 0 \\ &= c_t. \quad \therefore \{c_t\} \text{ is a martingale} \end{aligned}$$



Time Series Models

Sunday, September 18, 2011
12:08 PM

Consider model

$$y_{t+1} = Ax_t + C\varepsilon_{t+1}$$

$$y_t = Gx_t$$

$$1 \times 1 \quad 1 \times n \quad n \times 1$$

$$\varepsilon_{t+1} \sim N(0, I) \quad i.i.d.$$

$$E \varepsilon_{t+1} = 0, \quad E \varepsilon_{t+1} \varepsilon_{t+1}' = I_{m \times m}$$

$$y_{t+2} = Ax_{t+1} + C\varepsilon_{t+2}$$

$$= A(Ax_t + C\varepsilon_{t+1}) + C\varepsilon_{t+2}$$

$$= A^2x_t + AC\varepsilon_{t+1} + C\varepsilon_{t+2}$$

⋮
⋮

$$y_{t+j} = A^jx_t + \underbrace{A^{j-1}C\varepsilon_{t+1} + A^{j-2}C\varepsilon_{t+2} + \dots + C\varepsilon_{t+j}}_{\text{known at } t}$$

$$F_t y_{t+j} = A^jx_t$$

forecasting formula

↓ transpose

$$\sum_j \equiv E(y_{t+j} - F_t y_{t+j})(y_{t+j} - F_t y_{t+j})' = \begin{matrix} \text{variance-covariance} \\ \text{matrix of } j\text{-step} \\ \text{ahead predictors} \\ \text{errors} \end{matrix}$$

$$\begin{aligned} \sum_j &= E(A^{j-1}C\varepsilon_{t+1} + A^{j-2}C\varepsilon_{t+2} + \dots + C\varepsilon_{t+j}) \\ &\quad (A^{j-1}C\varepsilon_{t+1} + A^{j-2}C\varepsilon_{t+2} + \dots + C\varepsilon_{t+j})' \\ &= \underbrace{\dots}_{=I}, \quad \underbrace{\dots}_{=I}, \quad \dots, \quad \underbrace{\dots}_{=I}, \quad \dots \end{aligned}$$

$$= A^{j-1} C \underbrace{E[\varepsilon_{t+1} \varepsilon'_{t+1}]}_{\Sigma_{t+1}} C' (A^{j-1})' + \dots + C \underbrace{E[\varepsilon_{t+1} \varepsilon'_{t+1}]}_{\Sigma_{t+1}} C'$$

$$\Sigma_j = A^{j-1} C C' (A^{j-1})' + A^{j-2} C C' A^{j-2}' + \dots + C C'$$

As $j \rightarrow \infty$, Σ_j might diverge (if eigenvalues of A exceed 1 in absolute value)

or converge (if eigenvalues of A are all less than 1 in absolute value)

Impulse response function:

response of x_{t+j} to shock vector ε_{t+1} is

$$A^{j-1} C$$

response of y_{t+j} to shock vector ε_{t+1} is

$$G A^{j-1} C$$

Exercise: Compute present value of moving average

coefficients: $A^j C$

$$(+) \quad \sum_{j=0}^{\infty} \beta^{j-1} A^{j-1} C = (I - \beta A)^{-1} C$$

What question does (+) answer?

Well, note that

$$\sum_{j=0}^{\infty} \beta^j A^j C = C + \beta A C + \beta^2 A^2 C + \dots$$

\uparrow \uparrow \uparrow
 response response response

$$x_{t+1} \rightarrow \varepsilon_{t+1} \beta x_{t+2} + \varepsilon_{t+1} \beta^2 x_{t+3} + \varepsilon_{t+1}$$

= response of x_{t+1} to ε_{t+1}

i.e. "response of present value of x to ε_{t+1} "

Law of iterated expectation:

$$\begin{aligned} E_t x_{t+2} &= E_t (E_{t+1} x_{t+2}) \\ A^2 x_t &= E_t (A x_{t+1}) \\ &= E_t A (A x_t + C \varepsilon_{t+1}) \\ &= A^2 x_t \end{aligned}$$

Competitive equilibrium

Wednesday, February 03, 2010
5:52 PM

Introduction to general equilibrium

Commodities: c, l, h, k

household: preferences over c, l

technology: production function

endowments: of l & k

Household

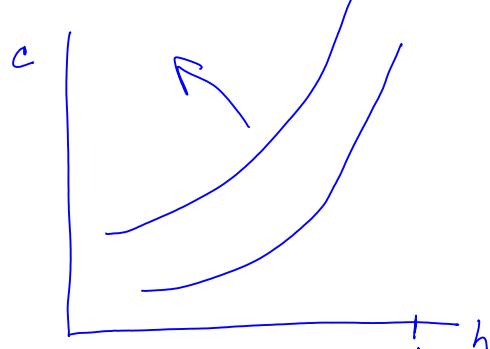
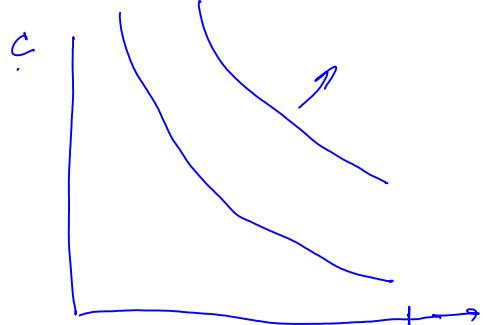
$u(c, l)$, $0 \leq l \leq 1$

$$M \uparrow \quad h = 1 - l, \quad l = 1 - h \\ \text{cons}^n \quad \uparrow \quad \text{leisure} \quad \uparrow \quad \text{hours of work}$$

$$\bar{U} = u(c, l) = u(c, 1-h)$$

$$d\bar{U} = u_c dc + u_l dl = 0 \quad \text{indifference curve}$$

$$\frac{dc}{dl} = -\frac{u_l}{u_c} = MRS$$



$$i \quad h$$

Examples:

$$(1) u(c, l-h) = \log c + \alpha(l-h) \quad \text{Prescott (2002)}$$

$$(2) u(c, l-h) = \log c - B h \quad \text{Hansen-Rogerson}$$

$$\downarrow \\ dU = \frac{1}{c} dc - B dh = 0$$

$$\Rightarrow \frac{dc}{dh} = c B \quad , \Rightarrow \frac{dc}{dl} = -c B$$

HH problem in a market economy:

$$\max_{c, h} u(c, l-h)$$

$$\text{s.t. } c \leq \omega(1-\gamma_h)h + z$$

ω = real wage,

γ_h = marginal labor tax rate

z = other income.

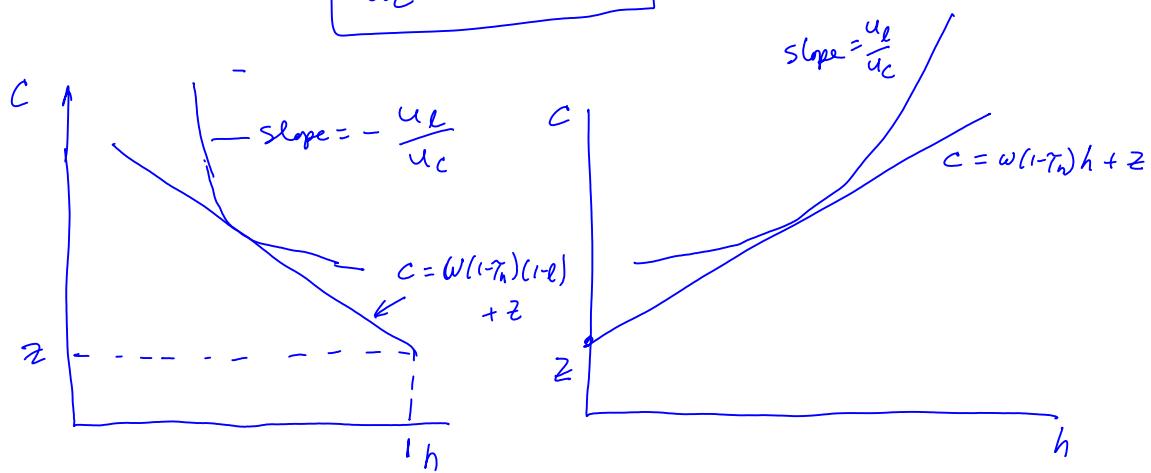
$$L = u(c, l-h) + \lambda [\omega(1-\gamma_h)h + z - c]$$

FONC:

$$c: u_c - \lambda \leq 0, = 0 \text{ if } c > 0$$

$$h: -u_h + \lambda \omega(1-\gamma_h) \leq 0, = 0 \text{ if } h > 0$$

at equality : $\frac{u_L}{u_C} = w(1-\gamma_h)$



~~Production Function:~~

$$Y = F(k, h)$$

assume constant returns to scale.

$$\lambda F(k, h) = F(\lambda k, \lambda h)$$

differentiate both sides w.r.t. h

$$\lambda F_h(k, h) = \lambda F_h(\lambda k, \lambda h)$$

or

$$(*) \quad F_h(k, h) = F_h(\lambda k, \lambda h)$$

this is "homogeneous of degree zero"

marginal products only depend on ratio

$$(h/k): \text{ set } \lambda = \frac{1}{k} \text{ in } (*) \text{ to get}$$

$$F_k(k, h) = F_h(1, \frac{h}{k})$$

Example:

$$F(k, h) = Ak^\alpha \cdot h^{1-\alpha}, \quad \alpha \in (0, 1)$$

$$\underbrace{F_k > 0, F_h > 0}_{\begin{array}{l} \uparrow \\ \text{positive marginal products} \end{array}}, \quad \underbrace{F_{kk} < 0, F_{hh} < 0}_{\begin{array}{l} \downarrow \\ \text{diminish returns} \end{array}}$$

Feasibility:

$$c + G \leq F(k, h)$$

\uparrow
govt purchases

G exogenous (given)

Planning problem: — Command economy

$$\max_{c, h} u(c, 1-h)$$

$$\text{s.t. } c + G \leq F(k, h)$$

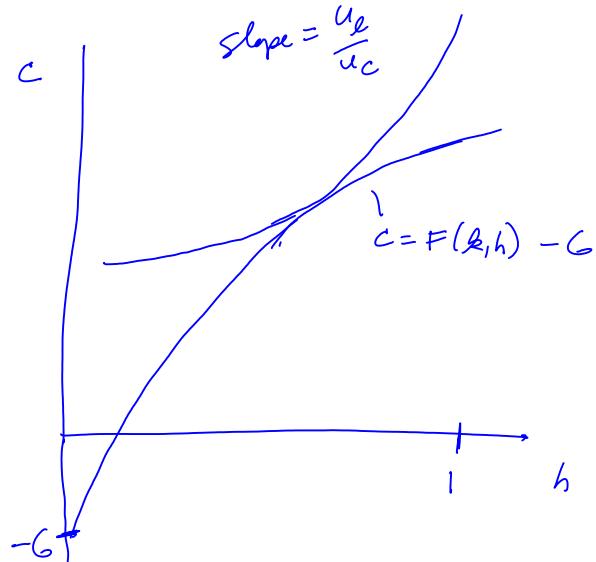
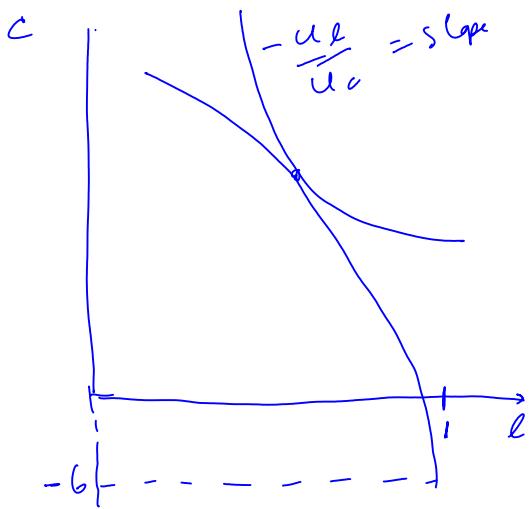
$$L = u(c, 1-h) + \lambda [F(k, h) - c - G]$$

FONC:

$$c: u_c - \lambda = 0$$

$$h: -u_h + \lambda F_h = 0$$

$$\Rightarrow \frac{u_e}{u_c} = F_h$$



Firm problem:

household owns capital & labor & rents them to the firm. Firm is competitive - operates as a price taker.

$$\max_{k, h} \tilde{p} Y - \tilde{r} k - \tilde{\omega} h$$

\tilde{p} - price in \$
 $\tilde{\omega}$ - wage in \$
 \tilde{r} - rental of capital in \$

$$Y = F(k, h)$$

$$\max_{k, h} \tilde{p} F(k, h) - \tilde{r} k - \tilde{\omega} h$$

FONC:

$$k: \tilde{p} F_k - \tilde{r} = 0$$

$$\Rightarrow \frac{\tilde{r}}{\tilde{p}} = F_k$$

$$n: \tilde{p} F_h - \tilde{\omega} = 0$$

$$\Rightarrow F_h = \tilde{\omega}/\tilde{p}$$

cell $\frac{\tilde{\omega}}{\tilde{p}} = \omega$ real wage = $\frac{\text{Goods}}{\text{units of labor}}$

$\frac{\tilde{r}}{\tilde{p}} = r$ real rental rate of capital
 $\sim \frac{\text{goods}}{\text{units of capital}}$

$$(++) \begin{cases} F_k(k, h) = r \\ F_h(k, h) = \omega \end{cases}$$

Note: F_k, F_h are homogeneous of degree zero -

given ω : $F_h(k, h)$ only determines h/k

given r : $F_k(k, h)$ only determines h/k

The size of the firm - the levels of k & h cannot be determined by these equations.

We'll have to use the eqs (++) carefully.

Digression: fading pure frontiers. Example

$$Y = Ak^\alpha h^{1-\alpha}$$

$$F_k = \alpha A k^{\alpha-1} h^{1-\alpha} = \alpha A \left(\frac{h}{k}\right)^{1-\alpha}$$

$$F_h = (1-\alpha) A k^\alpha h^{-\alpha} = (1-\alpha) A \left(\frac{h}{k}\right)^{-\alpha}$$

$$\Rightarrow r = \alpha A \left(\frac{h}{k}\right)^{1-\alpha}$$

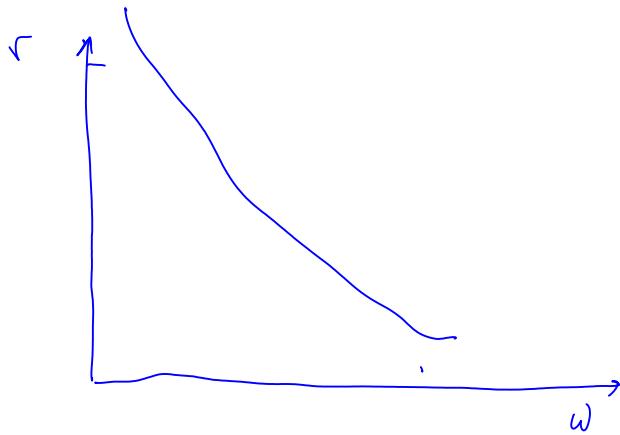
$$\omega = (1-\alpha) A \left(\frac{h}{k}\right)^{-\alpha}$$

$$\frac{\omega}{(1-\alpha)A} = z^{-\alpha} \quad z = \left(\frac{h}{k}\right)$$

$$\left(\frac{\omega}{(1-\alpha)A}\right)^{-\frac{1}{\alpha}} = z$$

$$r = \alpha A \left(\left(\frac{\omega}{(1-\alpha)A} \right)^{-\frac{1}{\alpha}} \right)^{1-\alpha} \quad \text{Factor price frontiers}$$

$$= \alpha A \left(\frac{\omega}{(1-\alpha)A} \right)^{\frac{1-\alpha}{\alpha}} \quad \frac{1-\alpha}{\alpha} = \frac{\alpha-1}{\alpha} \\ = 1 - \frac{1}{\alpha}$$



drew in matlab or
mathematica

end of dispersion

Competitive equilibrium (with $G=0$).

A feasible allocation is a k, h, c

that satisfy

$$c \leq F(k, h)$$

with k given.

A price system is a pair (ω, r) .

A c.e. is a feasible allocation and a price system such that

(1) Given (ω, r) , (c, h) solves the household problem.

(2) Given (ω, r) , (k, h) solve the firm's problem.

Remark: It is redundant to add "markets clear".

Example: Compute a c.e.

$$u(c, 1-h) = \ln c - \beta h$$

$$F(k, h) = Ak^\alpha h^{1-\alpha}$$

Method - solve planning problem, then

reverse engineer the required prices, - then verify c.e. condition holds.

Solve planning problem:

$$F_h = \frac{U_e}{U_c}$$

$$(1-\alpha) A \left(\frac{h}{B} \right)^{-\alpha} = c B$$

$\Rightarrow c = A B^\alpha h^{1-\alpha}$

$$(1-\alpha) A B^\alpha h^{-\alpha} = B A B^\alpha h^{1-\alpha}$$

$$B h = (1-\alpha)$$

$$(1) \quad h = \boxed{\frac{(1-\alpha)}{B}}$$

$$(2) \quad c = A B^\alpha \left(\frac{(1-\alpha)}{B} \right)^{1-\alpha}$$

$$(3) \quad w = (1-\alpha) A B^\alpha \left(\frac{(1-\alpha)}{B} \right)^{-\alpha}$$

$$(4) \quad r = \alpha A B^{\alpha-1} \left(\frac{(1-\alpha)}{B} \right)^{1-\alpha}$$

The FONC's for the firm's problem & the
hh's problem are satisfied.

Interest rates, pure theory of

Saturday, September 10, 2011

11:11 AM

Theory of interest rates.

A consumer has preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad u' > 0, u'' < 0$$
$$0 < \beta < 1$$

There is no uncertainty.

At time 0 , the consumer can buy or sell a claim to one unit of consumption at date $t \geq 1$ for a price g_t^0 .

Here 0 denotes the date of the "trade" (now) and $t > 0$ denotes the delivery date of the one unit of consumption. g_t^0 is the $t=0$ price of a zero coupon bond maturing at t .

Remark: We can convert g_0^0 to a t -period interest rate or yield to maturity using the formula

$$g_t^0 \equiv \frac{1}{(1 + f_{0,t})^t} \approx \exp(-t f_{0,t}).$$

A graph of $f_{0,t}$ against t is called the "term structure of interest rates" at $t=0$. $=$

Assume that the consumer owns an endowment stream $\{y_t\}_{t=0}^{\infty}$. The consumer can sell this endowment stream for the present value

$$W_0 = \sum_{t=0}^{\infty} q_t^0 y_t$$

"sum of price \times quantity"

We think of the consumer as a price-taker — he faces $\{q_t^0\}_{t=0}^{\infty}$ as an exogenous sequence.

We consider the following problem of a utility maximizing household who owns endowment $\{y_t\}_{t=0}^{\infty}$ and faces price system $\{q_t^0\}_{t=0}^{\infty}$ as a price taker.

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t. } \sum_{t=0}^{\infty} q_t^0 c_t \leq \sum_{t=0}^{\infty} q_t^0 y_t$$

β
present value
of consumption

\uparrow
present value of
endowment

To solve: form Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \theta \left[\sum_{t=0}^{\infty} g_t^0 (y_t - c_t) \right]$$

FONC:

$$c_t: \beta^t u'(c_t) = \theta g_t^0, \quad t=0, 1, \dots$$

$$\theta: \sum_{t=0}^{\infty} g_t^0 (y_t - c_t) \geq 0, = 0 \text{ if } \theta > 0$$

Because $u' > 0$ (no satiation) budget

constraint will be satisfied with equality and $\theta > 0$.

Thus, FONC for consumption implies that

$$(1) \quad \beta^t u'(c_t) = \theta g_t^0, \quad \theta > 0.$$

Competitive equilibrium: Consider an economy consisting of I consumers with preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i), \quad u' > 0, u'' < 0$$

and endowments

$$\{y_t^i\}_{t=0}^{\infty}, \quad i=1, \dots, I$$

Definitions:

A feasible allocation is a set of

sequences $\{c_t^i\}_{t=0}^\infty$ that satisfies

$$\sum_{i=1}^I c_t^i \leq \sum_{i=1}^I y_t^i \quad , t = 0, 1, \dots$$

A price system is a sequence $\{q_t^0\}_{t=0}^\infty$

A competitive equilibrium is a feasible allocation and price system such that

$\{c_t^i\}_{t=0}^\infty$ solves household i 's problem,

taking $\{q_t^0\}_{t=0}^\infty$ as given, for $i = 1, \dots, I$.

Example 1. Consider an economy with

I consumers each having identical endowment

sequences

$$y_t^i = y_t^1 \quad i = 2, \dots, I$$

$$t = 0, 1, \dots$$

Let's construct a competitive equilibrium using the "guess and verify" method.

Guess: no-trade - $c_t^i = y_t^i \quad i = 1, \dots, I$
 $t = 0, 1, \dots$

Guess: $q_t^0 = \beta^t u'(y_t^i)$

$$\theta = 1$$

Verify: $\beta^t u'(c_t^i) = q_t^0 \quad \checkmark$

$$\sum_{t=0}^{\infty} q_t^0 (y_t^i - c_t^i) = 0 \quad \checkmark$$

\therefore we have a no-trade competitive equilibrium.

Example 2: (Lucas's famous asset pricing model)

set $I = 1$ in above example. It still works. — this is a "representative consumer" construction.

Example 3: $I = 1$, $y_t^i = y_0$

then $\theta q_t^0 = \beta^t u'(y_0)$
 $c_t = y_0$ β constant

any $\theta > 0$ will work — choice of a numeraire. Only relative prices matter. Here $q_0^0 \propto \beta^t$

This is our " $R \beta = 1$ " specification.

Remark: remember formulas

$$q_t^0 \equiv \frac{1}{(1 + f_{0,t})^t} \approx \exp(-t f_{0,t})$$

for this to hold we must set $q_0^0 = 1$. We can do this by choosing $q_t^0 = \beta^t \frac{u'(c_t)}{u'(c_0)}$.

==

Example 4: $I = 1$

$$u(c_t) = \ln c_t , \Rightarrow u'(c_t) = \frac{1}{c_t}$$

$$y_t = y_0 \delta^t$$

competitive equilibrium: no-fuchs

$$\theta q_t^0 = \beta^t \frac{1}{c_t} = \beta^t \frac{\delta^{-t}}{c_0}$$

thus $q_t^0 \propto (\beta/\delta)^t$

we require $\left| \frac{\beta}{\delta} \right| < 1$

for wealth

$$\sum_{t=0}^{\infty} q_t^0 y_t \text{ to be finite.}$$

Example 5: $I = 2$,

$$y_t^1 = \{1, 0, 1, 0, \dots\}$$

$$y_t^2 = \{0, 1, 0, 1, \dots\}$$

Note $y_t^1 + y_t^2 = 1$

guess: (i) $c_t^1 = c^1, c_t^2 = c^2, t=0, 1, \dots$

(total consumption smooth), $c^1 + c^2 = 1$

(ii) $q_t^0 = \beta^t$

verify: $\theta' q_t^0 = \beta^t u'(c^1)$

$$\theta' q_t^0 = \beta^t u'(c^2)$$

✓ } because
 $\theta^1, \theta^2,$
 c^1, c^2
 are constants

to find: $c^1 \text{ and } c^2$

use budget constraint:

$$\sum_{t=0}^{\infty} \beta^t c^1 = (1 + \beta^2 + \beta^4 + \dots)$$

$$\frac{c_1}{(1-\beta)} = \frac{1}{(1-\beta^2)}$$

$$c_1 = \frac{(1-\beta)}{(1-\beta)(1+\beta)} = \frac{1}{1+\beta}$$

$$\sum_{t=0}^{\infty} \beta^t c^2 = (0 + \beta + 0 + \beta^2 + \dots)$$

as per ^{supt}, $\frac{c^2}{(1-\beta)} = \frac{\beta}{(1-\beta^2)}$

not a "square"

$$\int (1-\beta) = \frac{1}{(1-\beta^2)}$$

$$c^2 = \frac{\beta(1-\beta)}{(1-\beta)(1+\beta)} = \frac{\beta}{1+\beta}$$

$$c^1 + c^2 = \frac{1}{1+\beta} + \frac{\beta}{1+\beta} = 1 \quad \checkmark$$

total consumption equals endowment.

Here there is a lot of "borrowing & lending"

Example 6:

$$I = 2$$

$$y_t^1 = \begin{cases} 1 & \text{for } t=0, \dots, 10 \\ 0 & \text{for } t \geq 11 \end{cases}$$

$$y_t^2 = \begin{cases} 0 & \text{for } t=0, \dots, 10 \\ 1 & \text{for } t \geq 11 \end{cases}$$

Guess $c_t^1 = c^1$, $c_t^2 = c^2 \quad \forall t \geq 0$

$$g_t^0 \propto \beta^t.$$

as in example 5, the guess verifies the household's FONC (same agent)

To determine c^1, c^2 , use budget constraints

$$\sum_{t=0}^{\infty} \beta^t c^1 = \frac{1-\beta''}{1-\beta}$$

$$\frac{c^1}{1-\beta} = \frac{1-\beta''}{1-\beta} \Rightarrow c^1 = 1-\beta''$$

$$\sum_{t=0}^{\infty} \beta^t c^t = \frac{1-\beta''}{1-\beta}$$

$$\frac{c'}{1-\beta} = \frac{1-\beta''}{1-\beta} \Rightarrow c' = 1-\beta''$$

$$\sum_{t=0}^{\infty} \beta^t c^t = \frac{\beta''}{1-\beta} = \beta'' + \beta'^2 + \dots$$

$$\Rightarrow c^2 = \beta''$$

$$c' + c^2 = 1 - \beta'' + \beta'' = 1 \quad . \quad \checkmark$$

which consumer consumes more? It depends on β .

Permanent income, 3

Saturday, September 10, 2011
4:30 PM

Permanent income model -

"pure consumption loans model"

Example 1: There are two consumers with identical

preferences $\sum_{t=0}^{\infty} \beta^t u(c_t^i) , i=1,2$

$$u' > 0, u'' < 0, \beta \in (0, 1)$$

Each has intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i$$

where y_t^i is income of person i .

Assume that: person 1:

$$y_t^1 = \{0, 1, 0, 1, \dots\}$$

$$= \begin{cases} 0 & t \text{ even} \\ 1 & t \text{ odd} \end{cases} \quad t \geq 0$$

person 2:

$$y_t^2 = \{1, 0, 1, 0, \dots\}$$

$$= \begin{cases} 1 & t \text{ even} \\ 0 & t \text{ odd} \end{cases} \quad t \geq 0$$

$\rightarrow 1 \ 1 \ 1 \ \wedge \ 1 \ 0 \ 1 \ \vdash \ \wedge$

The total endowment for the two people is

$$y_e^1 + y_e^2 = \bar{y}_e = 1 \quad \forall t \geq 0$$

Apply permanent income result:

$$c^i = (1-\beta) \sum_{t=0}^{\infty} \beta^t y_e^i \quad , \quad c_e^i = c^i \quad \forall t .$$

for each person $i = 1, 2$

Person 1:

$$c^1 = (1-\beta) [0 + \beta \cdot 1 + \beta^2 \cdot 0 + \beta^3 \cdot 1 + \dots]$$

$$= (1-\beta) [\beta (1 + \beta^2 + \beta^4 + \dots)]$$

$$= (1-\beta) \frac{\beta}{(1-\beta^2)} = \beta \frac{(1-\beta)}{(1-\beta)(1+\beta)}$$

$$\star \quad c^1 = \frac{\beta}{1+\beta} = c_e^1 \quad \text{for all } t \geq 0$$

For person 2:

$$c^2 = (1-\beta) [1 + \beta \cdot 0 + \beta^2 \cdot 1 + \beta^3 \cdot 0 + \beta^4 \cdot 1 + \dots]$$

$$= (1-\beta) (1 + \beta^2 + \beta^4 + \dots)$$

$$= (1-\beta) \cdot \frac{1}{(1-\beta^2)} = (1-\beta) \frac{1}{(1-\beta)(1+\beta)}$$

$$\star \quad c^2 = \frac{1}{1+\beta}$$

$$\text{So } C_1 = \frac{\beta}{1+\beta}, \quad C_2 = \frac{1}{1+\beta} \quad (C_2 > C_1)$$

$$\text{and } C_1 + C_2 = \frac{\beta}{1+\beta} + \frac{1}{1+\beta} = 1$$

Compete asset trades that support this.

$$A_{t+1}^i = R(A_t^i + y_t^i - C_t^i), \quad i=1,2$$

$$R = \beta^{-1}$$

I initial $A_0^i = 0$ for both people.

Person 1:

$$A_1^1 = \beta^{-1}(0 + 0 - \frac{\beta}{1+\beta})$$

$$A_1^1 = -\frac{1}{1+\beta} = \text{borrowing of } -\frac{\beta}{1+\beta}, \text{ payback } \frac{1}{1+\beta} \text{ at time 1}$$

$$A_2^1 = \beta^{-1}\left(-\frac{1}{1+\beta} + 1 - \frac{\beta}{1+\beta}\right) \quad \text{time 1}$$

$$= \beta^{-1}\left(\frac{1+\beta}{1+\beta} - \frac{(1+\beta)}{1+\beta}\right) = 0$$

$$A_2^1 = 0 \quad \text{and so on}$$

debt oscillates between $-\frac{\beta}{1+\beta}$ and 0 .

Person 2:

$$A_1^2 = \beta^{-1}(0 + 1 - \frac{1}{1+\beta}) \quad t=0$$

$$= \beta^{-1} \left(\frac{1+\beta}{1+\beta} - \frac{1}{1+\beta} \right)$$

$A_1^2 = \frac{1}{1+\beta}$ lends $\frac{\beta}{1+\beta}$ at $t=0$ &
is repaid $\frac{1}{1+\beta}$ at $t=1$

$$A_2^2 = \beta^{-1} \left(\frac{1}{1+\beta} + 0 \cdot \frac{1}{1+\beta} \right) = 0$$

and so on.

This person 2's assets oscillate between
 $\frac{1}{1+\beta}$ and 0.

Consumers 1 and 2 borrow & lend with each other

Example 2:

Two consumers. $\sum_{t=0}^{\infty} \beta^t u(c_t^i)$, $0 < \beta < 1$
 $u' > 0, u'' < 0$

Consumer 1:

$$y_t^1 = \begin{cases} y_1 & t=0, \dots, T \\ 0 & t=T+1, T+2, \dots, \infty \end{cases}$$

Consumer 2:

$$y_t^2 = \begin{cases} 0 & t=0, \dots, T \\ y_2 & t=T+1, T+2, \dots, \infty \end{cases}$$

assume that $y_2 = \left(\frac{1}{\beta^{T+1}} - 1 \right) y_1$

$$\text{assume that } y_2 = \left(\frac{1}{\beta^{T+1}} - 1 \right) y_1$$

Apply permanent in one outcome

$$c_k^* = c^k = (1-\beta) \sum_{t=0}^{\infty} \beta^t y_k^t$$

Person 1:

$$c^1 = (1-\beta) y_1 \sum_{t=0}^T \beta^t = (1-\beta) \frac{(1-\beta^{T+1})}{1-\beta} \cdot y_1 \\ = (1-\beta^{T+1}) y_1$$

Person 2:

$$c^2 = (1-\beta) y_2 \sum_{t=T+1}^{\infty} \beta^t \\ = (1-\beta) \frac{\beta^{T+1}}{(1-\beta)} y_2$$

$$c^2 = \beta^{T+1} y_2$$

$$\text{But } y_2 = \left(\frac{1}{\beta^{T+1}} - 1 \right) y_1$$

so

$$c^2 = \beta^{T+1} \left(\frac{1}{\beta^{T+1}} - 1 \right) y_1$$

$$= (1 - \beta^{T+1}) y_1 = c^1$$

Compute asset level of person 1: $A_0^1 = A_0^2 = 0$

$$\begin{aligned}
 A_1' &= \beta^{-1} (A_0' + y_0' - c') \\
 &= \beta^{-1} (0 + y_1 - (1-\beta^{T+1}) y_1) \\
 &= \beta^{-1} (\beta^{T+1}) y_1 \\
 A_1' &= \beta^+ y_1
 \end{aligned}$$

$$\begin{aligned}
 A_2' &= \beta^{-1} (-\beta^T y_1 + y_1 - (1-\beta^{T+1}) y_1) \\
 &= \beta^{-1} (\beta^T + \beta^{T+1}) y_1 \\
 A_2' &= (\beta^{T+1} + \beta^+) y_1 \\
 \vdots \\
 A_{T+1}' &= [(1 + \beta + \beta^2 + \dots + \beta^T)] y_1 = \frac{1 - \beta^{T+1}}{1 - \beta} y_1
 \end{aligned}$$

$$\begin{aligned}
 A_{T+2}' &= \beta^{-1} \left[\frac{1 - \beta^{T+1}}{1 - \beta} y_1 - (1 - \beta^{T+1}) y_1 \right] \\
 &= \beta^{-1} \left[\frac{1 - \beta^{T+1}}{1 - \beta} - \frac{(1 - \beta)(1 - \beta^{T+1})}{(1 - \beta)} \right] y_1 \\
 &= \beta^{-1} \left[\frac{1 - \beta^{T+1} - (1 - \beta^{T+1}) + \beta(1 - \beta^{T+1})}{1 - \beta} \right] y_1 \\
 &= \beta^{-1} \left[\frac{\beta(1 - \beta^{T+1})}{1 - \beta} \right] y_1 \\
 &= \frac{1 - \beta^{T+1}}{1 - \beta} y_1 = A_{T+1}' \Rightarrow \text{roll over}
 \end{aligned}$$

his assets + just lives off interest earnings.

Param 2, does the opposite - he builds up debt -

Person 2 does the opposite - he builds up debt -
then rolls it over.

Homework 2

Friday, September 23, 2011
7:13 AM

1. An economy consists of two consumers, $i=1, 2$, each of whom orders consumption streams according to the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad \text{when } 0 < \beta < 1 \text{ and} \\ u' > 0 \text{ and } u'' < 0$$

each person has intertemporal budget

constraint

$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i. \quad \left[\text{Note: we are assuming that } R = \beta^{-1}. \right]$$

Consumer 1 has endowment stream

$$y_t^1 = \{1, 1, 0, 1, 1, 0, 1, 1, 0, \dots\}$$

while consumer 2 has stream

$$y_t^2 = \{0, 0, 1, 0, 0, 1, 0, 0, 1, \dots\}$$

a. Find the optimal consumption paths

$$\{c_t^i\}_{t=0}^{\infty} \text{ for } i=1, 2.$$

b. Compute $c_t^1 + c_t^2$ for $t=0, 1, 2, \dots$

c. For $t=0, 1, 2, \dots$, please compute

A'_{t+1} where

$$A'_{t+1} = \beta^{-1} [A'_t + y'_t - c'_t] , A'_0 = 0$$

d. For $t=0, 1, 2, \dots$, please compute

A''_{t+1} where

$$A''_{t+1} = \beta^{-1} [A''_t + y''_t - c''_t] .$$

2. Consider an economy consisting of two people, each of whom orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i) , i=1,2, u' > 0, u'' < 0$$

$$0 < \beta < 1$$

Person 1 has income stream

$$y_t^1 = \delta^t , t \geq 0 , \delta \in (0, 1).$$

Person 2 has income stream

$$y_t^2 = 1 - \delta^t , t \geq 0 , \delta \in (0, 1) .$$

Each person has the intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t [c_t^i - y_t^i] = 0$$

a. Find the optimal consumption path

$$\{c_e^i\}_{t=0}^{\infty} \text{ for } i=1, 2.$$

b. Compute $c_e^1 + c_e^2$ for $t=0, 1, 2, \dots$

c. For $t=0, 1, 2, \dots$, please compute

$$A_{t+1}^1 \text{ where}$$

$$A_{t+1}^1 = \beta^{-1} [A_t^1 + y_t^1 - c_t^1], \quad A_0^1 = 0.$$

d. For $t=0, 1, 2, \dots$, please compute

$$A_{t+1}^2 \text{ where}$$

$$A_{t+1}^2 = \beta^{-1} [A_t^2 + y_t^2 - c_t^2].$$

e. Given β with $0 < \beta < 1$, please

find a value of δ that makes consumers 1 and 2 consume identical amounts for $t=0, 1, 2, \dots$.

e. Assume the same two consumers with the same income processes mentioned above. But now assume that each consumer has the sequence of budget constraints

$$A_{t+1}^i = \beta^{-1} [A_t^i + y_t^i - c_t^i],$$

where $A_0^i = 0$ and

$$(*) \quad A_{t+1}^i \geq 0 \quad \text{for } i=0, 1, \dots$$

Here (*) is a no-borrowing constraint.

Please find the optimal consumption path

$$\{c_t^i\}_{t=0}^{\infty} \quad \text{for } i=1, 2.$$

- 3. A government wants to minimize the following measure of tax distortions:

$$\sum_{t=0}^{\infty} \beta^t D(T_t) \quad , \quad 0 < \beta < 1$$

where $D(T_t)$ is a measure of the costs of distortions at date t , T_t are total tax revenues, and $D' > 0$, $D'' > 0$.

The government has an exogenous stream of expenditures $\{G_t\}_{t=0}^{\infty}$ and faces the sequence of government budget constraints

$$B_{t+1} = R [B_t + G_t - T_t] \quad , \quad B_0 = 0$$

where B_{t+1} is government debt issued at t and due at $t+1$. The government can borrow or lend. $R = 1+r > 1$

is the gross rate of return on government debt. Assume that $R = \beta^{-1}$.

debt. Assume that $R = \beta^{-1}$.

Assume that $\lim_{S \rightarrow \infty} \beta^S D'(T_S) B_S = 0$

a condition to rule out "Ponzi schemes"

a. Consider the government expenditure process

$$G_t = \begin{cases} 1 & t \text{ even} \\ 0 & t \text{ odd} \end{cases}$$

Find the optimal setting for taxes $\{T_t\}_{t=0}^{\infty}$ for
 $t = 0, 1, 2, \dots$

b. Consider the expenditure process

$$G_t = \begin{cases} 1 & t \neq 10j, j = 0, 1, 2, \dots; t=0, 1, 2 \\ 10 & t = 10j, j = 0, 1, 2, \dots \end{cases}$$

here a "war" happens in periods 0, 10, 20, ...
and "peace" occurs the rest of the time.

Find the optimal level of $\{T_t\}_{t=0}^{\infty}$.

c. Does the analysis in this problem remind you
of any other problem or model you have seen?

If so, explain.

4. The unemployment rate at month t , u_t ,

Satisfies the difference equation.

$$(*) \quad u_{t+1} = \alpha (-u_t) + (1-\alpha) u_t$$

where α = rate of inflow into unemployment

φ = rate of exit from unemployment.

Definition: A steady state unemployment rate in satisfies

(*) with $\bar{u} = u_{t+1} = u_t$,

a. Compute a formula for the steady state unemployment rate \bar{u} as a function of λ and ϕ .

b. In 2007 in the U.S., $\delta = .0127$

and $\phi = .2579$. Compute the

implied steady state unemployment rate.

c. Suppose that in a recession year 20xx,

$\phi = .15$ and $\varphi = .0175$. These are monthly rates.

Compute the steady state unemployment rate

in year 2xxx.

d. Extra credit. Compute the average

duration of a job in 2^{k+1} . Compute the

average duration of unemployment in 2x^{xx}.

Trees

Wednesday, October 05, 2011
7:37 AM

Consider a two-state Markov chain

with state space $S = \{\bar{s}_1, \bar{s}_2\}$

and time-invariant transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

where $p_{ij} = P_{\text{rob}} \{ s_{t+1} = \bar{s}_j \mid s_t = \bar{s}_i \}$;

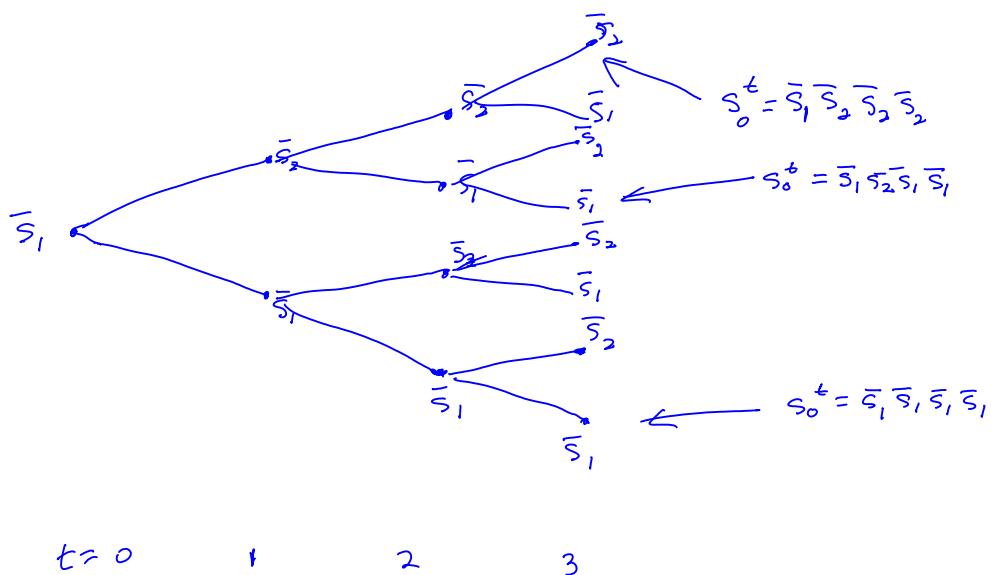
$$\text{and } \pi_0 = \begin{pmatrix} \pi_{0,1} \\ \pi_{0,2} \end{pmatrix} = \begin{pmatrix} P_{\text{rob}}(s_0 = \bar{s}_1) \\ P_{\text{rob}}(s_0 = \bar{s}_2) \end{pmatrix}$$

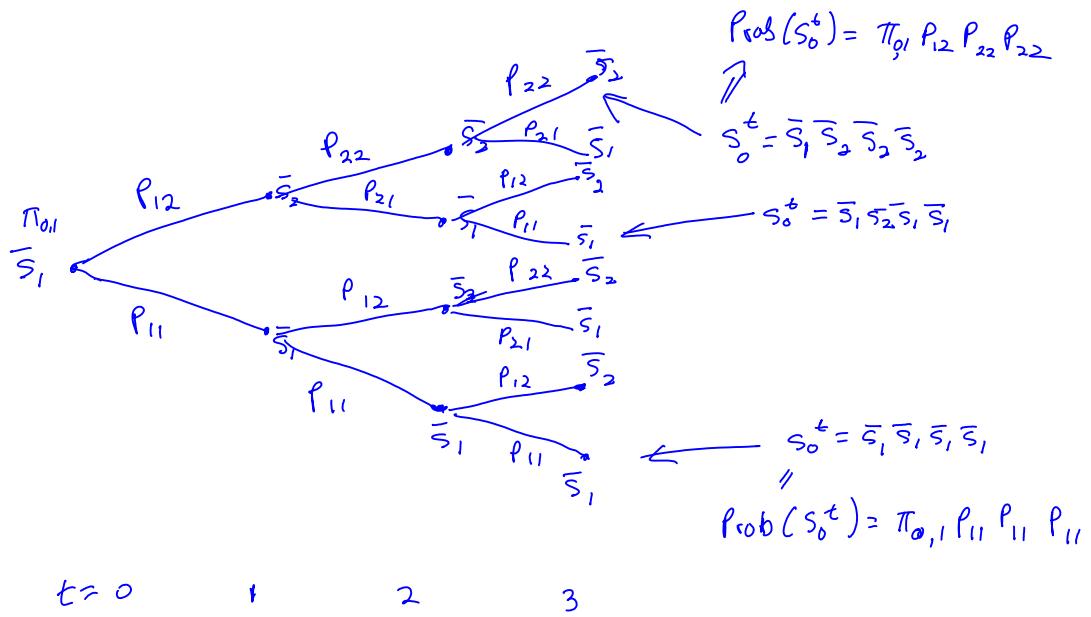
being the initial probability vector over states.

Here $\pi_{0,i} \geq 0$ and $\sum \pi_{0,i} = 1$ (i.e. the $\pi_{0,i}$'s are probabilities)

$p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ (i.e., the p_{ij} 's are probabilities)

for example. ($\bar{s}_1 = 0, \bar{s}_2 = 1$).





In this way, we build up probabilities over histories S_0^t (or s^t for shorthand)

$$\text{Prob}(s^t = \bar{s}^t) \equiv \pi_t(s^t)$$

$$\pi_t(\bar{s}_i, \bar{s}_j, \bar{s}_k, \bar{s}_l, \dots) = \pi_{0,i} p_{ij} p_{jk} p_{kl} \dots$$

Evidently $\sum_{s^t} \pi_t(s^t) = 1$ probability over a history of length t .
 $\pi_t(s^t) \geq 0$ $(s_0, s_1, s_2, \dots, s_t)$

t values of state at
 $t=0, 1, 2, \dots$

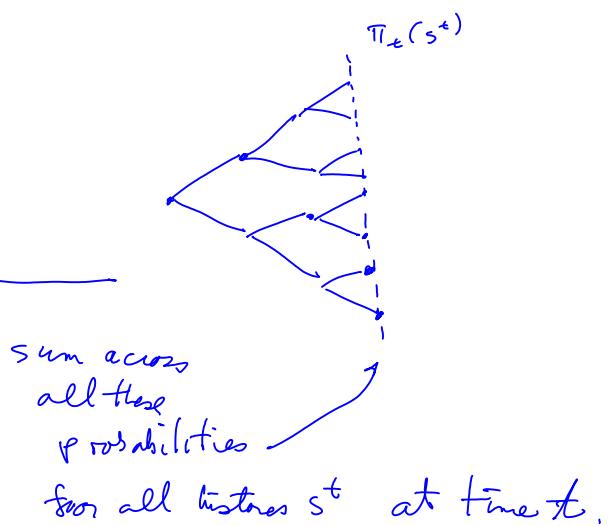
So for each $t \geq 0$, we have a collection
of histories $s^t \in S^{t+1} = S \times S \times \dots \times S$

$t+1$ times
Cartesian product

for each $s^t \in S^{t+1}$, we have probabilities

$$\pi_t(s^t) \geq 0$$

$$\sum_{s^t} \pi^t(s^t) = 1$$

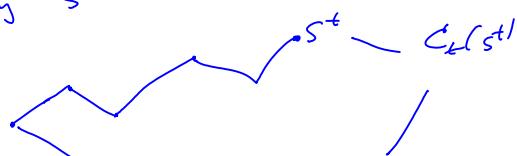


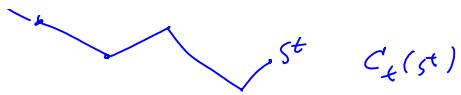
Modeling risk. At time $t=0$, the consumer observes s_0 but does not know subsequent realizations s_1, s_2, \dots . He knows that any history s^t with $\pi_t(s^t) > 0$ for $t > 0$ might happen. His discounted utility along one history s^t is

$$\sum_{t=0}^{\infty} \beta^t u(c_t(s^t)) \quad , 0 < \beta < 1$$

$u' > 0, u'' < 0$

where $c_t(s^t)$ is consumption at date t at history s^t





this we allow consumption at t to depend on the particular history realized at date t .

The consumer's expected utility at time 0 is

$$(*) \quad \sum_t \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

Here we are averaging discounted utilities
 $\beta^t u(c_t(s^t))$ across histories s^t
at date t , using the probabilities
 $\pi_t(s^t)$ as weights.

(*) is called "expected utility".

history-date contingent claim on consumption

$c_t(s^t)$ time - t
 history s^t

consumer values those for every $t \geq 0$, for every s^t

Note: a huge number of commodities

many t 's, and for each t , many s^t 's.

Complete markets: at time 0 the consumer

can buy or sell $c_t(s^t)$ at price

$q_t^0(s^t)$ = price of one unit of consumption at

t at history s^t

The consumer faces $q_e^o(s^t)$ as a price-taker.

Consumer problem:

$$\max_{\{c_t(s^t)\}} \sum_t \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

$$\text{s.t. } \sum_{t=0}^{\infty} \sum_{s^t} q_e^o(s^t) c_t(s^t) \leq W_0$$

Price \cdot quantity

time 0
wealth

One budget constraint

Formulate as a Lagrangian

$$L = \sum_k \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t) + \theta \left[W_0 - \sum_{t=0}^{\infty} \sum_{s^t} q_e^o(s^t) c_t(s^t) \right]$$

↑
Lagrange multiplier

FONC:

$$c_t(s^t) : \Leftrightarrow \beta^t u'(c_t(s^t)) \pi_t(s^t) = \theta q_e^o(s^t)$$

↑ ↑ given as
to be price taken

true since

determined
from budget
constraint

$$(*) \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t(s^t) = W_0$$

(*) $\forall (t, s^t)$ and $(**)$ determine

$c_t(s^t) \nabla (t, s^t)$ and θ
 ¶ "marginal utility of
 wealth"

Example: (1) Suppose that $q_t^0(s^t) = \beta^t \pi_t(s^t)$

(why? - just a guess - it might happen in
 a very special case)

then $(*) \Rightarrow$

$$\cancel{\beta^t u'(c_t(s^t)) \pi_t(s^t)} = \theta \cancel{\beta^t \pi_t(s^t)}$$

or $u'(c_t(s^t)) = \theta$ a constant

$$\Rightarrow c_t(s^t) = \bar{c} \quad \text{a constant } \forall t, \forall s^t$$

and $(**) \Rightarrow$

$$\bar{c} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) = W_0$$

$$\bar{c} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) = W_0$$

" "

$$= \underline{c}$$

$$\bar{c} \sum_{t=0}^{\infty} f^t = W_0$$

$$\bar{c} \frac{1}{1-\beta} = W_0$$

$$\boxed{\bar{c} = (1-\beta) W_0}$$

Complete consumption smoothing across time & across histories s^t .

Remark: Source of wealth: Consumer owns an endowment $y_t(s^t)$ of time t , history s^t consumption good.

His wealth W_0 is

$$W_0 = \sum_{t=0}^{\infty} \sum_{s^t} g_t^*(s^t) y_t(s^t)$$

Example: $S = \{0, 1\}$ only depends s_t , not history s^t

$$y_t(\cdot s^t) = y_t(s_t) = \begin{cases} \varepsilon & \text{if } s_t = 0 \\ 1-\varepsilon & \text{if } s_t = 1 \end{cases}$$

Thus, a sample endowment path is

$$\begin{array}{ccccccc} s_t & 0 & 0 & 1 & 0 & 1 & 1 \\ y_t & \varepsilon, \varepsilon, 1-\varepsilon, \varepsilon, 1-\varepsilon, 1-\varepsilon, \dots \end{array}$$

Example 2: (Lucas, 1978) - a pure endowment representative agent economy.

$$c_t(s^t) = y_t(s^t) \leftarrow \text{Exogenous}$$

representative consumer has to consume endowment in equilibrium. Our job: to find the

equilibrium price vector $q_t^*(s^t)$ that
reconciles a competitive, price-taking consumer
to consuming only his endowment each period.

\Rightarrow Substitute $c_t(s^t) = y_t(s^t)$ into (*) and
"store":

$$(+) \quad \beta^t u'(g_t(s^t)) \pi_t(s^t) = \theta q_t^*(s^t)$$

If prices satisfy (+), the consumer's FOMC
are satisfied. What about budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^*(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^*(s^t) y_t(s^t)$$

automatically for $c_t(s^t) = y_t(s^t)$

Note: (1) Any $\theta > 0$ works - we are
just scaling prices.

(2) multiplying all prices $q_t^*(s^t)$ by a
scalar > 0 just rescales θ .

"choice of numeraire" - only relative prices
matter.

- - - - -

Example 3: Two consumers with preferences
ordered by \succeq_i person i

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \quad , i=1,2$$

$u' > 0, u'' < 0$
 $\beta \in (0, 1)$

Note: no i in u or $\pi_t(s^t)$ or β

$$S = \{0, 1\}$$

Consumer i 's endowment streams

$$y_t^1(s_t) = s_t$$

$$y_t^2(s_t) = 1 - s_t$$

Note: $y_t^1(s_t) + y_t^2(s_t) = 1$ no aggregate risk

Try to find a competitive equilibrium price system

$$q_t^0(s^t) \text{ and allocation } \{c_t^1(s^t), c_t^2(s^t)\}_{t,s^t}$$

Definition: A feasible allocation is an allocation $\{c_t^1(s^t), c_t^2(s^t)\}_{t,s^t}$ that satisfies

$$c_t^1(s^t) + c_t^2(s^t) \leq y_t^1(s^t) + y_t^2(s^t)$$

$$\forall t, \forall s^t$$

Definition

A c.e. is a price system and feasible allocations such that given the price system, the allocation solves each household's optimum problem (i.e., it satisfies (*) and (**) for all $h=1, 2$)

(i.e., it satisfies (*) and (**) for each household).

How to find a competitive equilibrium. In general, a difficult problem: you must simultaneously solve all the conditions for household's optimality and feasibility. We'll study this later. For now we'll use guess & verify - guess an equilibrium price vector.

Example := make a lucky guess

$$g_t^0(s^t) = \beta^t \pi_t(s^t)$$

At this guess, FONC for i

$$\beta^t u'(c_t^i(s^t)) \pi_{t+1}(s^{t+1}) = \theta_i \beta^{t+1} \pi_t(s^t)$$

$$\Rightarrow u'(c_t^i(s^t)) = \theta_i \quad \text{for } i=1, 2$$

$$\Rightarrow c_t^i(s^t) = \bar{c}^i \quad \text{for } i=1, 2$$

constant consumption if t ad s^t for each consumer i .

Feasibility: $c^1 + c^2 = 1$

To determine: division of 1 between C^1 & C^2

Use budget constraints:

for i :

$$\infty \cdot - \infty$$

$$\bar{c}^i \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) = \sum_{t=0} \sum_{s^t} \beta^t \cdot y_t^i(s^t) \bar{\pi}_t(s^t)$$

σ

$$\bar{c}^i \frac{1}{1-\beta} = W_0^i$$

$$\begin{aligned} \text{Note: } W_0 + W_1 &= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \bar{\pi}_t(s^t) \\ &= \sum_{t=0}^{\infty} \beta^t \underbrace{\sum_{s^t} \pi_t(s^t)}_{\gamma_t^1 + \gamma_t^2 = 1} = \frac{1}{1-\beta}. \end{aligned}$$

this is as far as we can get - to

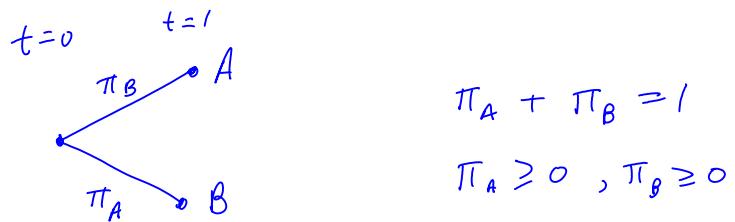
determine W_0^1 & W_0^2 separately, we need to know
the $\pi_t(s^t)$'s numerical values.

Trees, 2

Saturday, February 20, 2010
5:24 PM

Very simple example of a tree.

$$t = \{0, 1\} \quad \text{two periods}$$



Preferences:

$$u(c_0) + \beta u(c_1(A))\pi_A + \beta u(c_1(B))\pi_B$$

$$u' > 0, u'' < 0$$

expected utility. Randomness realized at date

1.

Endowment of consumer:

$$\text{at } t=0 : y_0$$

$$\text{at } t=1 : y_1(A) \text{ in state A}$$

$$y_2(B) \text{ in state B}$$

Commodities:

c_0 - consumption at date 0

$c_1(A)$ - consumption in state A at date 1

$c_1(A)$ - consumption in state A at date 1

$c_2(A)$ - consumption in state B at date 1

Prices:

q_0 = price of one unit of consumption at
 $t=0$

$q_1(A)$ = price of one unit of consumption at
 $t=1$ in state A

$q_1(B)$ = price of one unit of consumption
at $t=1$ in state B

e.g. state A - rain

state B - no rain

or

state A - Yankees win

state B - Yankees lose

etc ...

Consumer's problem:

Choose $\{c_0, c_1(A), c_1(B)\}$ to maximize

$$u(c_0) + \beta u(c_1(A)) \pi_A + \beta u(c_1(B)) \pi_B$$

$$\text{s.t. } q_0 c_0 + q_1(A) c_1(A) + q_1(B) c_1(B)$$

$$\leq g_0 c_0 + g_1(A) c_1(A) + g_1(B) c_1(B) \equiv W_0$$

Form Lagrangien:

$$L = u(c_0) + \beta u(c_1(A)) \pi_A + \beta u(c_1(B)) \pi_B$$

$$+ \theta [W_0 - g_0 c_0 - g_1(A) c_1(A) - g_1(B) c_1(B)]$$

↑
Lagrange multiplikator — preset values of consumption

FONC:

$$c_0 : u'(c_0) - \theta g_0 = 0 \quad (\text{interior solution because } u' > 0)$$

$$c_1(A) : \beta u'(c_1(A)) \pi_A - \theta g_1(A) = 0$$

$$c_1(B) : \beta u'(c_1(B)) \pi_B - \theta g_1(B) = 0$$

$$\theta : W - g_0 c_0 - g_1(A) c_1(A) - g_1(B) c_1(B) = 0$$

↑ budget constraint at equality
because $u' > 0$

First FONC \Rightarrow

$$\theta = u'(c_0)/g_0$$

Substitute into 2nd & 3rd FONC's

$$\beta u'(c_1(A)) \pi_A = \frac{g_1(A)}{g_0} u'(c_0)$$

$$\beta u'(c_1(B)) \pi_B = \frac{g_1(B)}{g_0} u'(c_0)$$

Asset pricing

Monday, February 22, 2010
9:31 AM

Asset pricing with redundant assets or

"arbitrage free pricing"

Suppose there are complete markets

$$s_t \in S \quad \forall t$$

$$s^t \in S^{t+1} \quad \forall t, \quad s^t = (s_t, s_{t-1}, \dots, s_0)$$

$$\pi_t(s^t) = \text{pris}(s^t)$$

Preferences

$$\sum_t f^t \sum_{s^t} u(c_t(s^t)) \pi_t(s^t)$$

Budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q^0_{t^0}(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q^0_{t^0}(s^t) y^0_t(s^t)$$

Suppose we add one (redundant) asset

I promise to pay one unit of consumption

for sure at a particular time T , e.g. $T=10$

One unit of the asset has price y_0 units of

the two remaining goods. — the consumer can

buy or sell unlimited amounts of the

asset at time 0. The

--- - - . . .
 consumer's budget constraint now becomes

$$\sum_t \sum_{s^t} q_{ft}^o(s^t) c_t(s^t) + v_0 \cdot B$$

$$\leq \sum_t \sum_{s^t} q_{ft}^o(s^t) y_t^o(s^t)$$

$$+ B \cdot \sum_{s^T} q_{fT}^o(s^T)$$

where $B =$ amount of the bond that he buys at $t=0$

After buying the bond, his entitlement to
 time T consumption is augmented by B
 for all histories s^T at T .

\Rightarrow budget constraint is

$$\begin{aligned} & \sum_{t=0}^{\infty} \sum_{s^t} q_{ft}^o(s^t) (c_t(s^t) - y_t^o(s^t)) \\ & \leq B \left(\sum_{s^T} q_{fT}^o(s^T) - v_0 \right) \end{aligned}$$

Stare:

$$\text{If } \sum_{s^T} q_{fT}^o(s^T) - v_0 > 0$$

Set B as large as possible " $B = +\infty$ "

Buy the bond for v_0 and "imposture" it
 self

→

$$\text{If } \sum_{S^T} q_{fT}^0(S^T) - V_0 < 0$$

$$\text{set } B \leq 0 \Rightarrow B = -\infty$$

Sell the bond, then buy & repackage

" $y_T(s^t)$ " commodities

$$\therefore \text{If } \sum_{S^T} q_{fT}^0(S^T) - V_0 \neq 0$$

If arbitrage profits: Perfectly sure, risk free profits. A "money machine". impossible in equilibrium. i.e.

$$(*) N_0 = \sum_{S^T} q_{fT}^0(S^T)$$

* is the price of a risk free bond. paying off for sure at T

Permanent income (stochastic)

Friday, September 30, 2011
9:59 AM

Permanent income model with random

income:

$$x_{t+1} = A x_t + C \varepsilon_{t+1}$$

$$y_t = G x_t \quad , \quad y_t = \text{"labor income"}$$

$$\varepsilon_{t+1} \sim N(0, I_m)$$

Budget constraint at $t \geq 0$:

$$(1) \quad F_{t+1} = \beta^{-1} [F_t + y_t - c_t]$$

/ ↑
 gross financial ↑ ↑
 interest assets income cons'n
 rate

"Consumption function":

$$(2) \quad c_t = (1-\beta) [F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}]$$

derive this by replacing $\sum_{j=0}^{\infty} \beta^j y_{t+j}$ by its

best forecast $E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}$.

Remark: This is valid when $u'(c_t)$ is linear in c_t .

otherwise it is an approximation that ignores effects of risk aversion.

(1) & (2) \Rightarrow

$$F_{t+1} = \beta^{-1} \left[F_t + y_t - (1-\beta) \left(F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right) \right]$$

$$= \beta^{-1} \left[(x - (1-\beta)) F_t + y_t - (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

$$(3) \quad F_{t+1} = F_t + \beta^{-1} \left[y_t - (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

decision rule: add difference between y_t and permanent income

$$(2) \Rightarrow \text{---} \quad \text{---}$$

$$C_{t+1} = (1-\beta) \left[F_{t+1} + E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} \right]$$

$$C_t = (1-\beta) \left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

\Rightarrow

$$C_{t+1} - C_t = (1-\beta) (F_{t+1} - F_t)$$

$$+ (1-\beta) \left[E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} - E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

$$= (1-\beta) \underbrace{\left[y_t - (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]}_{F_{t+1} - F_t}$$

(*)

$$+ (1-\beta) \left[E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} - E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

$$C_{t+1} - C_t = (1-\beta) \sum_{j=0}^{\infty} \beta^j (E_{t+1} y_{t+j+1} - E_t y_{t+j})$$

Proof:

$$c_{t+1} - c_t = (1-\beta) \beta^{-1} \left[y_t - (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \\ + (1-\beta) \left[E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} - E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

$$= (1-\beta) E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} + (\beta^{-1} - 1) y_t$$

$$- (1-\beta) \underbrace{[(1-\beta)\beta^{-1} + 1]}_{-(1-\beta)[\beta^{-1} - 1 + 1]} E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}$$

$$- \beta^{-1}(1-\beta)$$

\Rightarrow

$$c_{t+1} - c_t = (1-\beta) E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} + (\beta^{-1} - 1) y_t$$

$$\underbrace{- \beta^{-1}(1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}}$$

" "

$$\underbrace{- (1-\beta) E_t \sum_{j=0}^{\infty} \beta^{j-1} y_{t+j}}$$

$$\underbrace{- (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j+1} - (1-\beta)\beta^{-1} y_t}$$

\Rightarrow

$$c_{t+1} - c_t = (1-\beta) \sum_{j=0}^{\infty} \beta^j (E_{t+1} y_{t+j+1} - E_t y_{t+j+1})$$

qed.

Now,

$$E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} = G(I - \beta A)^{-1} x_{t+1}$$

$$E_t \sum_{j=0}^{\infty} \beta^j y_{t+j+1} = E_t y_{t+1} + \beta E_t y_{t+2} + \dots \\ = G(I - \beta A)^{-1} A x_t$$

1

$$\begin{aligned}
 & \sum_{j=0}^{\infty} \beta^j (E_{t+j} y_{t+j+1} - E_t y_{t+j+1}) \\
 &= G(I - \beta A)^{-1} x_{t+1} - G(I - \beta A) Ax_t \\
 &= G(I - \beta A)^{-1} [A x_t + C \varepsilon_{t+1} - A x_t] \\
 &= G(I - \beta A)^{-1} C \varepsilon_{t+1}
 \end{aligned}$$

Therefore:

$$c_{t+1} - c_t = (1-\beta) G \underbrace{(I - \beta A)^{-1} C}_{\text{Error Term}} \varepsilon_{t+1}$$

present value of impulse response coefficient

$$c_{t+1} - c_t = \bar{H} \cdot \varepsilon_{t+1}, \quad \bar{H} = (-\beta) G(I - \beta A)^{-1} C$$

Example: Friedman-Muth

$$y_t = z_t + u_t \xrightarrow{\text{transitory}} \\ \text{p} \quad \text{persistent} \\ \text{or} \\ \text{"persistent" income}$$

$$z_{t+1} = z_t + \sigma_1 \varepsilon_{1,t+1} \\ u_{t+1} = \sigma_2 \varepsilon_{2,t+1}, \quad \varepsilon_{t+1} = \begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix} \sim \mathcal{W}(0, I_{2 \times 2}) \\ \begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix} + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix} \\ y_t = [1 \ 1] \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$(I - \beta A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} \\ = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1 \end{pmatrix} \\ (I - \beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore C(I - \beta A)^{-1} = (1 - 1) \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix}$$

∴

$$c_t = (1 - \beta) \left[F_t + \frac{1}{1-\beta} z_t + u_t \right]$$

or

$$c_t = (1-\beta) F_t + \varepsilon_t + (1-\beta) u_t$$

↑

Marginal propensity to consume out of permanent income = 1

Marginal propensity to consume out of transitory income =
 $= (1-\beta)$

=====

$$\begin{aligned} G(I - \beta A)^{-1} C &\leq_{t+1} \\ &= \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{1}{1-\beta} \sigma_1 & \sigma_2 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix}$$

$$= \frac{1}{1-\beta} \sigma_1 \varepsilon_{1,t+1} + \sigma_2 \varepsilon_{2,t+1}$$

∴

$$c_{t+1} - c_t = \sigma_1 \varepsilon_{1,t+1} + (1-\beta) \sigma_2 \varepsilon_{2,t+1}$$

=====

What about savings in financial assets F_t ?

Earlier we showed

$$F_{t+1} - F_t = \beta^{-1} \left[y_t - (1-\beta) F_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

but using our forecasting formulas here \Rightarrow

$$F_{t+1} - F_t = \beta^{-1} [G x_t - (1-\beta) G(I - \beta A)^{-1} x_t]$$

$$= \beta^{-1} G [I - (1-\beta)(I - \beta A)^{-1}] x_t$$

Apply (*) to the Friedman-Muth example:

$$(1-\beta) G (I - \beta A)^{-1} x_t = [1 \quad (1-\beta)] \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$G I x_t = \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$\Rightarrow G (I - (1-\beta)(I - \beta A)^{-1}) x_t$$

$$= ([1 \quad 1] - [1 \quad 1-\beta]) \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$= [0 \quad \beta] \begin{pmatrix} z_t \\ u_t \end{pmatrix} \quad \text{multiply by } \beta^{-1} \Rightarrow$$

\Rightarrow

$$F_{t+1} - F_t = [0 \quad 1] \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$F_{t+1} - F_t = u_t$$

Thus: the consumer spends all of z_t , saves nothing.

He spends $(1-\beta)$ of u_t and saves β of u_t .

Incomplete vs complete

Tuesday, February 23, 2010
12:29 PM

Complete markets setting

$$\max_{\{c_t(s^t)\}_{t,s^t}} \sum_t \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

$$\text{s.t. } \sum_t \sum_{s^t} q_t^o(s^t) (c_t(s^t) - y_t(s^t)) \leq 0$$

$$\text{FONC: } \beta^t u'(c_t(s^t)) \cdot \pi_t(s^t) = \theta q_t^o(s^t)$$

divide for two histories at $t, t+1$

\Rightarrow

$$(*) \quad \frac{\beta u'(c_{t+1}(s^{t+1})) \pi_{t+1}(s^{t+1})}{u'(c_t(s^t)) \pi_t(s^t)} = \frac{q_{t+1}^o(s^{t+1})}{q_t^o(s^t)}$$

$$\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \equiv \pi_{t+1}(s^{t+1} | s^t)$$

\sim conditional probability

very easy to compute in our case
because we generated $\pi_t(s^t)$ from Markov
chain

$$\frac{q_{t+1}^o(s^{t+1})}{q_t^o(s^t)} \equiv q_{t+1}^t(s^{t+1} | s^t)$$

\sim one step ahead "pricing kernel"

Then we can write (*) as

Thus we can write (+) as

$$(+) \quad \beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_{t+1}(s^{t+1}|s^t) = q_{t+1}^t(s^{t+1}|s^t)$$

Remark:

$$\sum_{s^{t+1}|s^t} q_{t+1}^t(s^{t+1}|s^t) = \text{price at node } s^t$$

of a risk-free or sure
claim on consumption at time $t+1$

thus

$$\sum_{s^{t+1}|s^t} q_{t+1}^t(s^{t+1}|s^t) = (R_t(s^t))^{-1}$$

= reciprocal of gross one period
interest rate at node s^t

then (+) \Rightarrow

$$(+) \quad \beta R_t \sum_{s^{t+1}} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_t(s^{t+1}|s^t) = 1$$

Note: (+) \Rightarrow (#) but not (#) \Rightarrow (+)

(#) is weaker - holds "on average"

while (+) is "state by state"

Incomplete markets setting:

Consumer has preferences

$$\sum_t \beta^t \sum_{s^t} u(c_t(s^t)) \pi_t(s^t)$$

but now is subject to sequence of budget constraints:

$$A_{t+1}(s^t) = R_t(s^t) [y_t(s^t) - c_t(s^t) + A_t(s^{t-1})]$$

\uparrow
 $\underline{=}$ not s^{t+1}

\uparrow
 s^{t-1} not s^t

Form Lagrangian:

$$L = \sum_t \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

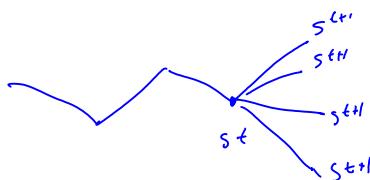
$$+ \sum_t \sum_{s^t} \lambda_t(s^t) [A_t(s^{t-1}) + y_t(s^t) - c_t(s^t) - R_t^{-1}(s^t) A_{t+1}(s^t)]$$

FONC:

$$(1) \quad c_t(s^t): \quad \beta^t u'(c_t(s^t)) \pi_t(s^t) = \lambda_t(s^t)$$

$$(2) \quad A_{t+1}(s^t) := \lambda_t(s^t) R_t^{-1}(s^t) + \sum_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1})$$

\uparrow
Sum over all s^{t+1} stemming from s^t



Substitute (2) into (1) \Rightarrow

$$R_t^{-1}(s^t) \beta^t u'(c_t(s^t)) \pi_t(s^t) = \beta^{t+1} \sum_{s^{t+1}|s^t} u'(c_{t+1}(s^{t+1})) \pi_{t+1}(s^{t+1})$$

$$\Rightarrow I = \beta R_t(s^t) \sum_{s^{t+1}|s^t} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_{t+1}(s^{t+1}|s^t)$$

|||

$$\left(\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \right)$$

or

$$I = \beta E_t R \frac{u'(c_{t+1})}{u'(c_t)}$$

This is a version of (H) under complete markets.

But (H) does not hold.

Punch lines:

(1) under complete markets

inter-temporal marginal rates of substitution

$$\beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_t(s^{t+1}|s^t)$$

are equated for all consumers able to
trade at relative prices $q_{t+1}^L(s^{t+1}|s^t)$

(2) Under incomplete markets with only risk-free
security with gross return $R_t(s^t)$,

only "average" inter-temporal rate of
substitution

- - - - -

$$\beta \sum_{s^{t+1}|s^t} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_{t+1}(s^{t+1}|s^t)$$

are equated.

Search theory

Saturday, October 15, 2011
11:08 AM

- First some facts about non-negative random variables.

$\rho \sim$ a nonnegative r.v. with p.d.f.

(probability density function) $f(\rho)$

and c.d.f. (cumulative density function)

$$F(\rho) = \int_0^{\rho} f(s) ds .$$

Assume $F(0) = 0$, $F(B) = 1$.

B is the upper bound on the random variable.

Mean or expectation of $\rho = \int_0^B \rho f(\rho) d\rho \equiv E(\rho)$

To get another useful formula for $E(\rho)$

remember integration by parts:

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

apply integration by parts to

$$\int_0^B \rho f(\rho) d\rho \text{ by taking}$$

$$u = \rho, dv = f(\rho) d\rho \Rightarrow$$

$$du = d\rho, v = F(\rho) \Rightarrow$$

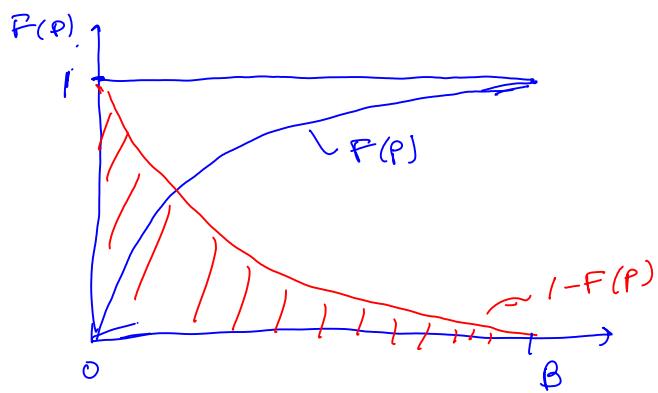
$$\int_0^B \rho f(\rho) d\rho = \rho F(\rho) \Big|_0^B - \int_0^B F(\rho) d\rho$$

$$= B - \int_0^B F(\rho) d\rho$$

(*) $= \int_0^B [-F(\rho)] d\rho = E(\rho)$

(*) is alternative formula for the mean of a nonnegative

R.V.



$$E\rho = \int_0^B (1-F(\rho)) d\rho$$

Consider two independent random variables ρ_1 and ρ_2 , both drawn from F . Event

$$\{\rho_1 < \rho\} \cap \{\rho_2 < \rho\}$$

↑ "intersection" or "and"

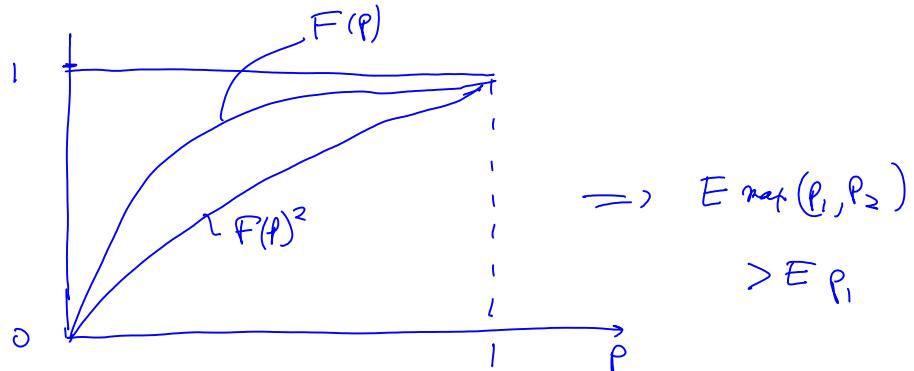
which means $\max(\rho_1, \rho_2) < \rho$

The probability that $\max(\rho_1, \rho_2) < \rho$ is evidently

$$F(\rho) \cdot F(\rho) = F(\rho)^2$$

\rightarrow \sim \sim \sim \sim \sim \sim

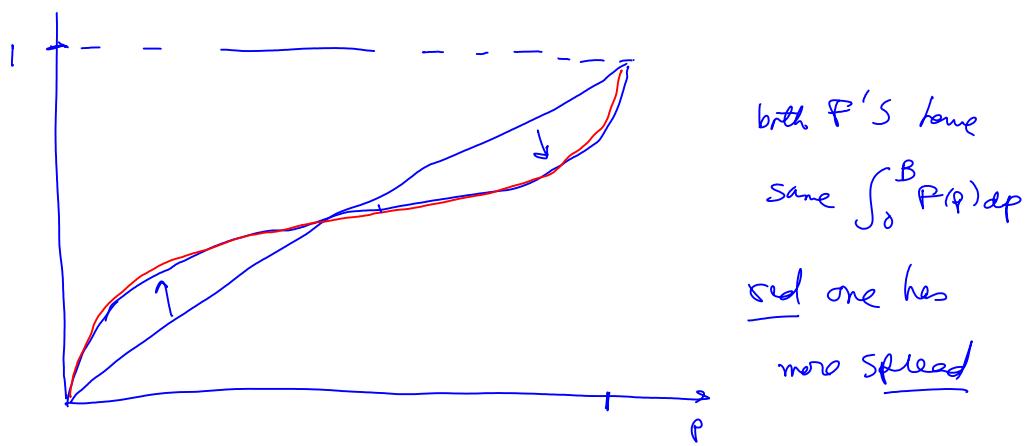
$$\text{Then } E \max(p_1, p_2) = \int_0^B [1 - F(p)]^2 dp$$



Mean preserving spread:

Keep $\int_0^B F(p) dp$ constant -

but move probabilities to both tails:



McCall search model.

Infinitely lived worker wants to maximize

$$E_p \sum_{t=0}^{\infty} \beta^t y_t \quad \beta \in (0,1)$$

expectation $\begin{cases} w & \text{if employed} \\ \dots & \end{cases}$

$$y_t = \begin{cases} c & \text{if unemployed.} \end{cases}$$

Worker starts out unemployed. Each period, a previously unemployed worker draws one, and only one, offer to work forever at wage w . The wage is drawn from a c.d.f. F , where $F(0) = 0$, $F(B) = 1$ for $B > 0$, $F'(w) = f(w)$

$$\begin{matrix} \uparrow \\ \text{c.d.f.} \end{matrix} \quad \begin{matrix} \uparrow \\ \text{density} \end{matrix}$$

There is no recall of past offers. Each period, the unemployed worker can either accept the current wage drawn from F , or reject it, collect $c > 0$ this period, and wait to draw another offer next period. Again, if he accepts the offer, he works forever at the wage w ; he neither quit nor be fired.

Problem: Find the worker's optimal strategy.

Solution: Let Q = optimal value of the problem for a worker who is about to draw a wage offer; i.e.,

$$Q = E_{-1} \sum_{t=0}^{\infty} \beta^t y_t$$

^ . . .

\uparrow expectation before wage is drawn.

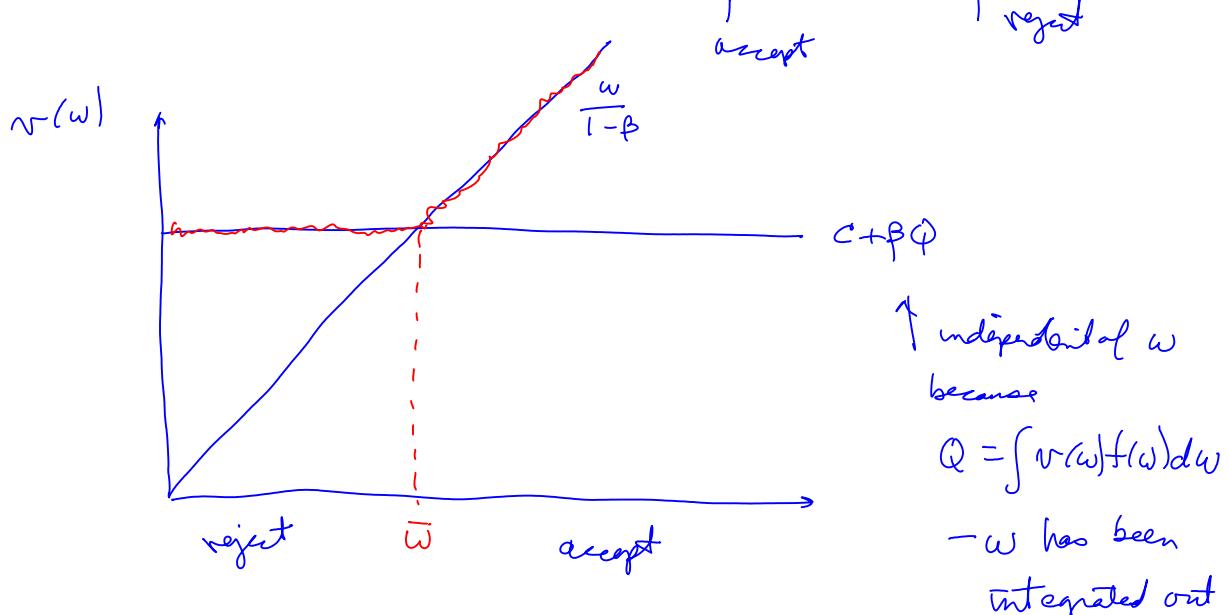
Let $v(w)$ be the optimal value of the problem for a previously unemployed worker who has just drawn offer w and is about to decide what to do.

Evidently,

$$Q = \int_0^B v(w) f(w) dw$$

Key observation:

$$(*) \quad v(w) = \max_{\{ \text{accept, reject} \}} \left\{ \frac{w}{1-\beta}, C + \beta Q \right\}$$



\uparrow independent of w
because
 $Q = \int v(w) f(w) dw$
 $-w$ has been
integrated out

(*) is called a Bellman equation. (Dynamic programming)

How can we solve (*)?

Method 1: Iterate on

\downarrow dummy to integrate

$$N_{j+1}(\omega) = \max \left\{ \frac{\omega}{1-\beta}, C + \beta \int_0^B N_j(\tilde{\omega}) f(\tilde{\omega}) d\tilde{\omega} \right\}$$

↑
next' pend's

start from $N_0(\omega) = 0$

This is guaranteed to converge.

Method 2:

$$N(\omega) = \max \left\{ \frac{\omega}{1-\beta}, C + \beta \int_0^B N(\tilde{\omega}) f(\tilde{\omega}) d\tilde{\omega} \right\}$$

evidently

$$\frac{\bar{\omega}}{1-\beta} = C + \beta \int_0^B N(\tilde{\omega}) f(\tilde{\omega}) d\tilde{\omega}$$

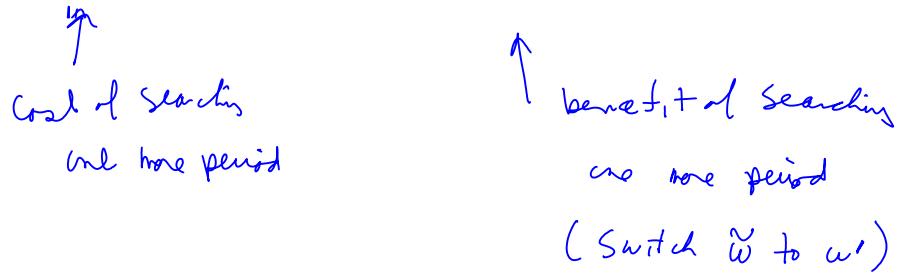
$$= C + \beta \int_0^{\bar{\omega}} \frac{\bar{\omega}}{1-\beta} f(\tilde{\omega}) d\tilde{\omega} + \beta \int_{\bar{\omega}}^B \frac{\tilde{\omega}}{1-\beta} f(\tilde{\omega}) d\tilde{\omega}$$

$$\frac{\bar{\omega}}{1-\beta} \left(\int_0^{\bar{\omega}} f(\tilde{\omega}) d\tilde{\omega} + \int_{\bar{\omega}}^B f(\tilde{\omega}) d\tilde{\omega} \right) = C + \beta \int_0^{\bar{\omega}} \frac{\bar{\omega}}{1-\beta} f(\tilde{\omega}) d\tilde{\omega} + \beta \int_{\bar{\omega}}^B \frac{\tilde{\omega}}{1-\beta} f(\tilde{\omega}) d\tilde{\omega}$$

$$\bar{\omega} \int_0^{\bar{\omega}} f(\tilde{\omega}) d\tilde{\omega} - C = \frac{1}{1-\beta} \int_{\bar{\omega}}^B (\beta \tilde{\omega} - \bar{\omega}) f(\tilde{\omega}) d\tilde{\omega}$$

add $\bar{\omega} \int_{\bar{\omega}}^B f(\tilde{\omega}) d\tilde{\omega}$ to both sides to get

$$(\bar{\omega} - C) = \frac{\beta}{1-\beta} \int_{\bar{\omega}}^B (\omega' - \bar{\omega}) f(\omega') d\omega'$$



Benefit of search:

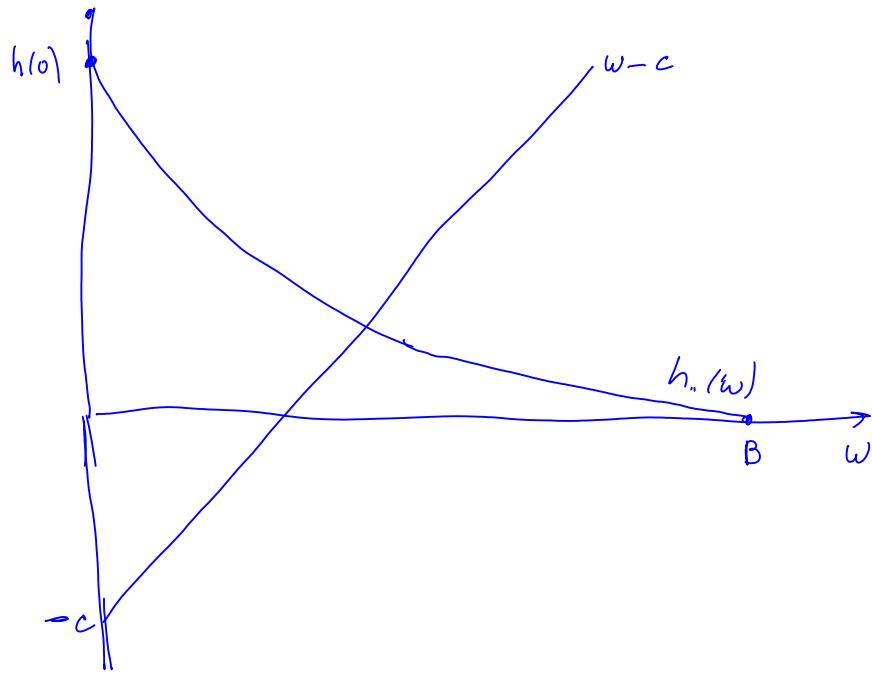
$$h(w) = \frac{\beta}{1-\beta} \int_w^B (w' - w) f(w') dw'$$

$$h(0) = \frac{\beta}{1-\beta} E(w) > 0$$

$$h(B) = 0$$

$$h'(w) = \frac{\beta}{1-\beta} [1 - F(w)] < 0$$

$$h''(w) = \frac{\beta}{1-\beta} F'(w) > 0$$



Fixing: probability of $\alpha > 0$ of being fired at the beginning of each period, after the first period in the job.

Let $\hat{v}(w)$ be the optimal value for a previously unemployed worker who has offer w in hand and is about to choose what to do.

If reject offer, get

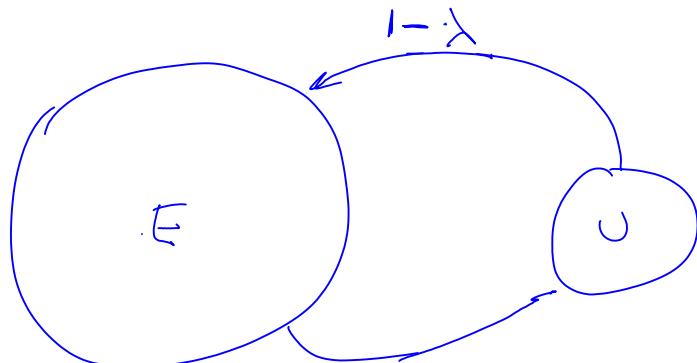
$$\hat{v}(w) = c + \beta \int \hat{v}(w') f(w') dw'$$

If accept offer, get

$$\hat{v}(w) = w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha [c + \beta \int \hat{v}(w') f(w') dw']$$

\uparrow today \uparrow not fired tomorrow \uparrow fired tomorrow \uparrow day after tomorrow

$$\tilde{v}(w) = \max_{\text{accept, reject}} \left\{ w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha [c + \beta E \hat{v}], c + \beta E \hat{v} \right\}$$



$$\lambda = \int_0^{\bar{w}} f(w') dw'$$

"prob stay
unemployed"

$$1 - \lambda = \int_{\bar{w}}^B f(w') dw' \\ = \text{prob } \underline{\text{leave}}$$

unemployed

Two quite different models of "financial crises"
and what government can do to contribute to them
or stop them.

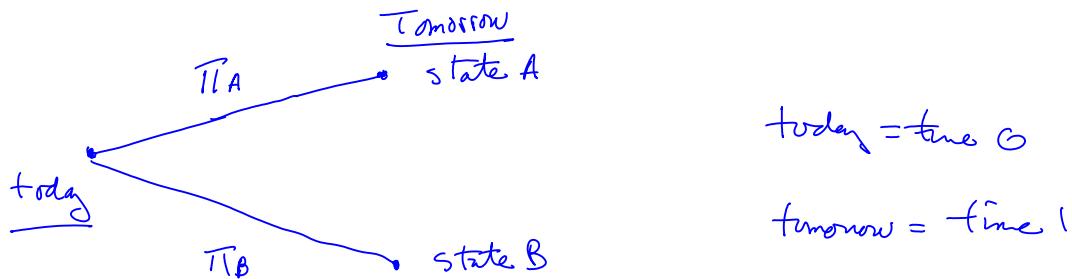
- Moral hazard (Kareken - Wallace)
- Deposit insurance and bank runs (Diamond - Dybvig)

Kareken - Wallace moral hazard analysis, 1977

Message: deposit insurance without regulation

Creates incentives for banks to take
too much risk.

A very simple two period ("today" and "tomorrow")
analysis that assumes two states tomorrow
and that takes as exogenous prices for
state contingent claims.



today = time 0

tomorrow = time 1

$$\Pi_A = \text{prob of state A}$$

$$\Pi_B = \text{prob of state B}$$

Exogenous prices at which a bank can trade:

p_A = price of one unit of consumption tomorrow if state A occurs, measured in time
○ good per good at time 1 in state A

p_B = price of one unit of consumption tomorrow if state B occurs, measured in time
○ good per good at time 1 in state B

Risk-free interest rate: $(1+r)$

$p_A + p_B = \frac{1}{1+r}$ = price of one unit of consumption tomorrow for sure in units of consumption at $t=0$

Model of a bank.

Today ($t=0$)

receives deposits D

choose a portfolio:

claims paying off at $t=1$ in state A: Q_A

claims paying off in state B at $t=1$: Q_B

Budget constraint: $p_A Q_A + p_B Q_B = D$

Assets	Liabilities	Balance sheet at $t=0$
\sim	\sim	$ $



$t=1$: • the state $S \in \{A, B\}$ is realized,

say state S.

- collect Q_S

- payoff depositors $(1+r)D$

- keep any profits $Q_S - (1+r)D$

- if $Q_S - (1+r)D < 0$, pay depositors Q_S .

Bank profits:

$$\text{Profit} = \max \{ Q_S - (1+r)D, 0 \}$$

Assume

(1) There is no deposit insurance

(2) Depositors will put D into the bank at $t=0$ only if

in state A $Q_A \geq (1+r)D$

in state B $Q_B \geq (1+r)D$

Guess and verify: the only portfolio that works is

$$Q_A = Q_B = \bar{Q} \quad \text{note: this is a risk-free portfolio - same } Q \text{ for states A \& B}$$

$$P_A \bar{Q} + P_B \bar{Q} = D$$

$$(P_A + P_B) \bar{Q} = D$$

$$\bar{Q} = \frac{D}{P_A + P_B}, \text{ but } \frac{1}{P_A + P_B} = 1+r$$

So,

$$\text{in state A: } Q_A = \bar{Q} = (1+r)D$$

$$\text{in state B: } Q_B = \bar{Q} = (1+r)D$$

expected profit of bank:

$$\Pi_A \underbrace{\max \{ Q_A - (1+r)D, 0 \}}_0 + \Pi_B \underbrace{\{ Q_B - (1+r)D, 0 \}}_0$$

$$= 0.$$

Remark: Any other portfolio would involve the bank taking on risk and would lead to

$$Q_S - (1+r)D < 0 \text{ in some states,}$$

causing all depositors to desert the bank.

Conclusion: without deposit insurance,

Banks only attract depositors if they have risk-free portfolios. Doing so gives the bank zero profits.

Remark: Free entry into the business of banking can rationalize the zero profits condition that banks offer deposit rate $(1+r) = (p_A + p_B)^{-1}$.

Deposit insurance:

The government offers deposit insurance that guarantees a return of $(1+r)$ to depositors. If the bank's assets can't cover $(1+r)D$ at $t=1$, the government makes up the difference.

Consider a bank portfolio with

$$Q_A > \bar{Q} \quad \text{and} \quad Q_B < \bar{Q}$$

$$\text{where } \bar{Q} = \frac{D}{p_A + p_B}$$

\Rightarrow when $S=B$ (state B occurs), then the bank defaults at $t=1$, activating the deposit insurer to pay the depositors the residual.

- With deposit insurance, the bank can attract depositors regardless of its portfolio, even if it is risky.

If it is risky.

- Key fact when there is deposit insurance,
it can happen that

$$D > p_A Q_A + p_B Q_B$$

$$D - (p_A Q_A + p_B Q_B) = \underbrace{\text{value of bank portfolio}}_{\text{value of bank}} - \underbrace{\text{value of deposit insurance to the bank}}$$

$$\text{when } Q_A > \bar{Q} = \frac{D}{p_A + p_B}$$

the deposit insurer is taking on risk - and is implicitly issuing claims to the depositors in the event that $s = B$.

- —

expected profits of the bank with $Q_A > \bar{Q}$, $Q_B < \bar{Q}$:

$$= \pi_A \cdot \max \left\{ \underbrace{Q_A - (1+\varepsilon) D}_{> 0}, 0 \right\} + \pi_B \cdot \max \left\{ \underbrace{Q_B - (1+\varepsilon) D}_{< 0}, 0 \right\}$$

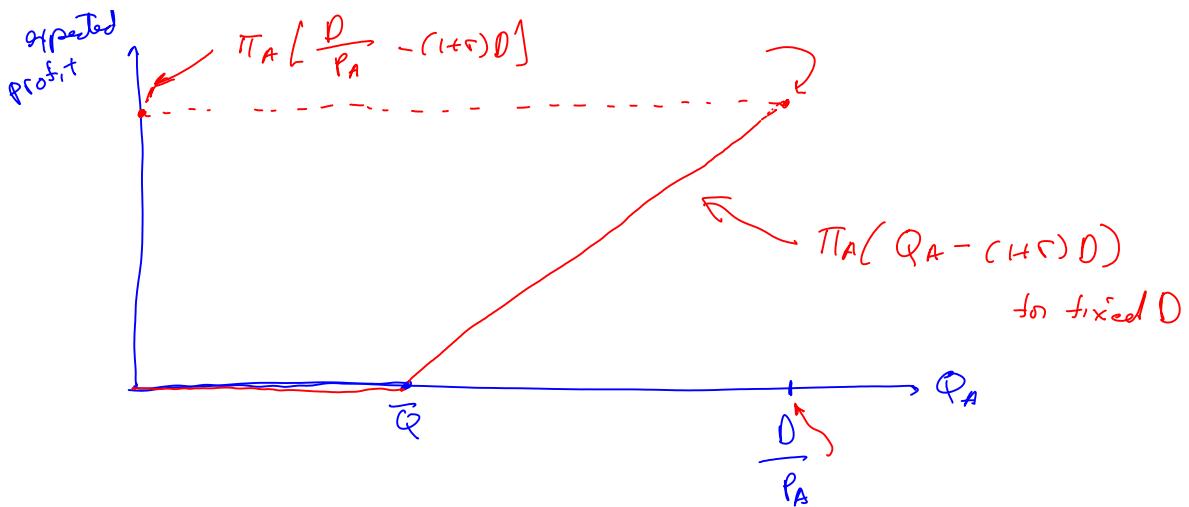
$$= \pi_A \max \left\{ Q_A - (1+\varepsilon) D \right\}$$

Highest profits occur where

$$Q_A = \frac{D}{p_A}, \quad Q_B = 0$$

(very risky port folio!)

$$\text{Expected } \pi_A = \pi_A \left[\frac{D}{p_A} - (1+\varepsilon) D \right] \rightarrow$$



Highest profit value for Q_A

$$Q_A = \frac{D}{P_A}, \quad Q_B = 0 \quad \text{when} \quad Q_A > \bar{Q}$$

Profits when $Q_A > \bar{Q}$

Expected profits =

$$\begin{aligned} & \pi_A \left[\frac{D}{P_A} - (1+r)D \right] \\ &= \pi_A D \left[\frac{1}{P_A} - \frac{1}{P_A + P_B} \right] > 0 \end{aligned}$$

Conclusion: set D arbitrarily large, and take on maximum risk. A force for banks to grow large and to expose the insurance agency to risks of big losses.

When $Q_A = \frac{D}{P_A}$ and $Q_B = 0$, the expected value of the losses of the insurance agency are

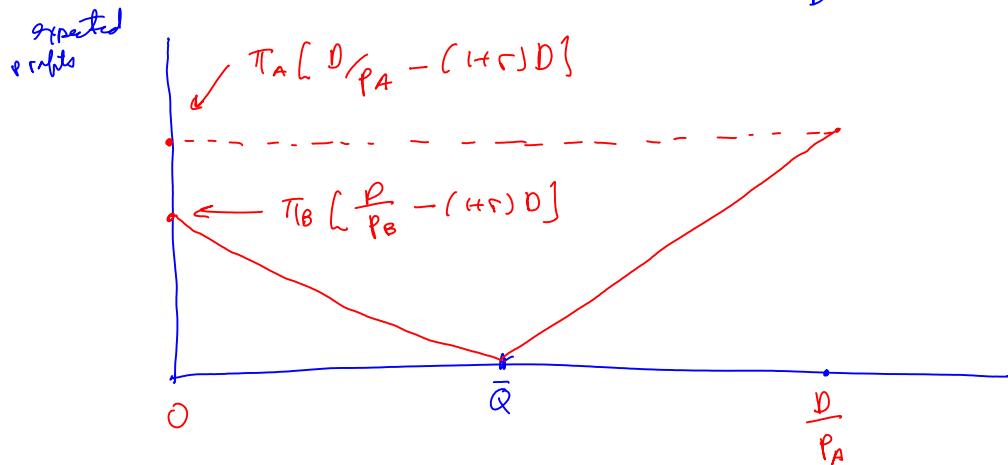
$$\underbrace{\pi_B (\gamma+1) D}_{\text{pay in state } B \text{ because } Q_B = 0.}$$

Now consider a bank portfolio with $Q_A < \bar{Q}$ and
 $Q_B > \bar{Q}$

expected profit of bank =

$$\begin{aligned} & \pi_A \cdot \max \{ Q_A - (1+r)D, 0 \} + \pi_B \max \{ Q_B - (1+r)D, 0 \} \\ &= 0 + \pi_B [Q_B - (1+r)D] \\ &= \pi_B \cdot \left[\frac{D - \frac{\pi_A Q_A}{P_A}}{P_B} - (1+r)D \right] \end{aligned}$$

highest profit: $Q_A = 0, Q_B = \frac{D}{P_B}$



Whether bank sets $Q_A = 0$ or $Q = \frac{D}{P_A}$ for a given D

depends on $\frac{\pi_B}{P_B}$ relative to $\frac{\pi_A}{P_A}$.

* bank wants to set D arbitrarily large when it has deposit insurance.

Conclusion: Banks maximize expected profits by

taking on much risk. (and becoming very large)

Remarks of Kaneen & Wallace:

- (1) Without deposit insurance and without regulation, bankruptcy does not occur because banks take on no risk. Banks take on no risk because if they did, depositors would desert them. Force for making banks be cautious - depositors watching banks' portfolios
- (2) If bank liabilities are insured, as under FDIC at a premium that is independent of portfolio risk (we set the premium to 0 in the above model) and if banks' portfolios are not regulated, then in some future states of the world there are numerous bankruptcies.

Thus, you cannot simultaneously favor

- deposit insurance (to avoid bank runs)
- unregulated banking.

Regulation of portfolios:

- capital requirements
- direct restrictions on portfolio composition

Similar forces at work:

articles by Jeffrey Sachs and Joe Stiglitz

Diamond-Dybvig model of bank panics

Themes and ideas:

- benefits of societal risk-sharing achieved through financial intermediaries
- private information that gives rise to incentive compatibility constraints ("truth telling")
- individuals' best responses to their conjectures about aggregate behavior
- multiple equilibria (interpretable as "bank run")
- government policy as a device to select unique (good) equilibrium (deposit insurance)

Model thus has many good features not present in the Diamond-Wallace model. But...

It neglects moral hazard.

Model justifies many of Bernanke's actions.

There is a large number D of potential depositors endowed with 1 unit of a good at time 0. The consumers live at $t = 0, 1, 2$, and are ex ante identical.

With probability $\pi \in (0, 1)$, a consumer wants to consume at $t=1$ and gets utility $u(c_1)$. and gets

Consume at $t=1$ and gets utility $u(c_1)$; and gets no utility from consumption c_2 at $t=2$;

With prob $(1-\pi)$, the consumer wants to consume at date $t=2$ and receive utility $u(c_2)$ from consuming c_2 . He gets no utility from c_1 at $t=1$.

Assume that $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$, $\gamma > 1$.

A fraction π of the population of size D is the "early" type of consumer, a fraction $(1-\pi)$ is the "late" type of consumer.

At $t=0$, consumers don't know their types. But at time $t=1$, all consumers learn their types. This knowledge is private: and known only to themselves

The 1 unit of the good at time 0 can be invested in a technology at time 0.

It will yield 1 unit of time 1 good if the project is terminated at $t=1$, and $(1+\rho)$ units of the time 2 good if left in the investment until time $t=2$. $\rho > 0$

If the good is withdrawn from the investment technology at time 1, it can be stored

until time 2 without loss or gain - i.e.

1 unit of time 1 good is transformed into 1 unit of time 2 good outside the investment

1 unit of time 2 good outside the investment technology.

Technology

time	0	1	2
input, output	1	→ 1	
1 → (1+r)			
1 → 1			

Remarks: (1) The preference structure is designed to capture "liquidity" shocks - random impulses for early withdrawal.

(2) The technology will $(1+r)$, $r > 0$ gross return if left in for two period, but gross return 1 if "liquidated prematurely" is designed to mimic the situation faced by financial intermediaries forced to make premature liquidations.

(3) Because there is a large number - think of a continuum of measure D - of ex ante identical consumers, there is scope for a social insurance arrangement.

A bank can play this role.

Autarky

First, let's consider what would happen if

the consumer was in autarky - isolated and alone. He would confront the investment technology on his own and choose c_1 and c_2 as follows:

- a) at $t=0$, he would invest 1 in the investment technology.
- b) at $t=1$, his type is revealed. If he is an early type, he would withdraw the entire amount and set $c_1=1$. If he is a late type, he leaves the investment in until maturity and enjoys $c_2 = (1+\epsilon)$ at $t=2$.

Note: it follows that

$$(*) \quad \frac{u'(c_1)}{u'(c_2)} = \frac{1}{(1+\epsilon)^{-\alpha}} \quad \text{at the autarky allocation} \\ c_1 = 1, c_2 = 1 + \epsilon$$

A benevolent planner:

Let $c_1 = \text{consn}$ of an early type at $t=1$
 $c_2 = \text{" " a late " } \text{ at } t=2$

Consider a benevolent planner who chooses c_1 & c_2 to maximize the ex-ante expected utility of a representative consumer

$$\pi u(c_1) + (1-\pi) u(c_2)$$

subject to the resource constraint

$$(1-\pi)c_2 = (1+r) \underbrace{[1 - \pi c_1]}_{\text{amount left in the investment technology}}$$

$$\pi c_1 + (1-\pi) \frac{c_2}{1+r} = 1$$

↑ ↑ ↗
faster faster because of private
of type 1 type 2 information

and "truth telling"

$$(*) \quad c_2 \geq c_1 \quad \text{constraint - needed to get late types to wait & leave}$$

Note: we could put

a Lagrange multiplier on $c_2 \geq c_1$, too. But we won't. Instead we'll...

Assume (*) satisfied, then check it later.

$c_2 \geq c_1$ is called an incentive compatibility constraint. (due to private information)

Formulate Lagrangian:

$$\begin{aligned} L = & \pi u(c_1) + (1-\pi) u(c_2) \\ & + \theta \left[1 - \pi c_1 - \frac{(1-\pi)}{(1+r)} c_2 \right] \end{aligned}$$

θ is Lagrange multiplier

$$\text{FONC: } u'(c_1) = \theta$$

$$u'(c_2) = \theta \frac{1}{1+r}$$

$$\Rightarrow \frac{u'(c_1)}{u'(c_2)} = (1+r) \quad , \quad \text{Note: this differs from the antonymy allocation } \Leftrightarrow$$

$$u'(c) = c^{-\delta} \Rightarrow$$

when $\delta \neq 1$
[but incentive constraint]

$$\left(\frac{c_2}{c_1}\right)^\gamma = (1+r)$$

satisfied only when
 $\gamma \geq 1$

$$c_2 = c_1 (1+r)^{\frac{1}{\gamma}}$$

Substitute into feasibility constraint to get

$$1 = \pi c_1 + \frac{(1-\pi)}{(1+r)} (1+r)^{\frac{1}{\gamma}} c_1$$

$$c_1 = \frac{1}{\pi + \frac{(1-\pi)}{(1+r)^{1-\frac{1}{\gamma}}}}$$

$$\gamma > 1 \Rightarrow c_1 > 1$$

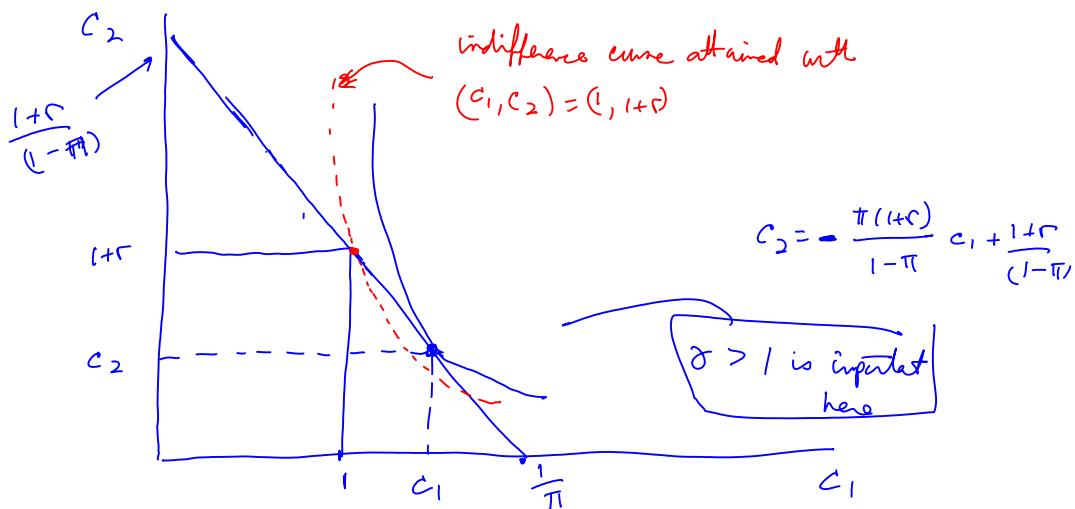
(where $(c_1, c_2) = (1, 1+r)$)
under "antiky")

$$c_2 = c_1 (1+r)^{\frac{1}{\gamma}}$$

$$= \frac{(1+r)^{\frac{1}{\gamma}}}{\pi + \frac{(1-\pi)}{(1+r)^{1-\frac{1}{\gamma}}}}$$

$$c_2 = \frac{(1+r)^{\frac{1}{\gamma}}}{(1+r)^{1-\frac{1}{\gamma}} \pi + (1-\pi)} < 1+r$$

See following indifference curve for welfare gain with $\gamma > 1$:



social intertemporal transformation

$$c_2 = -\frac{\pi(1+r)}{1-\pi} c_1 + \frac{(1+r)}{(1-\pi)}$$

rate: $\frac{\pi}{1-\pi} (1+r)$

$$c_1 = 1 \Rightarrow$$

$$c_2 = -\frac{\pi(1+r)}{1-\pi} + \frac{(1+r)}{(1-\pi)} = \frac{(1-\pi)}{1-\pi} (1+r) = 1+r$$

$\therefore (1, (1+r))$ is on the feasibility line

Note: the "truth telling" constraint

$c_2 > c_1$ is satisfied for us here when $\delta > 1$

because:

$$c_2 = (1+r)^{\frac{1}{\delta}} c_1 > c_1 \quad \checkmark$$

Thus: actually $c_1 = 1, c_2 = (1+r)$

risk-shy: $c_1 > 1, c_2 = (1+r)^{\frac{1}{\delta}} c_1 < (1+r)$

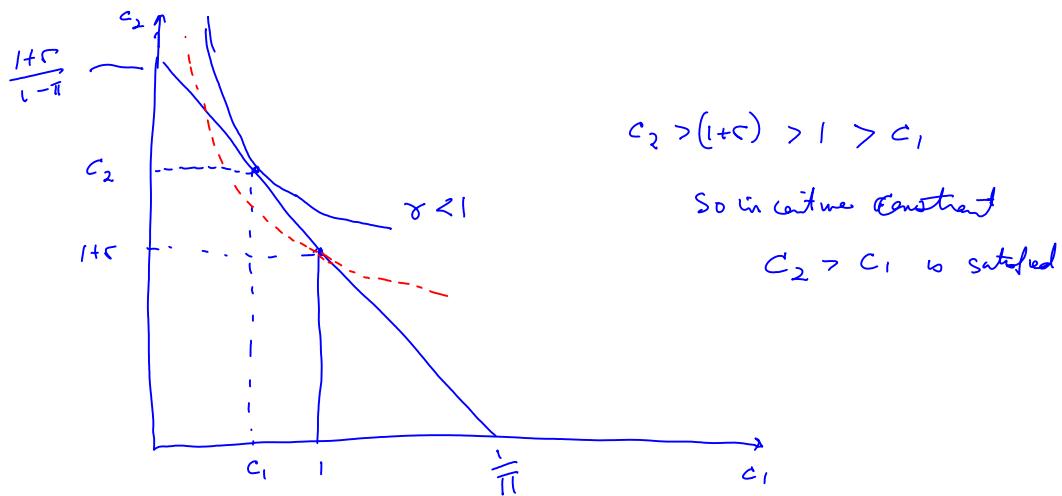
when $\delta > 1$

when $\delta < 1$, we have $c_1 < 1, c_2 = (1+r)^{\frac{1}{\delta}} c_1 > c_1$

so long as $r > 0, \delta > 0$

so incentive constraint still satisfied

When $\gamma < 1$ the picture is



Remark: $c_2 = (1+\epsilon)^{\frac{1}{\gamma}} c_1$ or $\frac{c_2}{c_1} = (1+\epsilon)^{\frac{1}{\gamma}}$

\Rightarrow bigger "spread" when $\gamma < 1$.

I implement this risk-sharing arrangement with a banking deposit contract. The contract imposes a sequential service constraint - "first come, first serve".

Let $q = c_1$ from the planning problem.

$$q = c_1 > 1$$

$$\boxed{q = c_1}$$

Let $D = \text{deposits}$

$W = \text{with draws at } t=1$.

Bank offers q to early withdrawers,
 $\max(D - qW, 0)(1+\epsilon)$ to late withdrawers.

If $W = \pi D$ if only early types withdraw

then payout to late withdrawers is $c(1+r)[D - qW]$, so return is

$$\begin{aligned}
 & \frac{D - qW}{(1-\pi)D} = \frac{D - D\pi q}{(1-\pi)D} (1+r) \\
 & = \frac{(1 - \pi q)(1+r)}{(1-\pi)} \\
 & = \frac{(1 - \pi c_1)(1+r)}{(1-\pi)} = c_2 \text{ in option of} \\
 & \text{pleasing problem.}
 \end{aligned}$$

If you think no one else will run, this will be

Your expected return $c_2 \geq q > c_1$, so "you won't want to run."

gross return in period 1 with drawal

$$q \text{ if } D - qW \geq 0 ; \quad [W = w. \text{ withdraw}]$$

but if $qW > D$, can't pay everyone q; then pay

- q to the first x people in line,

$$\text{where } qx = D \Rightarrow x = \frac{D}{q} < D$$

$$\Rightarrow \text{because } q > 1$$

$$\frac{1}{q} = \frac{x}{D} = \text{fraction of people getting paid}$$

$$\bullet 0 \text{ to fraction } (1 - \frac{1}{q})$$

If a ^{late type person}
late type person expects everyone to run to the bank in
 period 1, the ^{expected} return to you from running is

$$q \cdot \frac{1}{q} + 0 \cdot (1 - \frac{1}{q}) = 1$$

If one expects everyone to run to the bank in period 1,

the expected return from your not running is 0.

∴ if you expect everyone to run, you
run too.

How to cure the threat of a bank run.

Government deposit insurance: if government promises to pay everyone who withdraws early $q > 1$, then there will be no runs. The government will have to pay nothing out in equilibrium — (though it achieves the equilibrium by promising to pay it out off the equilibrium).

Diamond Dybvig mention one big caveat:

moral hazard. — not included in the analysis

Remarks about $\gamma < 1 \Rightarrow C_1 < 1, C_2 = (1+r)^{\frac{1}{\gamma}} C_1$,
case.

Bank contract promises to pay $q < 1$
early withdraws. Two observations

- If $W = D$ (everyone withdraws,

$$D - qW = D(1-q) > 0$$

the bank can pay everyone.

- If everyone a fraction $1-\varepsilon$ withdraws
and ε don't, the return to the late types
who leave in is
$$\frac{1}{1-\varepsilon} - \frac{1}{1-\varepsilon} \cdot \frac{1}{1+r} > 1$$

$$(1+\varepsilon) \frac{D - g(1-\varepsilon)D}{\varepsilon} = \frac{(1+r)D [1 - g(1-\varepsilon)]}{\varepsilon}$$

becomes arbitrarily large as $\varepsilon \rightarrow 0$

\therefore no late type person ever wants to withdraw.

Conclusion: with $\delta < 1$ (and $\gamma, c_1 < 1$),

there can be no runs with or without deposit insurance.

Homework 3

Thursday, October 27, 2011
11:47 AM

1. Please read the op ed article by Jeffrey Sachs.

Jeff Sachs calculations (his article criticizing
the Treasury plan)

Consider an asset that pays off

$X > 0$ with probability $\pi > 0$

and

$Y = 0$ with probability $(1-\pi)$

A risk neutral investor would be willing to

pay

$$\pi \cdot X + (1-\pi) Y = \pi X \quad \text{for the asset}$$

Now suppose that the government offers to

lend the buyer α , what he pays for the
asset. The loan has interest rate $r=0$ and
is "non-recourse"- the borrower can walk away
from the loan costlessly if the asset does
not pay off. Here $\alpha \in (0, 1)$

How much would a risk-neutral investor
be willing to pay for the asset?

let $P =$ amount he is willing to pay

he borrows αP and puts up $(1-\alpha)P$ of his own money.

His expected payoff is

$$\pi(X - \alpha P) + (1-\pi) \max(0, Y - \alpha P)$$

\uparrow repay loan "o

payout in
good state

$$= \pi(X - \alpha P)$$

In terms of his own money, the expected payoff of $\pi(X - \alpha P)$ costs him $(1-\alpha)P$

so to equate the expected payoff to the cost, set

$$\pi(X - \alpha P) = (1-\alpha)P$$

$$\Rightarrow \alpha P = \frac{\pi}{\pi \alpha + (1-\alpha)} \cdot X$$

Note that $\lim_{\alpha \uparrow 1} P(\alpha) = X$ —

a. Use these formulas to replicate Jeff Sachs' calculations.

b. Without the government-supplied leverage, the asset is worth

πX , while with the leverage it is worth

$$\frac{\pi}{\pi\alpha + (1-\alpha)} X .$$

The expected value of the loan to the government is

$$\underbrace{\pi \cdot \alpha \underbrace{P}_{\substack{\text{payoff in} \\ \text{good states}}}}_{\substack{\cdot \\ \text{good states}}} - \underbrace{\alpha P}_{\substack{\text{from loan}}}$$

$$= -\alpha P (1-\pi) = -\frac{\alpha \pi (1-\pi)}{\pi \alpha + (1-\alpha)} X < 0$$

Note that the value paid by the investor + the value of the loan to the government is

$$\begin{aligned} & \frac{\pi X}{\pi \alpha + (1-\alpha)} - \frac{\alpha \pi (1-\pi)}{\pi \alpha + (1-\alpha)} X \\ &= \frac{\pi [1 - \alpha(1-\pi)]}{\pi \alpha + (1-\alpha)} X = \pi X = \text{price of} \\ & \quad \text{asset without} \\ & \quad \text{the government loan.} \end{aligned}$$

Remark: Sach's point is that the Treasury plan transfers $\frac{\alpha \pi (1-\pi)}{\pi \alpha + (1-\alpha)}$ from the Treasury to the original owner of the asset that offers payoff $(X, 0)$ in the good state, bad state.

2. Consider the Kaseben-Wallace model of moral hazard.

Recall that when the bank takes maximal risk by investing all of its deposits in the state A claim, it exposes the insurance agency to a lot of risk.

When $Q_A = \frac{D}{P_A}$ and $Q_B = 0$, the expected value of the losses of the insurance agency are

$$\pi_B \underbrace{(r+1) D}_{\text{right in state } B \text{ because } Q_B=0}.$$

Can you think of a scheme for pricing deposit insurance that

- a. deters the bank from excessive risk-taking, and
- b. protects the deposit insurance agency from expected losses?

The idea is that the deposit insurance agency will charge the bank a fee for deposit insurance with the fee depending on (i) its total deposits, and (ii) its investment portfolio.

Growth model, 1

Thursday, October 27, 2011
3:20 PM

Some useful background

Euler's homogeneous function theorem

$$t^n F(x, y) = F(tx, ty) \quad \begin{matrix} F \text{ is} \\ \text{homogeneous of} \\ \text{order } n \end{matrix}$$

$$\text{let } x' = tx, y' = ty$$

Differentiate both sides with respect to $t \Rightarrow$

$$\begin{aligned} n t^{n-1} F(x, y) &= \frac{\partial F}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial t} \\ &= x \frac{\partial F}{\partial x'} + y \frac{\partial F}{\partial y'} \end{aligned}$$

$$n t^{n-1} F(x, y) = x \frac{\partial F}{\partial (xt)} + y \frac{\partial F}{\partial (yt)}$$

$$\text{set } t=1 \Rightarrow$$

$$n F(x, y) = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y}$$

If $n=1$ ("homogeneous of degree 1" or
"constant returns to scale")

$$F(x, y) = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y}$$

$\frac{\partial F}{\partial x}$ $\frac{\partial F}{\partial y}$
 marginal product of marginal product of
 x y

$x \frac{\partial F}{\partial x}$ is competitive factor payment to x

$x \frac{\partial F}{\partial x}$ is competitive factor payment to x

Let $F(x, y)$ be homogeneous of degree n .

Then $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are homogeneous of degree $n-1$

Proof:

$$(*) t^n F(x, y) = F(tx, ty)$$

$$x' = tx, y' = ty$$

Differentiate both sides of (*) w.r.t. x

$$t^n \frac{\partial F}{\partial x}(x, y) = \frac{\partial F}{\partial x'} \cdot \frac{\partial x'}{\partial x} = \frac{\partial F}{\partial x'} t$$

$$(*) t^{n-1} \frac{\partial F}{\partial x}(x, y) = \frac{\partial F}{\partial (xt)}(tx, ty)$$

(*) is homogeneous of degree $n-1$.

q.e.d.

Now

$$\frac{\partial F}{\partial x}(x, y) = \frac{\partial F}{\partial (xt)}(tx, ty)$$

$$\text{set } t = \frac{1}{y}$$

$$\frac{\partial F}{\partial x}(x, y) = \frac{\partial F}{\partial (\frac{x}{y})}\left(\frac{x}{y}, 1\right)$$

$$\text{let } F\left(\frac{x}{y}, 1\right) \equiv f\left(\frac{x}{y}\right)$$

$$\rightarrow \frac{\partial F}{\partial x} \leftarrow r' / x$$

$$\Rightarrow \frac{\partial F}{\partial x}(x, y) = f'\left(\frac{x}{y}\right)$$

$$F(x, y) = y f\left(\frac{x}{y}\right)$$

$$\frac{\partial F}{\partial x} = y f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} = f'\left(\frac{x}{y}\right) \quad (+)$$

$$\begin{aligned} \frac{\partial F}{\partial y} &= f\left(\frac{x}{y}\right) - y f'\left(\frac{x}{y}\right) \left(-\frac{x}{y^2}\right) \\ &= f\left(\frac{x}{y}\right) - f'\left(\frac{x}{y}\right) \left(\frac{x}{y}\right) \end{aligned} \quad (++)$$

we'll use (+) and (++) later.

Malthus & Darwin

Mathematical growth model.

$$Y_t = z F(L_t, N_t)$$

\uparrow \uparrow \uparrow
 output land labor

- F is homogeneous of degree 1
- $L_t = L$ is fixed

$$C_t = Y_t \quad (\text{all production is consumed}) \quad \text{over time}$$

$$N_{t+1} = N_t \cdot g\left(\frac{C_t}{C^*}\right)$$

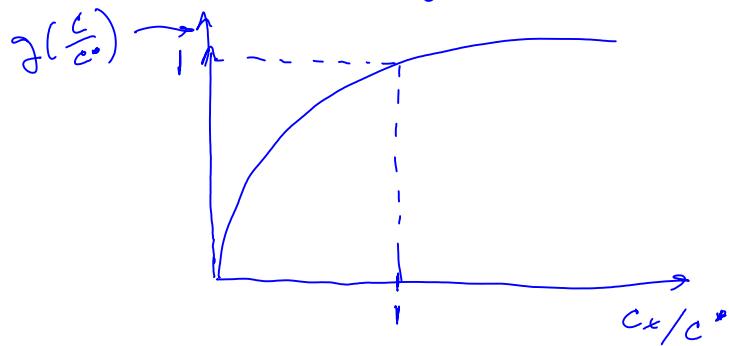
$$c_t = \frac{C_t}{N_t} = \text{consumption per capita}$$

c^* a constant

$$g(1) = 1, \quad g' > 0, \quad g'' < 0$$

we'll assume that

$$g\left(\frac{c}{c^*}\right) = \left(\frac{c}{c^*}\right)^\gamma \quad \text{where } 0 < \gamma < 1$$



$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^\gamma$$

interpretation: c^* is subsistence consumption.

$$c_t > c^* \Rightarrow \frac{N_{t+1}}{N_t} > 1 \quad \text{population grows}$$

$$c_t < c^* \quad \frac{N_{t+1}}{N_t} < 1 \quad \text{population shrinks}$$

$$\text{Note: } C_t = Y_t = z F(L, N_t)$$

$$\Rightarrow c_t = \frac{C_t}{N_t} = z F\left(\frac{L}{N_t}, 1\right) \equiv z f(l_t)$$

$$l_t = \frac{L}{N_t} \quad \text{land / labor ratio}$$

Parameterize f :

$$z f(l) = z l^\alpha, \alpha \in (0, 1) \quad (= z \left(\frac{L}{N_t}\right)^\alpha)$$

$$g\left(\frac{c_t}{c^*}\right) = \left(\frac{c_t}{c^*}\right)^\delta, \delta \in (0, 1)$$

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^\gamma = \left(\frac{z \left(\frac{L}{N_t}\right)^\alpha}{c^*}\right)^\gamma$$

$$\Rightarrow \frac{N_{t+1}}{N_t} = \frac{z^\gamma \left(\frac{L}{N_t}\right)^{\alpha\gamma}}{(c^*)^\gamma}$$

$$\Rightarrow (*) \quad N_{t+1} = \left(\frac{z}{c^*}\right)^\gamma L^{\alpha\gamma} N_t^{1-\alpha\gamma}$$

Look for a steady state $N_{t+1} = N_t = \bar{N}$

$$\bar{N} = \left(\frac{z}{c^*}\right)^\gamma L^{\alpha\gamma} \bar{N}^{1-\alpha\gamma}$$

$$\Rightarrow (+) \quad \bar{N} = L / \left(\frac{c^*}{z}\right)^{1/\alpha}$$

$$(+)\quad \text{and} \quad c_t = c^*.$$

Effect of an increase in productivity z from

$$(+) \text{ to } (++) \Rightarrow \bar{N} \text{ increases, but } c_t = \frac{c_t}{N_t} = c^*$$

stays at subsistence.

Pessimistic? Realistic?

To get dynamics - take logs

$$\log N_{t+1} = \gamma (\log z + \alpha \log L) + (1-\alpha\gamma) \log N_t$$

or

$$m_{t+1} = \phi_0 + \phi_1 m_t \quad , \quad m_t = \log N_t$$

this is of the simple form we've analyzed
often before

$$x_{t+1} = A x_t \quad , \quad x_t = \begin{pmatrix} 1 \\ m_t \end{pmatrix} .$$

\Rightarrow

$$x_t = A^t x_0 \quad , \quad x_t \text{ converges}$$

because $\alpha\gamma \in (0,1)$.

Solow growth model

- Make N_t exogenous: $N_{t+1} = (1+n) \cdot N_t$, N_0 given
- Other factor of production is now capital - K_t
not land.

$$K_{t+1} = (1-\delta) K_t + X_t \quad , \quad \delta \in (0,1)$$

$$K_{t+1} = (1-\delta)K_t + X_t \quad , \quad \delta \in (0,1)$$

$$X_t = zF(K_t, N_t) \quad \begin{matrix} \text{constant returns to} \\ \text{scale} \end{matrix}$$

$$\text{(*)} \quad C_t + X_t = zF(K_t, N_t)$$

$$\frac{K_{t+1}}{N_t} = (1-\delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}$$

$$\frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = (1-\delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}$$

$$(†) \quad k_{t+1} (1+n) = (1-\delta)k_t + x_t \quad , \quad x_t = \frac{X_t}{N_t}$$

$$\text{Solve } \underline{\text{assumes}} \quad C_t = (1-s)zF(K_t, N_t)$$

$$c_k = \frac{C_t}{N_t} = (1-s)z f(k_t)$$

$$\text{where } f(k_t) = F\left(\frac{K_t}{N_t}, 1\right)$$

$$(*) \Rightarrow X_t = zF(K_t, N_t) - C_t$$

$$= s zF(K_t, N_t)$$

$$\Rightarrow x_t = s z f(k_t)$$

thus (†) becomes

$$k_{t+1} (1+n) = (1-\delta)k_t + s z f(k_t)$$

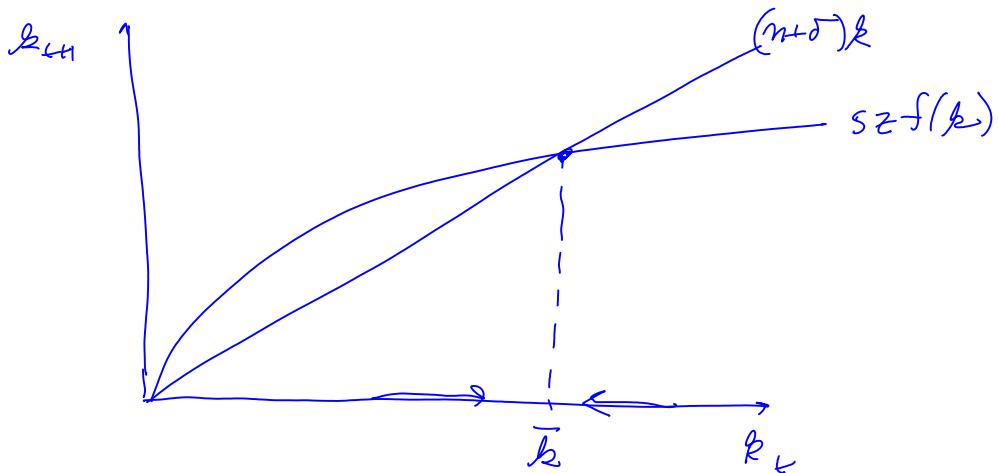
$$(**) \quad k_{t+1} = \frac{s}{1+n} z f(k_t) + \frac{(1-\delta)}{1+n} k_t$$

look for a fixed point

$$\bar{k} = \frac{s}{1+n} z f(\bar{k}) + \frac{(1-\delta)}{1+n} \bar{k}$$

$$\left[\frac{1+\delta}{1+n} - \frac{(1-\delta)}{1+n} \right] \bar{k} = \frac{s}{1+n} z f(\bar{k})$$

$$(m+\delta) \bar{k} = s z f(\bar{k})$$



Use this as a machine to study effects

on \bar{k} of (a) increase in z

(b) increase in s

(c) increase in m

Return to dynamics:

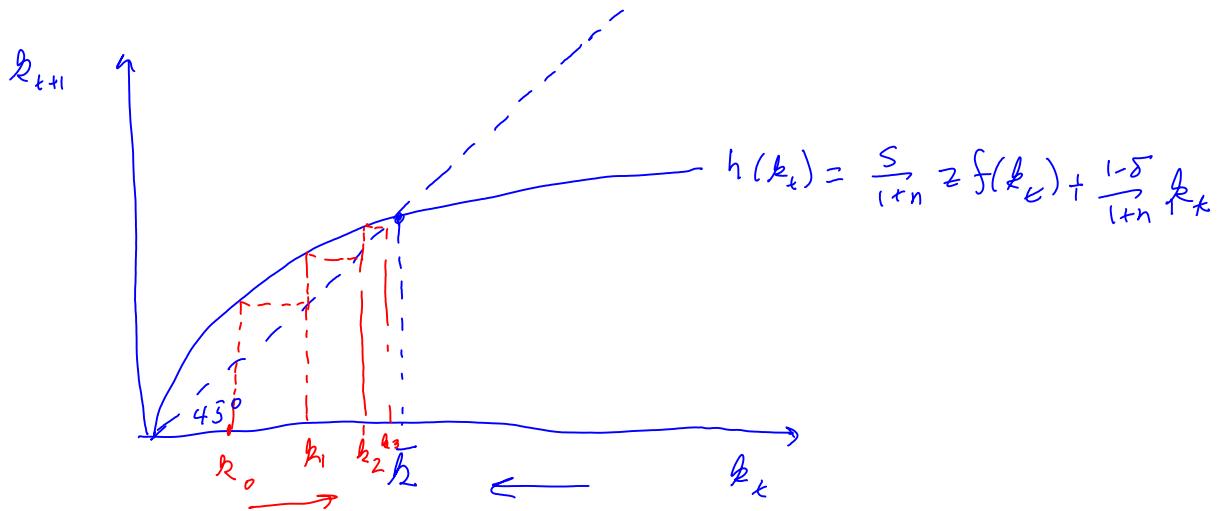
$$k_{t+1} = \frac{s}{1+n} z f(k_t) + \frac{(1-\delta)}{1+n} k_t$$

$$\equiv h(k_t) \quad ; \quad h(0) = \frac{s}{1+n} z f(0) = 0$$

$$h'(k_t) \geq \frac{S}{1+n} z + f'(k_t) + \frac{(1-\delta)}{1+n}$$

$$h'(0) = +\infty \quad \text{if } f'(0) = +\infty$$

$$\lim_{k \rightarrow \infty} h'(k) = \frac{(1-\delta)}{1+n} \quad \text{if } \lim_{k \rightarrow \infty} f'(k) = 0$$



this graph shows the dynamics - convergence to steady state

Mention real business cycle theory. $\sim z_t$ stochastic.

Growth model, 2

Thursday, October 27, 2011
8:23 AM

Growth model

Solow model took savings rate s to be a constant -
let's change that - make it the consequence of
a purposeful choice of a representative consumer.

Growth model: Technology

$$Y = F(K, N) \text{ linearly homogeneous} \quad \text{eg. } AK^\alpha N^{1-\alpha}$$

$$\lambda Y = F(\lambda K, \lambda N)$$

$$\Rightarrow F(k, 1) = f(k) \text{ where } k = K/N$$

$$F(K, N) = N f(k)$$

$$\frac{\partial F}{\partial K}(K, N) = f'(k)$$

$$\frac{\partial F}{\partial N}(K, N) = f(k) - k f'(k)$$

$$C_t + k_{t+1} = f(k_t) + (1-s)k_t \quad \text{gross dep per person}$$

P refines .

$$u(c, 1-n) = u(c) \quad (\text{no disutility of labor})$$

$$u'(c) > 0, u''(c) < 0, u'(0) = +\infty$$

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Planning problem:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t. } c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

to given

Digression: briefly optimal reading: we can solve the consumer's problem using dynamic programming. Let $v(k_0)$ be the optimal value function - i.e.

$$v(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

The Bellman equation is

$$v(k_t) = \max_{\{c_t, k_{t+1}\}} \{ u(c_t) + \beta v(f(k_t) + (1-\delta)k_t - c_t) \}$$

$$\text{FONC: } u'(c_t) - \beta v'(k_{t+1}) \leq 0 \Rightarrow \text{if } k_{t+1} > 0$$

Brennan-Schimme:

$$v'(k) = u'(c) (f'(k) + (1-\delta))$$

$$\Rightarrow u'(c_t) = \beta u'(c_{t+1}) [f'(k_{t+1}) + (1-\delta)]$$

Euler eqn. ↑ End of digression

Formulate as Lagrangian, finite T :

$$L = \sum_{t=0}^T \beta^t \{ u(c_t) + \lambda_t [f(k_t) + (1-\delta)k_t - c_t - k_{t+1}] \}$$

$$\text{FONC: } \{c_t, k_{t+1}\}, \lambda_t$$

$$c_t: u'(c_t) - \lambda_t = 0$$

$$k_{t+1}: -\lambda_t + \beta \lambda_{t+1} [f'(k_{t+1}) + (1-\delta)] , t=0, \dots, T-1$$

$$\dots \quad \alpha^T \lambda_1 - \dots - \lambda_0 = 0$$

$$k_{T+1}: -\beta^T \lambda_T \leq 0 \quad = 0 \text{ if } k_{T+1} > 0.$$

$$\stackrel{?}{=} \beta^T \lambda_T k_{T+1} = 0$$

$$\text{infinite horizon} \quad \lim_{T \rightarrow \infty} \beta^T \lambda_T k_{T+1} = 0$$

Rearrange the FOC's

$$u'(c_t) = \lambda_t$$

$$(1) \quad \lambda_t = \beta \lambda_{t+1} [f'(k_{t+1}) + (1-\delta)]$$

$$k_{t+1} + c_t = f(k_t) + (1-\delta)k_t$$

Look for steady state

$$(1) \Rightarrow \beta^{-1} \equiv (1+\rho) = f'(k) + (1-\delta)$$

$$\left[\begin{array}{l} \sigma (\rho + \delta) = f'(k) \quad \text{modified golden rule:} \\ c + \delta k = f(k) \end{array} \right] \quad \xrightarrow{\text{talk about in terms of}} \text{preferences/technology}$$

Solving by shooting: take finite horizon

$$u'(c_t) = \lambda_t \quad (1)$$

$$\lambda_t = \beta \lambda_{t+1} [f'(k_{t+1}) + (1-\delta)] \quad (2)$$

$$k_{t+1} + c_t = f(k_t) + (1-\delta)k_t \quad (3)$$

$$\beta^T \lambda_T k_{T+1} = 0 \quad (4)$$

Boundary conditions: k_0 given, $\beta^T \lambda_T k_{T+1} = 0 \Rightarrow k_{T+1} = 0$

Shooting method:

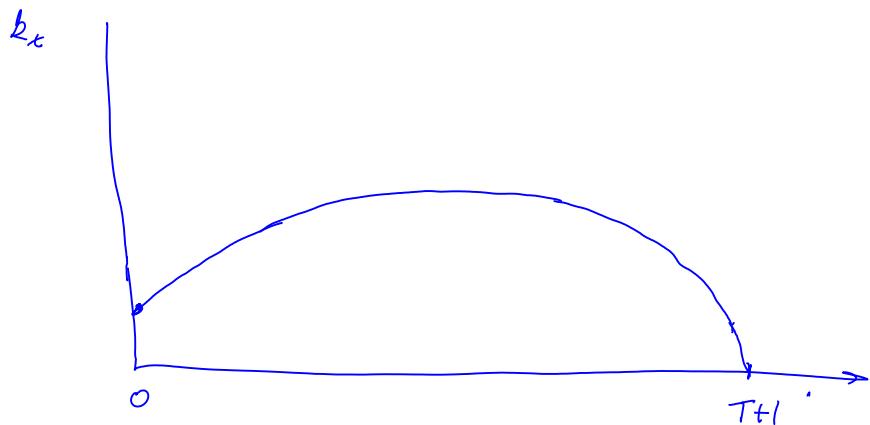
Set $\lambda_0 \Rightarrow c_0$. Then (3) $\Rightarrow k_1$. Then (2) \Rightarrow

$\lambda_1 \Rightarrow c_1$. & so on

Check to see if "target" $k_{T+1} = 0$ is hit.

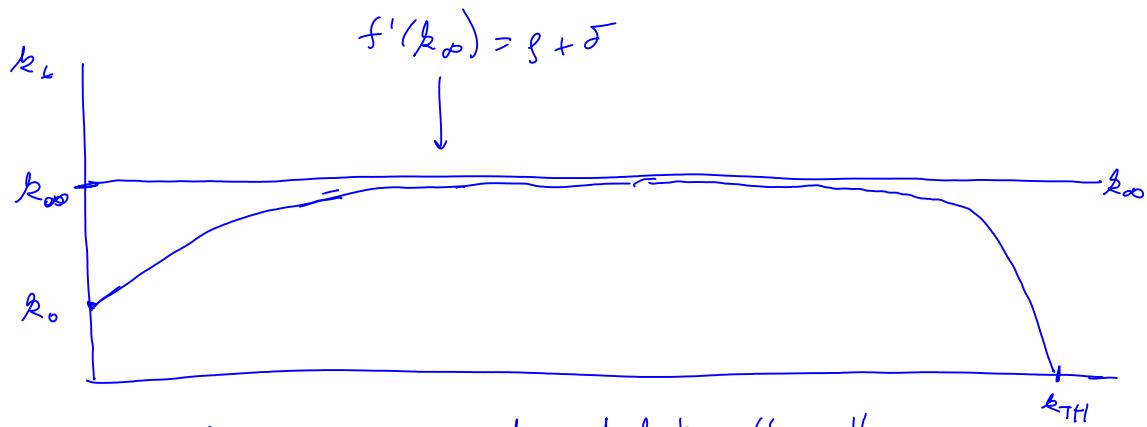
If $k_{T+1} > 0$, "eat more" by lowering λ_0 .

+ try again.



how lengthen T . $k_t \rightarrow k_\infty$ where $f'(k_\infty) = g + \delta$

Digression: "turnpike" theorem idea.



For large $T+1$, - spend most of time "near"
the "turnpike" k_∞

 end of digression

• Interpretation of solution in terms of forces impeding
consumption smoothing. or --

"more about the shooting algorithm"

-- .

Write as

$$u'(c_t) = \beta u'(c_{t+1}) [f'(k_{t+1}) + (-\delta)]$$

$$1 = \beta \frac{u'(c_{t+1})}{u'(c_t)} \overset{\sigma}{\Phi}_{t+1}$$

$$\overset{\sigma}{\Phi}_{t+1} \equiv f'(k_{t+1}) + (-\delta) \quad (\text{just a definition of } \overset{\sigma}{\Phi}_{t+1})$$

For example, let

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln c & \text{if } \gamma = 1 \end{cases}$$

$$\begin{aligned} \frac{u'(c_{t+1})}{u'(c_t)} &= \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \\ &= \left(\frac{c_t}{c_{t+1}} \right) \quad \text{if } \gamma = 1 \end{aligned}$$

Then we have, if $\gamma = 1$,

$$\begin{aligned} 1 &= \beta \left(\frac{c_t}{c_{t+1}} \right) [f'(k_{t+1}) + (-\delta)] \\ c_{t+1} &= \left(\beta [f'(k_{t+1}) + (-\delta)] \right) c_t \end{aligned}$$

where $f' > 0, f'' < 0$

$$f'(\bar{k}) = \varphi + \delta$$

$$\begin{aligned} \therefore \beta [f'(\bar{k}) + (-\delta)] &= \beta [\varphi + \delta + (-\delta)] \\ &= \beta [\varphi] = 1 \end{aligned}$$

$$\text{so } c_{t+1} = c_t \quad \text{when } k_{t+1} = \bar{k} \quad (\text{steady state})$$

Also, $k_{t+1} < \bar{k} \Rightarrow$

$$\beta [f'(k_{t+1}) + (1-\delta)] > 1 \text{ because } f'' < 0$$

(lower k , higher f')

$k_{t+1} > \bar{k} \Rightarrow$

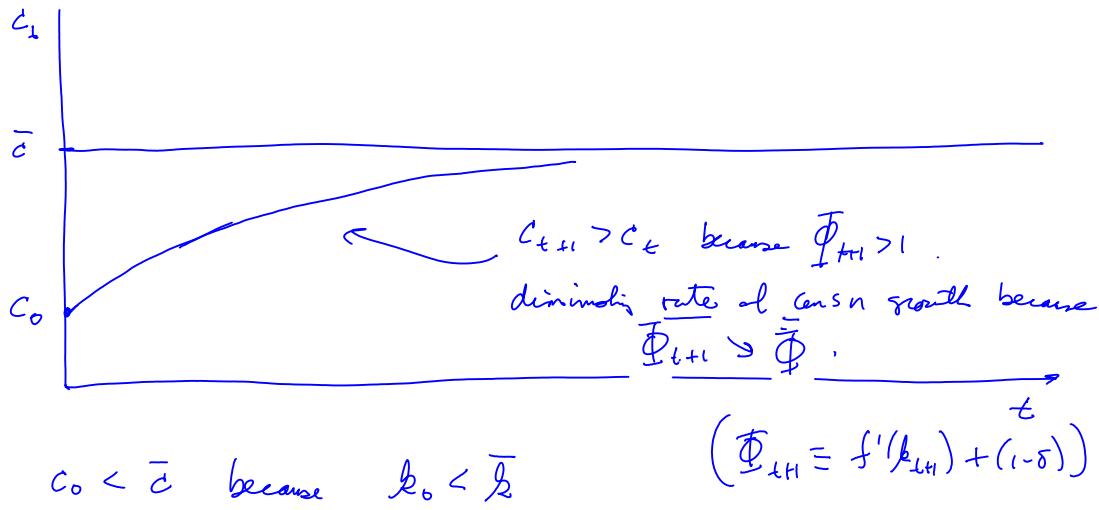
$$\beta [f'(k_{t+1}) + (1-\delta)] < 1$$

steady state consn:

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \Rightarrow$$

$$\bar{c} + \bar{k} = f(\bar{k}) + (1-\delta)\bar{k}$$

$$\bar{c} = f(\bar{k}) - \delta\bar{k} \quad \text{or} \quad \bar{c} + \delta\bar{k} = f(\bar{k})$$



$c_0 < \bar{c}$ because $k_0 < \bar{k}$

$$\left(\Phi_{t+1} = f'(k_{t+1}) + (1-\delta) \right)$$

$$c_{t+1} = \left(\beta [f'(k_{t+1}) + (1-\delta)] \right) c_t \quad \text{when } \gamma = 1$$

$$\text{or} \quad 1 = \left(\frac{c_{t+1}}{c_t} \right)^{\gamma} \beta [f'(k_{t+1}) + (1-\delta)]$$

$$\Leftrightarrow \left(\frac{c_{t+1}}{c_t} \right)^{\gamma} = \beta [f'(k_{t+1}) + (1-\delta)]$$

$$c_{t+1} = \left[\beta [f'(k_{t+1}) + (1-\delta)] \right]^{\frac{1}{\gamma}} c_t \quad \text{for } \gamma = 1$$

$$c_{t+1} = (\Phi_{t+1})^{\frac{1}{\gamma}} c_t$$

$$c_{t+1} = (\phi_{t+1})^{\frac{1}{\gamma}} c_t$$

γ gives the elasticity of consn growth with respect to ϕ_{t+1} .

$$c_{t+1} > c_t : \text{because } \phi_{t+1}^T > 1$$

diminishing rate of consumption growth because

$$\phi_{t+1} \rightarrow \text{over time } \phi_{t+1} \rightarrow \beta [f'(k) + (-\delta)]$$

Summary: Why not complete consumption smoothing:

because $\phi_{t+1} \neq 1$ always - only in
a steady state

Decentralization: Price system $\{q_t, \bar{w}_t, r_t\}_{t=0}^{\infty}$ at date 0.

Household problem:

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) , \quad 0 < \beta < 1$$

where $U(c_t, l_t)$, $u_c > 0, u_{cc} < 0, u_l > 0, u_{ll} < 0$

$$\text{leisure } l_t = 1 - m_t$$

$$0 \leq l_t \leq 1$$

$$\text{special case } U(c_t, l_t) = u(c_t) + v(l_t)$$

Budget constraint:

$$\sum_{t=0}^{\infty} q_t (c_t + n_t) = \sum_{t=0}^{\infty} w_t m_t + \sum_{t=0}^{\infty} r_t k_t$$

\uparrow rented as labor \uparrow rented on capital

$b - r_k n_k + r$

$$k_{t+1} = (1-\delta)k_t + r_t$$

$$\sum_{t=0}^{\infty} (w_t n_t + r_t k_t) = \sum_{t=0}^{\infty} q_t (c_t + k_{t+1} - (1-\delta)k_t)$$

$\underbrace{q_t}_{r_t}$

Form Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t) + \sum_{t=0}^{\infty} (\lambda_t (1-n_t) + \phi_t n_t)$$

Lagrange multiplier on $1-n_t \geq 0$ Lagrange multiplier
on $n_t \geq 0$

$$+ M \left[\sum_{t=0}^{\infty} (w_t n_t + r_t k_t) - \sum_{t=0}^{\infty} q_t (c_t + k_{t+1} - (1-\delta)k_t) \right]$$

FONC:

$$c_t: \beta^t U_c(c_t, 1-n_t) - M q_t = 0, t=0, 1, \dots$$

$$n_t: -\beta^t U_n(c_t, 1-n_t) + M w_t + \phi_t - \lambda_t = 0, t=0, 1, \dots$$

$$\text{Complementary slackness: } \lambda_t (1-n_t) = 0$$

$$\phi_t n_t = 0$$

$$\Rightarrow \beta^t U_e(t) = M w_t + \phi_t - \lambda_t$$

$$\Rightarrow \beta^t U_e(t) = M w_t \quad \text{if } 0 < n_t < 1$$

$$\beta^t U_e(t) < M w_t \quad \text{if } n_t = 1$$

$$\beta^t U_e(t) > M w_t \quad \text{if } n_t = 0$$

$$k_t: r_t + q_t (1-\delta) - q_{t+1}, t=1, 2, \dots$$

or

$$q_t = q_{t+1} (1-\delta) + r_{t+1}$$

Summarizing:

$$q_t = q_{t+1} (1-\delta) + r_{t+1} \quad \text{max cond for } k_t$$

$$\max q_t = \beta^t u_c(t) \quad \text{max cond for } c_t$$

$$\max w_t = \beta^t u_e(t) \quad \text{if } 0 < n_t < 1$$

$$> \beta^t u_e(t) \quad \text{if } n_t = 1$$

$$< \beta^t u_e(t) \quad \text{if } n_t = 0$$

We'll assume we have the $n_t = 1$ condition prevailing

- e.g. $u_e(c_t, l_t) = 0$ - because, e.g.

$$u(c_t, l_t) = u(c_t)$$

Firm: Maximize

$$\sum_{t=0}^{\infty} \left(q_t [c_t + r_t] - r_t K_t - w_t N_t \right)$$

* sales

$$\text{s.t. } c_t + r_t = F(K_t, N_t) \quad \begin{array}{l} \text{law fixed by firm} \\ \text{created hired by firm} \end{array}$$

$$\lambda F(K, L) = F(\lambda K, \lambda L) \quad \begin{array}{l} \text{constant returns to scale} \\ (\text{Euler's theorem}) \end{array}$$

$$F(K, L) = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L$$

This firm problem is

$$\max_{\{K_t, L_t\}} \sum_{t=0}^{\infty} \left\{ (q_t F_K - r_t) K_t + (q_t F_L - w_t) N_t \right\}$$

Firm hires K_t & N_t as a pure factor -
 i.e. $c_t = \dots = 0$

takes q_t, r_t, w_t as given.

Zero profit condition for firm inel points K_t, N_t :

$$q_t F_K - r_t = 0$$

$$q_t F_N - w_t = 0$$

constant returns to scale \Rightarrow

$$F_K(K, N) = F_K\left(\frac{K}{N}, 1\right) = f'\left(\frac{K}{N}\right)$$

$$F_N(K, N) = F_N\left(\frac{K}{N}, 1\right) = f\left(\frac{K}{N}\right) - f'\left(\frac{K}{N}\right) \cdot \frac{K}{N}$$

Note: size of firm is indeterminate — double size of firm leaves these marginal conditions intact.

The "number" of firms is thus indeterminate.

Defns: A pure system $\{q_t, w_t, r_t\}_{t=0}^{\infty}$.

A feasible allocation $\{L_t, C_t\}_{t=0}^{\infty}$, $C_t \geq 0, L_{t+1} \geq 0$

Defn: A C.R. is a pure system & feasible allocation such that

(a) given pure system, allocation solves firm problem

(b) given pure system, allocation solves HH problem

To compute equilibrium: guess & verify.

- (1) Solve planning problem for $\{\tilde{c}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$
- (2) Guess $q_t = \beta^t u'(c_t) \in \beta^t U_c(c_t, 1-n_t)$
 $w_t = (f(\tilde{k}_t) - \lambda_t f'(\tilde{k}_t)) q_t$
 $r_t = f'(\tilde{k}_t) \cdot q_t$

- (3) Verify that the firm's & HH's FOC's
are satisfied ..

Key conclusion: For this economy, the allocation that solves the planning problem is a competitive equilibrium.

This is the end of the lecture. The below is entirely optional advanced material.

To formulate a recursive competitive equilibrium, first write sol'n of planning problem as

$$\begin{aligned}\tilde{k}_{t+1} &= h(\tilde{k}_t) & k_0 &= \tilde{k}_0 \\ w_t &= w(\tilde{k}_t) \\ r_t &= f'(\tilde{k}_t)\end{aligned}$$

Formulate Bellman eqn for HH

$$\begin{aligned}v(k_t, \tilde{k}_t) &= \max_{c_t} \{ u(c_t) + \beta v(k_{t+1}, \tilde{k}_{t+1}) \} \\ \text{s.t. } c_t + k_{t+1} &= w(\tilde{k}_t) \cdot 1 + r(\tilde{k}_t) k_t \\ \tilde{k}_{t+1} &= h(\tilde{k}_t), \\ k_0, \tilde{k}_0 &= k_0 \quad \text{given}\end{aligned}$$

Firm problem - sequence of static problems (no forecasts)

$$\max_{\tilde{m}_e, \tilde{k}_e} \left\{ f(\tilde{k}_e) m_e - w(\tilde{k}_e) m_e - r(\tilde{k}_e) \tilde{k}_e \right\}$$

— end of optimal material.

Taxes and government expenditures

Sunday, March 28, 2010
5:24 PM

Government expenditures in the growth model.

Feasibility becomes

$$c_t + \gamma_t + g_t = f(k_t)$$

\uparrow
government expenditures per capita

$$k_{t+1} = (1-\delta) k_t + x_t$$

or

$$c_t + k_{t+1} + g_t = f(k_t) + (1-\delta) k_t$$

$\{g_t\}_{t=0}^{\infty}$ an exogenous sequence.

Government budget constraint (at time 0)

$$\sum_{t=0}^{\infty} g_t g_t = \sum_{t=0}^{\infty} g_t \gamma_{ct} c_t + \sum_{t=0}^{\infty} \tau_{kt} \gamma_t k_t \\ + \sum_{t=0}^{\infty} w_t \gamma_{nt} n_t + \sum_{t=0}^{\infty} q_t \gamma_{ht}$$

γ_{ct} - consumption tax γ_{kt} - tax on earnings from capital γ_{nt} - tax on earnings from labor	$\left. \begin{array}{l} \text{potentially} \\ \text{distortionary taxes} \end{array} \right\}$
---	---

γ_{ht} - lump sum tax (non distortionary)

Definition: The household partially affects the amount of

taxes he pays for a distortion tax by possibly altering his choice of $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$.

The household's choice of $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ leaves unaffected the amount it pays in non-distortion taxes.

Household problem:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$\begin{aligned} & \sum_{t=0}^{\infty} q_t [(1+\gamma_{ct}) c_t + k_{t+1} - (1-\delta) k_t] \\ & = \sum_{t=0}^{\infty} w_t (1-\gamma_{nt}) n_t + \sum_{t=0}^{\infty} r_t (1-\gamma_{kt}) k_t - \sum_{t=0}^{\infty} q_t \gamma_{ht} \end{aligned}$$

s.t. q_0 given

Here we are simplifying by having $n_t = 1$ inelastically supplied.

Formulate Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} & + M \left[\sum_{t=0}^{\infty} w_t (1-\gamma_{nt}) \cdot 1 + \sum_{t=0}^{\infty} r_t (1-\gamma_{kt}) k_t \right. \\ & \left. - \sum_{t=0}^{\infty} q_t \gamma_{ht} - \sum_{t=0}^{\infty} q_t ((1+\gamma_{ct}) c_t + k_{t+1} - (1-\delta) k_t) \right] \end{aligned}$$

FONC:

(we are free to set $M=1$ by scaling prices)

$$c_t: \beta^t u'(c_t) - \mu q_t (1+\gamma_{ct}) = 0 \quad , t=0, 1, \dots$$

$$n_t: \sim (1-\gamma_{nt}) \cdot 1 + a (1, \dots) = a \quad \rightarrow \quad t=1 \sim$$

$$k_t: r_t(1-\gamma_{kt}) + g_t(1-\delta) = g_{t+1} = 0, \quad t=1, 2, \dots$$

$$(1) \Rightarrow \frac{\beta^t u'(c_t)}{1+\gamma_{kt}} = g_t$$

$$(2) \quad g_t = g_{t+1}(1-\delta) + (1-\gamma_{kt+1}) r_{t+1}$$

Firm problem: Same as before - firm pays no taxes

households pay all taxes

$$\Rightarrow r_t = g_t f'(k_t) \quad \text{where } k_t \in \underline{k} \text{ for firm and } \bar{k} \text{ for economy in equilibrium.}$$

$$w_t = g_t [f(k_t) - f'(k_t) k_t]$$

Defn: A government policy is a $\{g_t\}_{t=0}^\infty$ and

tax sequences $\{\gamma_{ct}, \gamma_{mt}, \gamma_{kt}, \gamma_{ht}\}_{t=0}^\infty$.

A feasible allocation is a $\{c_t, k_{t+1}, g_t\}_{t=0}^\infty$ that satisfies

$$c_t + k_{t+1} + g_t = f(k_t) + (1-\delta) k_t, \quad t \geq 0$$

to given.

A competitive equilibrium (with taxes) is

a government policy, a price system $\{g_t, r_t, w_t\}_{t=0}^\infty$,

and a feasible allocation such that

- Given the allocation and the price system, the government policy satisfies its budget constraint.

- b) Given the price system and the government policy, the allocation solves the household's problem.
- c) Given the price system and the govt policy, the allocation solves the firm's problem.

Computing an equilibrium:

Remark: in general, can't solve a planning problem for the allocation.

Instead - impose equilibrium conditions in the form of govt budget constraint & FONC's.

for the household & firm and attack them with the "shooting" algorithm:

We have

$$q_t = \frac{\beta^t u'(c_t)}{1 + \gamma_{ct}}$$

$$q_t = q_{t+1} (1-\delta) + (1-\gamma_{kt+1}) r_{t+1}$$

$$\overset{\circ}{q}_t = q_{t+1} [(1-\delta) + (1-\gamma_{kt+1}) f'(k_{t+1})]$$

\Rightarrow

$$\frac{\beta^t u'(c_t)}{(1+\gamma_{ct})} = \beta \frac{\beta^{t+1} u'(c_{t+1})}{(1+\gamma_{ct+1})} \left[(1-\delta) + (1-\gamma_{kt+1}) f'(k_{t+1}) \right]$$

\cap

$$(1/\gamma_{ct}) = \beta (1/\gamma_{ct+1}) \left[(1-\delta) + (1-\gamma_{kt+1}) f'(k_{t+1}) \right]$$

$$(1) \quad u'(c_t) = \beta u'(c_{t+1}) \frac{(1+\gamma_{ct})}{(1+\gamma_{ct+1})} \left[(1-\delta) + (1-\gamma_{ct+1}) f'(k_{t+1}) \right]$$

$$(2*) \quad c_t + k_{t+1} + g_t = f(k_t) + (1-\delta) k_t$$

$$\begin{aligned} & k_0 \text{ given} \\ & \{g_t, \gamma_{ct}, \gamma_{kt}\}_{t=0}^{\infty} \text{ given} \end{aligned}$$

To solve \$(1), (2*)\$ for \$\{c_t, k_{t+1}\}_{t \geq 0}^{\infty}\$

Assume that \$g_t \rightarrow \bar{g}\$, \$\gamma_{ct} \rightarrow \bar{\gamma}_c\$, \$\gamma_{kt} \rightarrow \bar{\gamma}_k\$ and

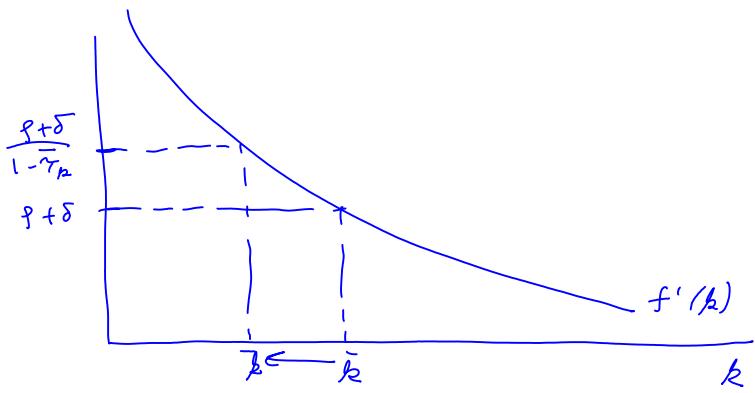
look for steady state - \$(*) \Rightarrow\$

$$c_t = c_{t+1} = \bar{c}$$

$(*) \Rightarrow$

$$1 = \beta \left[(1-\delta) + (1-\bar{\gamma}_k) f'(\bar{k}) \right]$$

$$\Rightarrow \frac{\bar{g} + \delta}{1 - \bar{\gamma}_k} = f'(\bar{k})$$



raising $\bar{\gamma}_k > 0$ causes steady state \bar{k} to be lower

$$(1) \quad u'(c_t) = \beta u'(c_{t+1}) \frac{(1+\gamma_{ct})}{(1+\gamma_{ct+1})} \left[(1-\delta) + (1-\gamma_{ct+1}) f'(k_{t+1}) \right]$$

$$(2*) \quad c_t + k_{t+1} + g_t = f(k_t) + (1-\delta)k_t$$

Solve (1), (2*) by "shooting" to \bar{k} steady state.

guess at c_0 , solve (2*) for k_1 , solve (1) for c_1 , ...
continue & hope you hit \bar{k} .

More general utility specification

$$U(c, 1-n) = \frac{c^{1-\varphi}}{1-\varphi} + B \frac{(1-n)^{1-\varphi}}{1-\varphi}$$

$$\log c \text{ if } \varphi=1 \quad \ln(1-n) \text{ if } \varphi \neq 1$$

$$\sum_{t=0}^{\infty} \beta^t \left\{ U(c_t, 1-n_t) + \mu \left[\sum_{t=0}^{\infty} r_t (1-\gamma_{bc}) k_t \right. \right. \\ \left. \left. + w_t (1-\gamma_{ne}) n_t - q_t \gamma_{ht} - q_k (1+\gamma_{ck}) c_t \right. \right. \\ \left. \left. + (1-\gamma_{ik}) q_t [k_{t+1} - (1-\delta)k_t] \right] \right\}$$

$$c_t: \beta^{-t} U_{1t} = \mu q_t (1+\gamma_{ck})$$

$$n_t: \beta^{-t} U_{2t} \leq \mu w_t (1-\gamma_{ne}), = 1 \quad 0 < \mu < 1$$

$$\begin{cases} U_{1t} = c_t^{-\varphi} \\ U_{2t} = B (1-n_t)^{-\varphi} \end{cases}$$

To solve:

$$\frac{(1-\gamma_{ik})}{(1+\gamma_{ck})} U_1(c_t, 1-n_t) = \beta (1+\gamma_{ck+1})^{-1} U_1(c_{t+1}, 1-n_{t+1})$$

$$(1 + \gamma_{c_k})$$

$$\star [((1 - \gamma_{i,t+1})(1 - \delta) + (1 - \gamma_{k,t+1}) F_k(k_{t+1}, n_{t+1})]$$

$$\frac{U_2(c_{t+1}, n_{t+1})}{U_1(c_t, n_t)} = \frac{(1 - \gamma_{n_t})}{(1 + \gamma_{c_k})} F_n(k_t, n_t)$$

To get steady-state:

solve

$$1 = \beta [(1 - \delta) + \frac{(1 - \gamma_k)}{(1 - \gamma_i)} F_k(\bar{k}, \bar{n})]$$

$$\frac{U_2}{U_1} = \frac{(1 - \gamma_n)}{(1 + \gamma_c)} F_n(\bar{k}, \bar{n})$$

$$\bar{k} = \frac{\bar{c}}{\bar{n}}$$

$$1 = \beta [(1 - \delta) + \frac{(1 - \gamma_k)}{(1 - \gamma_i)} f'(\bar{k})]$$

$$\Rightarrow \bar{x}$$

$$\frac{U_2}{U_1} = \frac{(1 - \gamma_n)}{(1 + \gamma_c)} [f(\bar{k}) - \bar{k} f'(\bar{k})]$$

or

$$(*) \frac{\beta (1 - \bar{n})^{-\varphi}}{\bar{c}^{-\varphi}} = \frac{(1 - \gamma_n)}{(1 + \gamma_c)} [f(\bar{k}) - \bar{k} f'(\bar{k})]$$

$$(*) \bar{c} + \bar{g} + \delta \bar{k} = \bar{n} f(\bar{k})$$

$\bar{k} = \frac{\bar{c}}{\bar{n}}$

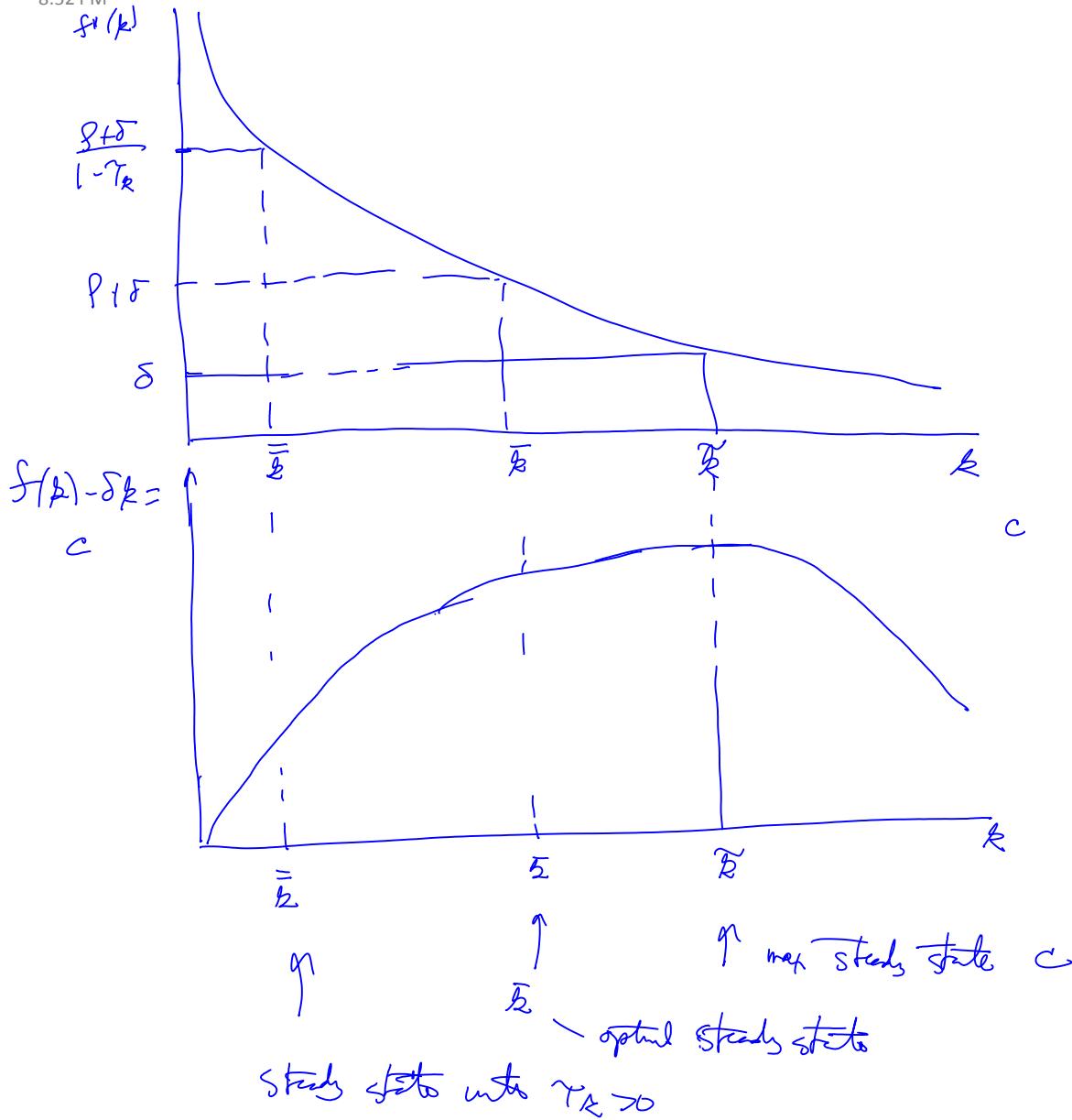
given \tilde{k} , there are two fins in \bar{m} and \bar{c}

because $\tilde{k} = \hat{k} \cdot \bar{m}$

Steady states

Sunday, March 28, 2010

8:52 PM



Use these graphs to prove that steady state c is lower for steady states with $\gamma_k > 0$ than with $\gamma_k = 0$.

Foresight versus surprise

Sunday, March 28, 2010
8:45 AM

Recall the optimal growth model with an infinitely lived consumer who orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad u' > 0, u'' < 0$$

for example $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0$

The technology is

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0,1)$$

c_t = consumption per capita

g_t = government purchases per capita

k_t = capital per unit (labor)

In class, we studied the following two scenarios.

(i) Before $t=0$, the economy was resting in the steady state k_t appropriate for a situation in which g had always been $= 0$. i.e.

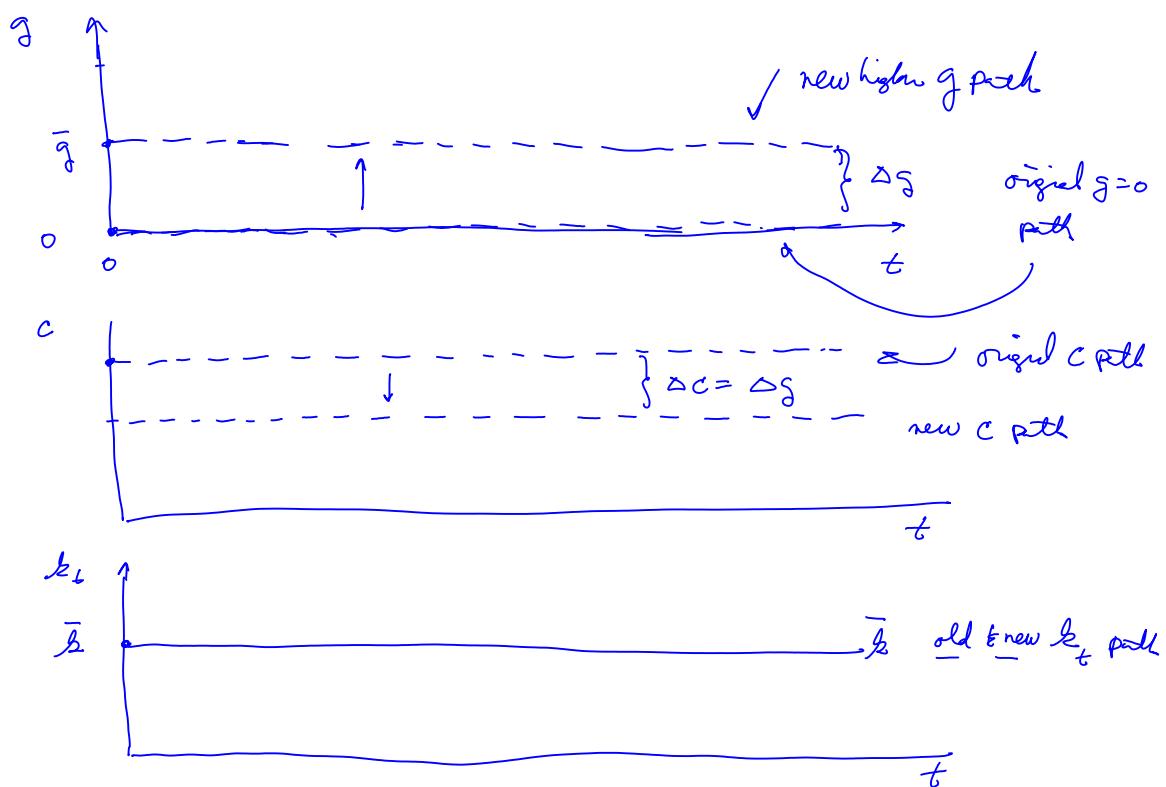
$$k_0 = \bar{k} \quad \text{where} \quad f'(\bar{k}) = \beta + \delta$$

At $t=0$, government expenditures per capita

suddenly and unexpectedly rise to $g_t = \bar{g} > 0$ if $t \geq 0$.

Assume that g is entirely financed by lump sum taxes. We saw that the effects of the increase

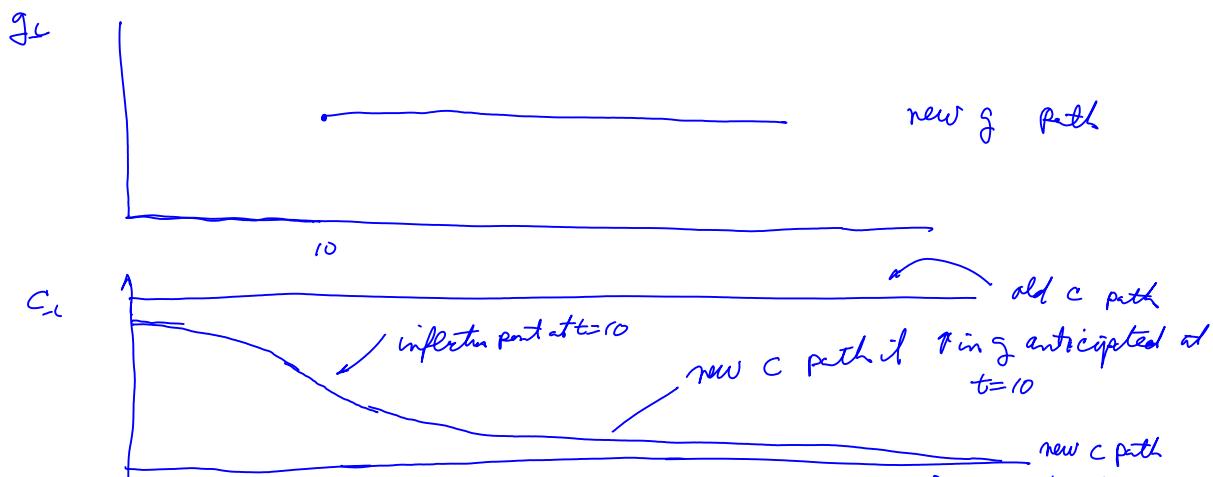
taxes. We saw that the effects of the increase in \bar{g} were as follows:

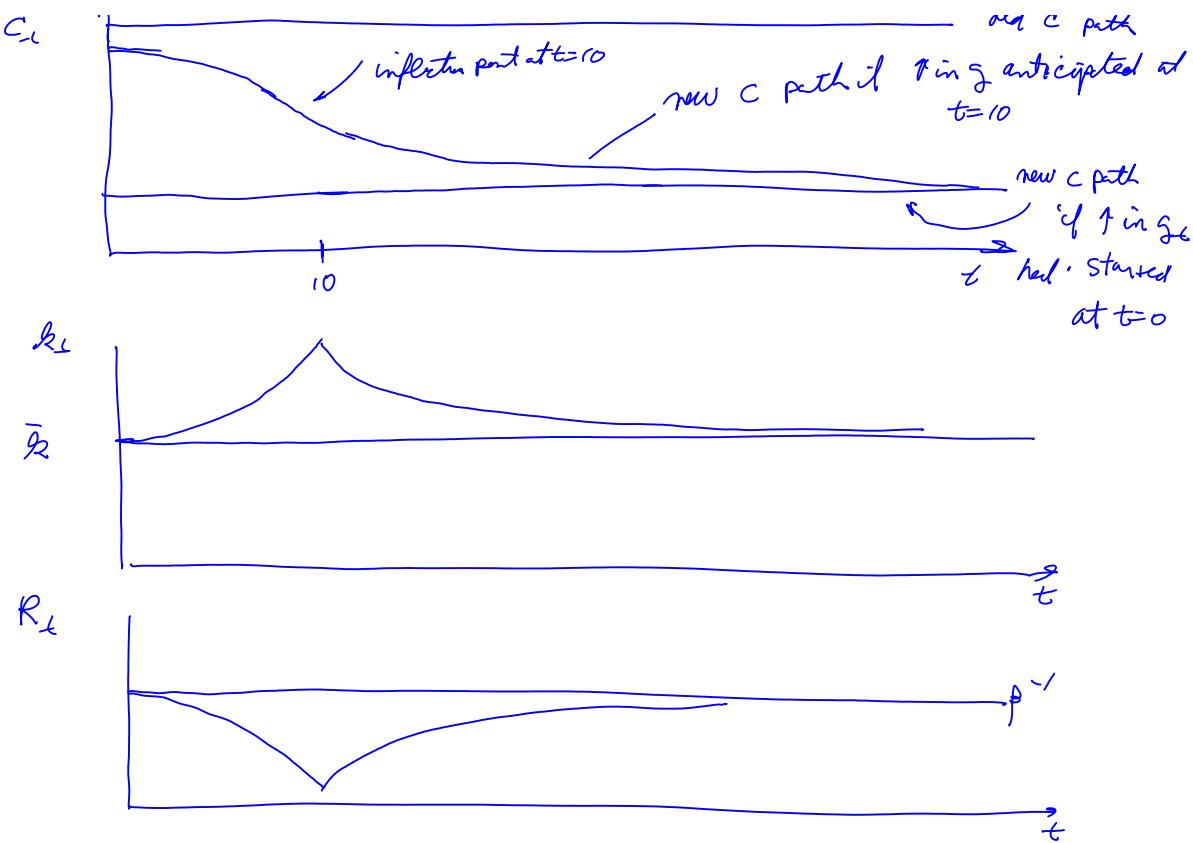


But if instead, we model the increase in \bar{g} not as a surprise but as foresight, the path is different.

If at $t=0$ the path of g_t changes unanticipated from 0 for all t to $\bar{g}_t = \begin{cases} 0 & t=0, \dots, 9 \\ \bar{g} > 0 & t=10, \dots \end{cases}$

we saw that the consequence was as follows :





The above graph shows the path of c_t , $t \geq 0$, when the f in g_t is anticipated to occur 10 periods in the future. Here c_t drops immediately at $t=0$ and then declines smoothly to the same steady state level it would have had the increase in g_t occurred at $t=0$ instead of $t=10$.

A general principle is illustrated by this example.

The effects of an increase in g_t at t depend on whether the change is unanticipated or whether it had been anticipated in advance.

Homework 4

Tuesday, April 06, 2010
10:09 AM

Homework 4 .

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The first several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t \log c_t , \quad 0 < \beta < 1$$
$$\beta = \frac{1}{1+\rho}, \quad \rho > 0$$

where c_t is consumption per worker.

The technology is .

$$y_t = f(k_t) = z k_t^\alpha, \quad 0 < \alpha < 1$$
$$z > 0$$

where y_t is output per unit labor and

k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta)k_t + x_t, \quad 0 < \delta < 1$$

The government finances its expenditures by

levying a flat rate tax $\tau_c t$ on the value of

consumption goods purchased at t , a flat rate

tax of $\tau_m t$ on the value of labor earnings at t ,

a flat rate tax τ_k on earnings from capital

a flat rate tax $\gamma_{k,t}$ on earnings from capital at t , and a lump sum tax of $\gamma_{h,t}$ in the t consumption goods per worker at time t .

Let $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ be a pure system.

1. Define a competitive equilibrium with taxes and government purchases.
2. Assume that $g_t = 0$ for all $t \geq 0$. Assume that all taxes are zero for all $t \geq 0$. Find a steady state level of the capital stock per worker. Find the level of consumption per worker c at the steady state level of capital stock per worker.
3. Find the level of steady state c as a function of steady state k . Suppose now that even though $g_t = 0 \ \forall t \geq 0$, the government can tax ($\gamma_k > 0$) or subsidize ($\gamma_k < 0$) returns on capital by setting a constant over time tax $\gamma_{k,t} = \gamma_k \ \forall t \geq 0$ on capital. The government rebates the revenues it raises (if $\gamma_k > 0$) to the public by setting lump sum taxes negative ($\gamma_h < 0$);

public by setting lump sum taxes negative ($\gamma_h < 0$); or the government raises the revenues to subsidize capital (if $\gamma_k < 0$), by setting a positive lump sum tax $T_h > 0$. Please find a level of the tax or subsidy on capital γ_k that maximizes the steady state level of consumption per unit labor. Is γ_k positive, negative, or zero?

4. Suppose that you live in an economy where in the past g_t was always = 0 and where for a long time the ^{was set} tax or subsidy on capital γ_k at the problem 3 rate that maximizes steady state $C = \text{consumption per worker}$. Suppose that γ_k had been set at this rate for such a long time that you are at a steady state at time 0, so that $k_0 = \text{steady state } k$ you computed in problem 3. Suppose that $g_t = 0 \forall t \geq 0$.

- a. Starting from k_0 , describe the allocation that would be chosen by a planner who solves the problem

$$\text{max}_{\{C_t\}} \sum_{t=0}^{\infty} \beta^{-t} \log C_t$$

$$\text{max } \{\bar{c}_t, \bar{k}_{t+1}\}_{t=0}^{\infty} \quad \leftarrow \quad \sigma =$$

$$c_t + k_{t+1} = z k_t^\alpha + (1-\delta) k_t.$$

Find the steady state level of capital per worker and consumption per worker. Compare with the initial level of k_0 .

- b. Use your answer to (a) to discuss whether the \bar{k} set in problem 3 is too high or too low.

5. Suppose now that $g_t = 0 \forall t$ and that all taxes are zero for all time. Let $\bar{k}(z=1)$ be the steady state level of capital associated with a given level of $z=1$. Let $k_0 = \bar{k}(z=1)$. Suppose that at time 0 z suddenly and unexpectedly jumps to $z=1.5$.

- a. Find the new steady state level of capital.
 b. Describe the transition path for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ as k moves from $\bar{k}(z=0)$ to $\bar{k}(z=1)$.

Describe the behavior of

$$R_{t+1} = f'(k_{t+1}) + (1-\delta)$$

along the transition path, and how c_{t+1}/c_t
 $\therefore \dots -1.1 \dots +1 \dots R$.

is associated with R_{t+1} .

6. Suppose that at time $t=0$ the economy is in a steady state associated with a level of $g=0$ always and no taxes of any kind.

Let \bar{k} be the level of capital per worker and $\bar{c} = f(\bar{k}) - \delta \bar{k}$ the level of consumption per worker in this initial steady state.

Let $k_0 = \bar{k}$. Now assume that at time $t=0$ the government begins spending $g_t = \bar{g} \quad \forall t \geq 0$ where $\bar{g} = \frac{1}{4} [f(k_0) - \delta k_0]$.

The government finances its purchase by imposing lump sum taxes.

(a) Find the new steady state level of capital and consumption per worker.

(b) Describe the transition path of consumption and capital to the new steady state.

(c) Explain the behavior of c_{t+1}/c_t and $R_{t+1} = f'(k_{t+1}) + (1-\delta)$ along a transition path to a new steady state

(d) Prove that the timing of lump sum taxes is irrelevant.

Term structure

Saturday, April 11, 2009
11:39 AM

Term structure of interest rates.

Deterministic model:

q_t^0 = price of time t consumption good in terms of time 0 consumption good.

$$\sim \frac{\text{time 0 goods}}{\text{time } t \text{ good}} \sim \text{units}$$

$$q_t^0 = q_0^0 \cdot \frac{q_1^0}{q_0^0} \cdot \frac{q_2^0}{q_1^0} \cdots \frac{q_t^0}{q_{t-1}^0}$$

$$= \frac{1}{m_{0,1} m_{1,2} \cdots m_{t-1,t}} , q_0^0 = 1$$

where $m_{t-1,t} = \frac{q_t^0}{q_{t-1}^0} \sim \frac{\text{time } t-1 \text{ goods}}{\text{time } t \text{ goods}}$

where $m_{t-1,t} = \frac{\frac{q_t^0}{q_{t-1}^0}}{\frac{q_t^0}{q_{t-1}^0}} \approx \frac{\text{time } t-1 \text{ good}}{\text{time } t \text{ good}}$

$$m_{t-1,t} = \exp(-r_{t-1,t}) \approx \frac{1}{1+r_{t-1,t}}$$

$$-\log m_{t-1,t} = r_{t-1,t} \quad \left(\log \frac{1}{1+r_{t-1,t}} \approx -r_{t-1,t} \right)$$

$r_{t-1,t}$ = one period net yield between $t-1, t$.

Time 0
Price of a t -period pure discount bond promising 1 unit of consumption at t ,

$$q_t^0 = m_{0,1} m_{1,2} \cdots m_{t-1,t}$$

$$= \exp(-r_{0,1}) \exp(-r_{1,2}) \cdots \exp(-r_{t-1,t})$$

$$(*) \quad g_t^0 = \exp\left(-(r_{0,1} + r_{1,2} + \cdots + r_{t-1,t})\right)$$

definition of t period yield:

$$g_t^0 = \exp(-t r_{0,t}) \quad \begin{matrix} \uparrow \\ "from" \end{matrix} \quad \begin{matrix} \uparrow \\ "to" \end{matrix}$$

or

$$(**) \quad -\frac{1}{t} \log g_t^0 = r_{0,t}$$

(*) and (**) \Rightarrow

$$r_{0,t} = \frac{1}{t} \underbrace{(r_{0,1} + r_{1,2} + r_{2,3} + \cdots + r_{t-1,t})}_{\text{average of } t \text{ one-period yields over the horizon.}}$$

$r_{t-1,t}$ a "forward" one-period yield
 $\text{for } s < t-1.$

More generally,

$$r_{t,t+\gamma} = \frac{1}{\gamma} \underbrace{(r_{t,t+1} + r_{t+1,t+2} + \cdots + r_{t+\gamma-1,t+\gamma})}_{\text{average of short yields from } t \text{ to } t+\gamma}$$

"long" γ -period yield at t

"expectation theory" of the term structure of interest rates

Specifying theory about g_t^0 - from growth model

$$g_t^0 = M \beta^t \frac{u'(c_t)}{1 + \tau c_t}$$

then

$$m_{t-1,t} = \frac{u'(c_t)}{u'(c_{t-1})} \cdot \left(\frac{1 + r_{t-1}}{1 + r_{t-1}} \right) = \exp(-r_{t-1,t})$$

↑

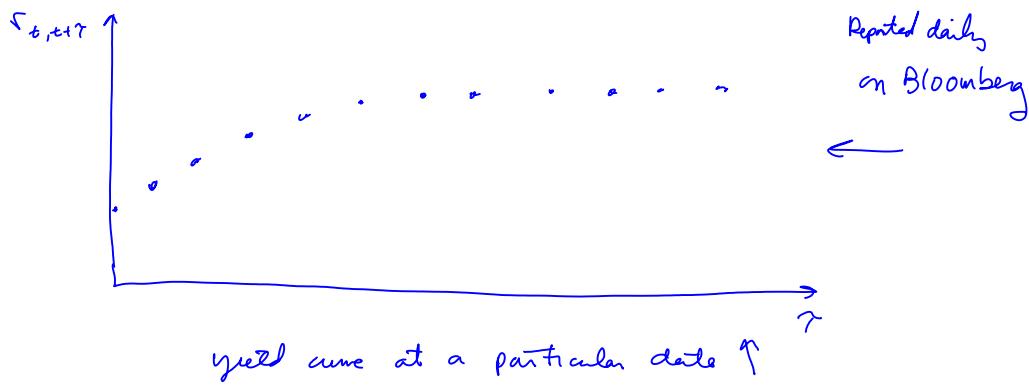
specific theory about "discount factor" $m_{t-1,t} = \exp(-r_{t-1,t})$

Remark: Consider the "expectations theory" formulae

$$r_{t,t+\gamma} = \frac{1}{\gamma} (r_{t,t+1} + r_{t+1,t+2} + \dots + r_{t+\gamma-1,t+\gamma})$$

The yield curve at t is a plot of

$r_{t,t+\gamma}$ against horizon γ .



yield curve at a particular date ↑

How the yield curve tells us that the market expects short rates $r_{S-1,S}$ to rise over time -

why? — well, note, for example, that

$$r_{t,t+2} = \frac{1}{2} [r_{t,t+1} + r_{t+1,t+2}]$$

$$r_{t,t+1} = \frac{1}{2} [r_{t,t+1} + r_{t,t+1}]$$

⇒

$$r_{t,t+2} - r_{t,t+1} = \frac{1}{2} [r_{t+1,t+2} - r_{t,t+1}]$$

↑

two year — one year
yield yield

∴

∴

$$r_{t,t+2} - r_{t,t+1} > 0 \Rightarrow r_{t+1,t+2} > r_{t,t+1}$$

↑
upward
sloping
yield curve

↑
one period yields are
expected to rise

Link to growth model:

remember

$$(*) \quad m_{t-1,t} = \frac{\beta u'(c_t)}{u'(c_{t+1})} \cdot \left(\frac{1 + \gamma_{c,t}}{1 + \gamma_{c,t+1}} \right) = \exp(-\epsilon_{t-1,t})$$

Recall the FONC from the growth model

$$u'(c) = \beta u'(c_{t+1}) R_{t+1}$$

$$\text{where } R_{t+1} = \frac{(1 + \gamma_{c,t})}{(1 + \gamma_{c,t+1})} \left[(1 - \delta) + (1 - \gamma_{k,t+1}) f'(k_{t+1}) \right]$$

Evidently (•) (shift forward one period)

$$u'(c_t) = \beta u'(c_{t+1}) \frac{(1 + \gamma_{c,t})}{(1 + \gamma_{c,t+1})} \exp(+r_{t,t+1})$$

so evidently

$$\exp(r_{t,t+1}) = (1 - \delta) + (1 - \gamma_{k,t+1}) f'(k_{t+1})$$

$$\approx (1 + r_{t,t+1})$$

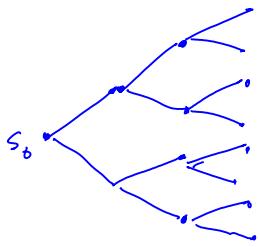
$$\text{and } R_{t+1} = \frac{1 + \gamma_{c,t}}{1 + \gamma_{c,t+1}} \exp(r_{t,t+1})$$

Stochastic model:

Recall the "pricing in trees" model

where $s_t \in S \sim \text{"state space"} , t=0,1,2,\dots$

$$s^t = (s_t, s_{t+1}, \dots, s_0) \quad \text{a history}$$



each "node" is a history s^t

we formulated a model with a competitive price

$$q_t^0(s^t) = \text{price of one unit of consumption at date } t \\ \text{contingent on history } s^t - \text{measured in} \\ \text{units of time } \frac{\text{good}}{\text{time } t, \text{ history } s^t \text{ good}}$$

Write m_t^1 — choice of time 0 goods as remainder

$$q_t^0(s^t) = q_0^0(s_0) \frac{q_1^0(s^1)}{q_0^0(s_0)} \frac{q_2^0(s^2)}{q_1^0(s^1)} \dots \frac{q_t^0(s^t)}{q_{t-1}^0(s^{t-1})}$$

$$= q_0^0(s_0) m_{0,1}(s^1) m_{1,2}(s^2) \dots m_{t-1,t}(s^t)$$

$$\text{where } \frac{q_t^0(s^t)}{q_{t-1}^0(s^{t-1})} = m_{t-1,t}(s^t) \\ = \frac{\text{goods at } t-1, s^{t-1}}{\text{goods at } t, s^t} \\ = \text{price of a unit of cons' at } t, s^t \\ \text{in units of time } t, \text{ history } s^{t-1} \text{ goods}$$

One period risk-free yield $r_{t-1,t}(s^{t-1})$:

$$\sum_{s_t \in S} m_{t-1,t}(s_t, s^{t-1}) \equiv \exp(-r_{t-1,t}(s^{t-1}))$$

$$\begin{array}{c} \uparrow \text{cost to} \\ \text{buy one unit of consumption} \\ \text{in all states} \end{array} \approx \frac{1}{1 + r_{t-1,t}}$$

in all states $s_t \in S$
 next period starting from
 history s^{t-1} at time $t-1$.

- - - - - - - - - - - - - - -

Remark: relationship between risk-free yield of different horizons is now more complicated than given by the "pure expectations" theory that we derived for the non-stochastic case.

For example, the two period risk-free yield $r_{t-1, t+1}(s^{t-1})$

$$\exp(-2r_{t-1, t+1}) = \sum_{s_t \in S} \sum_{s_{t+1} \in S} m_{t-1, t}(s_t | s^{t-1}) m_{t, t+1}(s_{t+1} | s_t, s^{t-1})$$

\uparrow
 Sum over
 all
 "intermediate"
 states

\uparrow
 sum these
 to get
 consumption for
 sure at date t

Remark: formulas must be modified for inflation.

- The simplest way is to reinterpret q_{bt}^0 as the price of one dollar at t in time 0 dollars and $m_{t-1,t}$ as the price of one dollar at t in time $t-1$ dollars. Then the above formulas apply directly to the yields reported on Bloomberg.
- Interpreting q_{bt}^0 as a "nominal" (dollar) pricing function in this way breaks the link $q_{bt}^0 = M \beta^t \frac{u'(c_t)}{1 + \gamma c_t}$

that we established, for example, in the growth model. Instead, we would have something like

$$\tilde{q}_t^0 = \underbrace{\mu \beta^+ u'(c_t)}_{\tilde{q}_t^0 = \text{"real" price function}} \frac{p_0}{p_t}$$

where p_t = price level at time t

$\sim \frac{\text{time } t \text{ dollars}}{\text{time } t \text{ goods}}$

Note that inflation ($\frac{p_t}{p_0} > 1$)

causes the price of time t dollars
in terms of time 0 dollars to decrease.

q_t^0 = real price system

$\tilde{q}_t^0 = \frac{p_0}{p_t} q_t^0$ = nominal price system

represent \tilde{q}_t^0 as

$$\tilde{q}_t^0 = q_t^0 \frac{p_0}{p_1} \frac{p_1}{p_2} \cdots \frac{p_{t-1}}{p_t}$$

$$= q_t^0 \left(m_{0,1} \frac{p_0}{p_1} \right) \left(m_{1,2} \frac{p_1}{p_2} \right) \cdots \left(m_{t-1,t} \frac{p_{t-1}}{p_t} \right)$$

represent $m_{t-1,t} = \exp(-\epsilon_{t-1,t})$

$$\frac{p_{t-1}}{p_t} = \exp(-\pi_{t-1,t})$$

$$\Downarrow$$

$$\cdots \quad 0 \quad \cdots / \pi \quad \backslash$$

$$P_t = P_{t-1} \exp(\pi_{t-1,t})$$

$$\pi_{t-1,t} = \log\left(\frac{P_t}{P_{t-1}}\right)$$

Thus

$$\tilde{q}_{0,t}^0 = q_{0,t}^0 \exp(-(\epsilon_{0,1} + \pi_{0,1})) \exp(-(\epsilon_{1,2} + \pi_{1,2})) \dots \exp(-(\epsilon_{t-1,t} + \pi_{t-1,t}))$$

$$= q_{0,t}^0 \exp\{-\tau \tilde{r}_{0,t}\}$$

$$\text{where } \tilde{r}_{0,t} = \frac{1}{t} [\tilde{r}_{0,1} + \tilde{r}_{1,2} + \dots + \tilde{r}_{t-1,t}]$$

↑

nominal yield to maturity on t -period
zero coupon risk free nominal bond

$$\tilde{r}_{t-1,t} = r_{t-1,t} + \pi_{t-1,t}$$

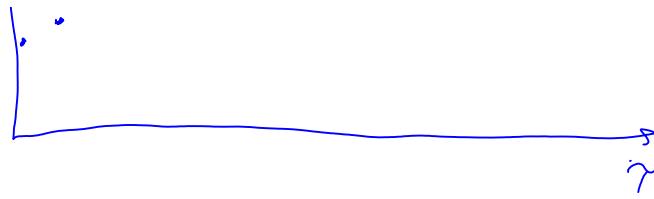
$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{nominal} & \text{real} & \text{expected} \\ \text{one-period rate} & \text{one-period} & \text{inflation from} \\ & \text{rate} & t-1, t \end{array}$$

our theory of $r_{t-1,t}$ ← real yield.

$$\exp(-r_{t-1,t}) = \beta \frac{u'(c_t)}{u'(c_{t-1})} \quad \text{when} \\ 1 + \gamma_{ct} = 1 + \gamma_{ct-1}$$

Each day Bloomberg plots

$$\tilde{r}_{t-1,t} \uparrow \dots \dots \dots$$



nominal yield to maturity on γ -period bonds.

Remark: people read these to make inferences

about (a) expected inflation

(b) paths of expected real short rates of interest.

Irving Fisher Theory:

$$\tilde{r}_{t,t+\gamma} = r_{t,t+\gamma} + \pi_{t,t+\gamma}$$

$$\pi_{t,t+\gamma} = \frac{1}{\gamma} \left[\pi_{t+1} + \dots + \pi_{t+\gamma-1, t+\gamma} \right]$$

Incomplete mkts

Friday, April 09, 2010
10:47 AM

Household problem: choose $c_0, c_{1(1)}, c_{1(2)}, b$
to maximize

$$u(c_0) + \beta u(c_{1(1)}) \pi + \beta u(c_{1(2)}) (1-\pi)$$

$$\text{s.t. } c_0 + b \leq y_0$$

$$c_{1(1)} \leq y_{1(1)} + R(1)b$$

$$c_{1(2)} \leq y_{1(2)} + R(2)b$$

$$c_0 \geq 0, c_{1(1)} \geq 0, c_{1(2)} \geq 0$$

$R(1)$ is gross return on single asset in state 1

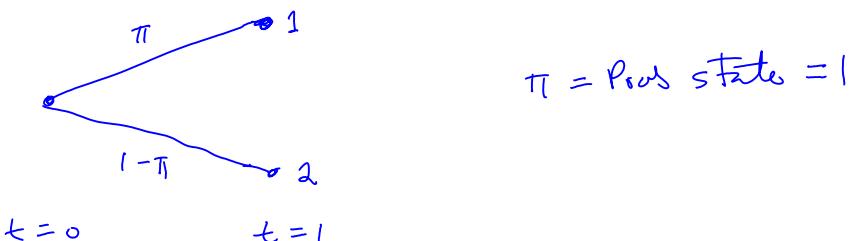
$R(2)$ " " " " " " " " 2

$$u' > 0, u'' < 0, u'(0) = +\infty$$

The asset is risky when $R(1) \neq R(2)$

The asset is risk free when $R(1) = R(2)$

There is only one asset available even though there are two states 1 and 2 at $t=1$



Form Lagrangian:

$$L = u(c_0) + \beta u(c_{1(1)}) \pi + \beta u(c_{1(2)}) (1-\pi)$$

$$\begin{aligned}
& + \theta_0 [y_0 - b - c_0] \\
& + \theta_1 [y_1(1) + R(1)b - c_1(1)] \\
& + \theta_2 [y_1(2) + R(2)b - c_1(2)] \\
& - \eta_1 c_1(1) - \eta_2 c_1(2) - \eta_0 c_0
\end{aligned}$$

$\max L$ w.r.t. $c_0, c_1(1), c_1(2), b$

$\min_{\text{FONC}} L$ w.r.t. $\theta_0, \theta_1, \theta_2, \eta_1, \eta_2, \eta_0$

FONC:

$$c_0: u'(c_0) - \theta_0 - \eta_0 = 0, \quad \eta_0 c_0 = 0$$

$$c_1(1): \beta u'(c_1(1)) \pi - \theta_1 - \eta_1 = 0, \quad \eta_1 c_1(1) = 0$$

$$c_1(2): \beta u'(c_1(2)) (1-\pi) - \theta_2 - \eta_2 = 0, \quad \eta_2 c_1(2) = 0$$

$$b: -\theta_0 + \theta_1 R(1) + \theta_2 R(2) = 0$$

seek an interior soln (use $u'(0) = +\infty$ to justify this)

$$\Rightarrow \eta_0 = \eta_1 = \eta_2 = 0$$

FONC's imply:

$$\theta_0 = u'(c_0)$$

$$\theta_1 = \beta u'(c_1(1)) \pi$$

$$\theta_2 = \beta u'(c_1(2)) (1-\pi)$$

$$u'(c_0) = \beta u'(c_1(1)) R_1 \pi + \beta u'(c_1(2)) R_2 (1-\pi)$$

$$1 = \beta \mathbb{E} \frac{u'(c_1)}{R}$$

or

random variable

$$I = \beta \mathbb{E} \frac{u'(c_1)}{u'(c_0)} R$$

random variable

mathematical expectation
w.r.t. distribution of (C_1, R)

or

$$(*) I = \beta \left[\frac{u'(y_0) + R(1)b)R(1)\pi + u'(y_1(2) + R(2)b)R(2)(1-\pi)}{u'(c_0)} \right]$$

given $y_0, y_1(1), y_1(2), R(1), R(2)$

$(*)$ is one equation in the unknown b .

Solving $(*)$ for $b = b(y_0, y_1(1), y_1(2), R(1), R(2))$

gives the consumer's demand curve for b .

Note: we seek a soln for $(*)$ s.t.

$$y_1(1) + R(1)b \geq 0$$

$$y_1(2) + R(2)b \geq 0$$

$$y_0 - b \geq 0$$

A popular special case: the asset is risk-free,

meaning $R(1) = R(2)$ (same return in all states)

Then $(*)$ becomes

$$(**) I = \beta R \left[\frac{u'(y_1(1) + Rb)\pi + u'(y_1(2) + Rb)(1-\pi)}{u'(c_0)} \right]$$

$$(*) \quad L - \pi u'(\hat{y}_0 - b)$$

to be solved for b s.t.

$$y_1(1) + Rb \geq 0$$

$$y_1(2) + Rb \geq 0$$

$$y_0 - b \geq 0$$

General Equilibrium version. Suppose there are two consumers, one whose demand b solves $(*)$ and a second whose demand \hat{b} solves

$$(**) \quad l = \beta R \frac{[u'(\hat{y}_1(1) + R\hat{b})\pi + u'(\hat{y}_1(2) + R\hat{b})(1-\pi)]}{u'(\hat{y}_0 - \hat{b})}$$

where $\hat{y}_0, \hat{y}_1(1), \hat{y}_1(2)$ is person 1 's endowment, and R is the same gross-return on risk-free asset as in $(*)$.

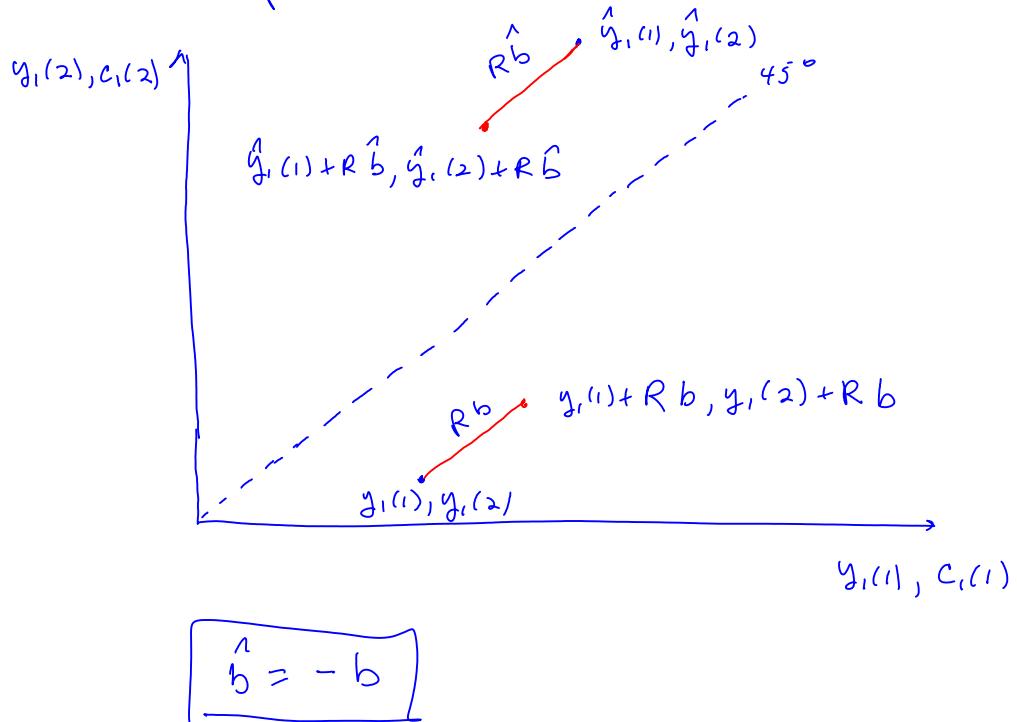
An incomplete market equilibrium is an R, b, \hat{b} that satisfy $(*)$, $(**)$, and

$$(*** \quad b + \hat{b} = 0)$$

Remark: $(***)$ says that the asset is "in zero net supply", so this is an economy in which one person's asset is $+1$. $\exists l \in \mathbb{R}^+$

The other person's debt.

Graphical depiction of what trades in the risktree asset accomplish:



remark: trading the risk free bond allows the households to trade consumption between times 0 and 1 but not across states at time 1.

The consumers are restricted to moving along lines parallel to the 45° line with length $R^b, R\bar{b}$ governed by b, \bar{b} .

Note: In general case (we allow $R(1) \neq R(2)$) we have

$$(+)\quad 1 = \mathbb{E} m_j R$$

$$\text{where } m_j = \frac{\beta u'(c_{i^j})}{u'(c_0)}$$

is the (random) marginal rate of substⁿ for consumer j .

Remark: m_j is called consumer j 's stochastic discount factor.

But we do not equate

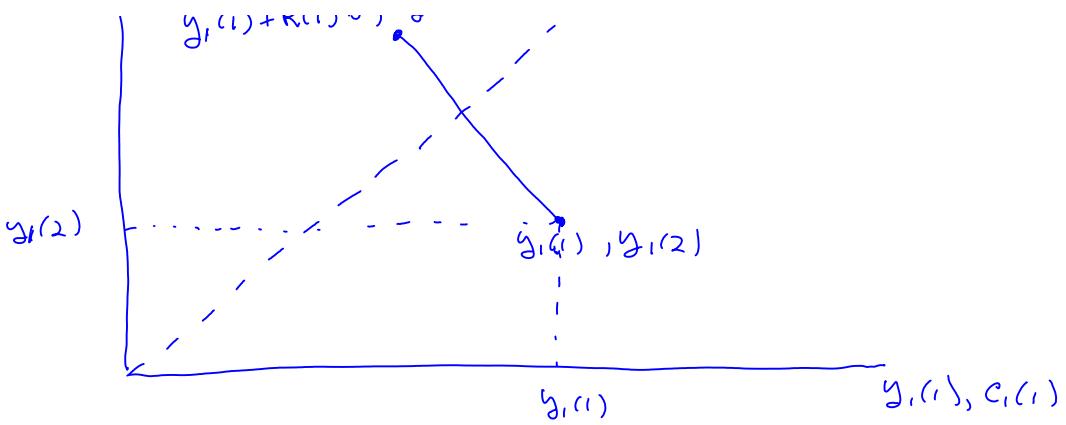
m_i^j across consumers, as does occur with complete markets and common probabilities across agents.

Remark: If we have as many assets as states, we might be able to conclude that

$$m_j = m_i \text{ for all } i, j. \quad (\text{complete markets})$$

Redraw the above graph when $R(1) \neq R(2)$





If here $b > 0$ and $R(1) < 0, R(2) > 0$

or $b < 0$ and $R(1) > 0, R(2) < 0$

$$d c_1(1) = R(1) d b$$

$$d c_1(2) = R(2) d b$$

$$\Rightarrow \frac{d c_1(2)}{d c_1(1)} = \frac{R(2)}{R(1)}$$

this is the direction in the $c_1(1), c_1(2)$ plane

that trades of b attain.

Remark: Consider the key asset pricing equation

$$\mathbb{E} m R = 1$$

$$\text{or } \int_{\omega \in \Omega} m(\omega) R(\omega) f(\omega) = 1 \quad (*)$$

where $m(\omega)$ is the s.d.f., $R(\omega)$ is the return

in state $\omega \in \Omega$, and $f(\omega) \geq 0$ is the probability

density of ω , $\int f(\omega) = 1$.

Rewrite (1) as

$$R_f^{-1} \int R(\omega) \left[\frac{m(\omega) f(\omega)}{\int m(\tilde{\omega}) f(\tilde{\omega}) d\tilde{\omega}} \right] d\tilde{\omega}$$

or

$$R_f^{-1} \int R(\omega) \hat{f}(\omega) d\omega = 1 \quad (\Rightarrow)$$

$$\text{where } \int m(\tilde{\omega}) f(\tilde{\omega}) d\tilde{\omega} \equiv E_m \equiv R_f^{-1}$$

\sim reciprocal of risk free rates of return.

because

$$\begin{aligned} 1 &= \int R_f m(\omega) f(\omega) d\omega = R_f \int m(\omega) f(\omega) d\omega \\ &\Rightarrow R_f^{-1} = \int m(\omega) f(\omega) d\omega \end{aligned}$$

Thus (1) becomes

$$(1) \quad \int R(\omega) \hat{f}(\omega) d\omega = R_f$$

"risk neutral" probability:

$$\hat{f}(\omega) = \frac{m(\omega)}{\int m(\tilde{\omega}) f(\tilde{\omega}) d\tilde{\omega}} f(\omega)$$

Note: $\hat{f}(\omega) \geq 0$ and $\int \hat{f}(\omega) = 1$

so $\hat{f}(\omega)$ is a probability density

(2) $c_{\omega_1} + c_{\omega_2} + \dots +$

(+) says that expected return of any asset evaluated at the "risk-neutral" probabilities equals the risk-free return.

$\omega \in \Omega$ state of the world

$f(\omega)$ ~ prob of ω (density)

$$\int f(\omega) d\omega = 1, f(\omega) \geq 0$$

$x(\omega)$ - random payoff (next period)

household problem: choose $c_0, c_1(\omega), b$ to
maximize

$$u(c_0) + \beta \int u(c_1(\omega)) f(\omega) d\omega$$

$$\text{s.t. } c_0 + p \cdot b \leq y$$

$$c_1(\omega) \leq x(\omega) \cdot p$$

$$u' > 0, u'' < 0, u'(0) = +\infty, y_0 > 0$$

Lagrangian L :

$$L = u(c_0) + \beta \int u(c_1(\omega)) f(\omega) d\omega$$

$$+ \theta [y - c_0 - p \cdot b]$$

$$+ \int \lambda(\omega) [x(\omega) b - c_1(\omega)] d\omega$$

$\theta, \lambda(\omega)$ are Lagrange multipliers.

FONC:

$$c_0: u'(c_0) - \theta = 0$$

$$\sim \dots \sim \dots \sim \dots \sim \dots$$

$$c_1(\omega) : \beta u'(c_1(\omega)) f(\omega) - \lambda(\omega) = 0$$

$$b: -\theta p + \int \lambda(\omega) x(\omega) d\omega = 0$$

$$\theta p = \int \lambda(\omega) x(\omega) d\omega$$

$$u'(c_0) p = \beta \int u'(c_1(\omega)) x(\omega) f(\omega) d\omega$$

$$p = \int \underbrace{\beta \frac{u'(c_1(\omega))}{u'(c_0)}}_{m(\omega)} x(\omega) f(\omega) d\omega$$

$m(\omega)$ — one for each person who holds (or issues) the asset.

$$(*) p = \int m(\omega) x(\omega) f(\omega) d\omega$$

↑ ↑ ↑
price s.d.f. payoff probability

Remark: There are as many s.d.f.'s as there are

$$m(\omega) = \frac{\beta u'(c_1(\omega))}{u'(c_0)} \quad \text{across agents.}$$

m
a random variable.

how many & how far apart the s.d.f.'s are depends on whether markets are complete or incomplete

1. . . - . + 1.

and how incomplete they are.

note: (*) is

$$p = \int m(\omega) x(\omega) f(\omega) d\omega$$

$$p = E[m x]$$

$$\sigma^2 = E[m R] \quad \text{where} \quad R \equiv \frac{x}{p} \quad (\text{a gross return})$$

Remark: if $x(\omega) = 1$ for ω ,

$p = E[m]$ — price of a risk-free asset

Remark: if we have discrete states $j=1, \dots, J$

formula (*) becomes

$$p = \sum_j p_j x_j f_j$$

↑
s.d.f. payoff in state j

probability of state j

Remark: If we have multiple securities

$i=1, \dots, I$, we have

$$p_i = \int m(\omega) x_i(\omega) f(\omega) d\omega, \quad i=1, \dots, I$$

↑
payoff on security i

$$\underset{I \times I}{P} = \int_{\Omega} m(\omega) \underset{I \times I}{X}(\omega) f(\omega) d\omega$$

on

$$(*) \quad P = E[m X]$$

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_I \end{pmatrix}, \quad X = \begin{pmatrix} x_1(\omega) \\ x_2(\omega) \\ \vdots \\ x_n(\omega) \end{pmatrix}$$

Multiple m 's satisfy $(*)$ in general with incomplete markets. Goal: deduce some restrictions that $(*)$ puts on data on P and X for all possible m 's satisfying $(*)$. This will guide us about what is a good or acceptable theory of a s.d.f.

One way to proceed: we don't have data on m , but we do on X .

I imagine constructing linear least squares projections

$$(†) \quad m = a + x' b + e$$

$$\underset{I \times 1}{m} \quad \underset{I \times 1}{a} \quad \underset{I \times I}{x} \underset{I \times 1}{b} \quad \underset{I \times 1}{e}$$

where $E[e|X] = 0_{I \times I}$ " $e \perp X$ ",
 $E[e] = 0$ " $e \perp 1$ "

$e \perp X$ identifies $a + x'b$ as projection

$e \perp x$ identifies $a + x'b$ as projection
of m on $[1, x]$.

$$(+) \Rightarrow E m = a + E x' b$$

$$\Rightarrow m - E m = (x' - E x') b + e$$

$$(x - E x)(m - E m) = (x - E x)(x' - E x') b' \\ + (x - E x)e$$

Take E on both sides

$$E(x - E x)(m - E m) = E(x - E x)(x' - E x') b \\ + 0$$

$$\text{cov}(x, m) = \text{cov}(x, x) b$$

or

$$b = [\text{cov}(x, x)]^{-1} \text{cov}(x, m)$$

$$a = E m - E x' b$$

we have no data on m , but we have data
on $\rho = E m x$

Note:

$$\text{cov}(x, m) = E m x - E m E x$$

$$= \rho - E m E x \\ \uparrow \quad \uparrow \quad \uparrow \text{"know"} \\ \text{"know"} \quad \text{don't}$$

know — (unless risk-free
rate is observed)

∴

$$b = [\text{cov}(x, x)]^{-1} [p - E_m Ex]$$

For a given guess about E_m , we can

estimate b from $\text{cov}(x, x) \in Ex \in p$.

Note:

$$m = a + x'b + e$$

$$(++) \quad \text{var } m = \text{var}(x'b) + \text{var } e \geq \text{var}(x'b)$$

because $e \perp x \Leftrightarrow \text{var } e \geq 0$

(++) is a version of the Cauchy-Schwarz
inequality

(+++) \Rightarrow

$$(++) \quad \sigma(m) \geq [\text{var}(x'b)]^{1/2}$$

$$\text{where } b_0 = [\text{cov}(x, x)]^{-1} [p - E_m Ex]$$

(++) gives us the Hansen-Jagannathan bound on $\sigma(m)$

a lower bound on $\sigma(m)$'s for a given E_m .

If instead of payoffs, we use data on

If instead of payoffs, we use data on

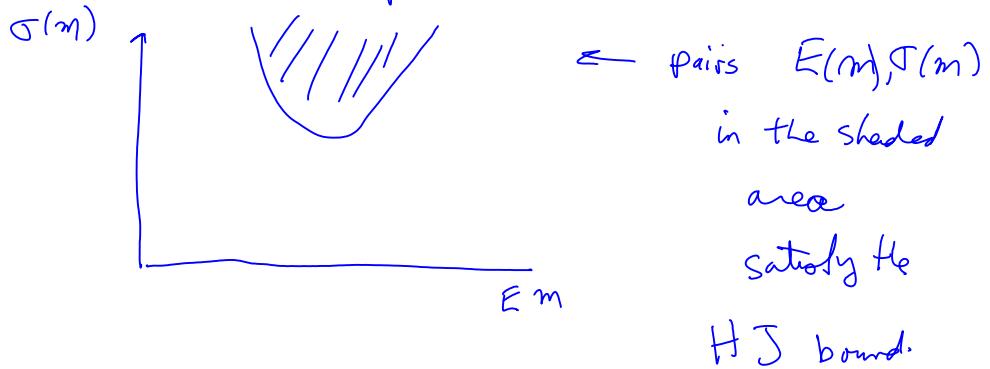
$$\text{returns} \quad R_i = \frac{x_i}{p_i}, \quad R = \begin{pmatrix} R_1 \\ \vdots \\ R_I \end{pmatrix}$$

we would have

$$(1) \quad b = [\text{cov}(R, R)]^{-1} [1 - E_m E R]$$

$$(2) \quad \sigma(m) \geq [\text{var}(R' b)]^{1/2} = \sqrt{b' \text{cov}(R, R) b}$$

We can use this to compute a set of m's



Example of a theoretical s.d.f.

$$m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}, \quad C_t \text{ - consumption}$$

$$c_t = \log C_t, \quad C_t = \exp(c_t)$$

$$u'(C_{t+1}) = C_{t+1}^{-\gamma} \quad u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$$

$$= (\exp C_{t+1})^{-\gamma} = \exp(-\gamma c_{t+1})$$

$$\therefore \exp(-\gamma c_{t+1})$$

$$\therefore m_{t+1} = \beta \frac{\exp(-\gamma c_{t+1})}{\exp(-\gamma c_t)}$$

$$m_{t+1} = \beta \exp(-\gamma(c_{t+1} - c_t))$$

Common statistical model:

$$c_{t+1} = c_t + \mu_c + \sigma_\varepsilon \varepsilon_{t+1}$$

$$\varepsilon_{t+1} \sim N(0, 1)$$

log of consumption is a random walk with drift μ

C_t consumption is a "geometric random walk with drift"

then

$$(1) \quad m_{t+1} = \beta \exp(-\gamma(\mu_c + \sigma_\varepsilon \varepsilon_{t+1}))$$

$$(2) \quad \log m_{t+1} = \log \beta - \gamma(\mu_c + \sigma_\varepsilon \varepsilon_{t+1})$$

Wikipedia remark:

a random variable X is said to be log normal

if $\log X \sim N(\mu, \sigma^2)$

↑ mean variance of $\log X$

Fact:

$$(1) \quad E X = \exp(\mu + \frac{1}{2} \sigma^2)$$

$$\text{std}(X) = \exp(\mu + \frac{1}{2} \sigma^2) \sqrt{e^{\sigma^2} - 1}$$

∴

1 1 1 ..

$$(f) \quad \frac{std(x)}{E(x)} = \sqrt{e^{\sigma^2} - 1}$$

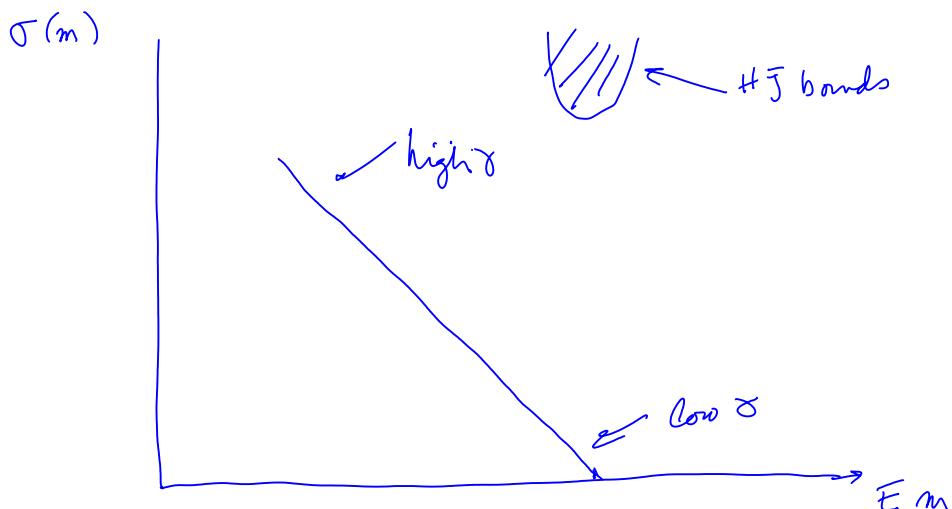
Apply (f) and (f) to (e), (d)

$$\log m \sim N(\log \beta - \gamma \mu_c, \sigma^2 \sigma_e^2)$$

$$\Rightarrow E_m = \beta \exp \left[\gamma (-\mu_c + \frac{1}{2} \sigma^2 \sigma_e^2) \right]$$

$$\frac{\sigma(m)}{E(m)} = \sqrt{e^{\sigma^2 \sigma_e^2} - 1}$$

Tallarini: Plot these for various values of γ
with estimated μ_c, σ_e from G.S. time series.



Remark: recall

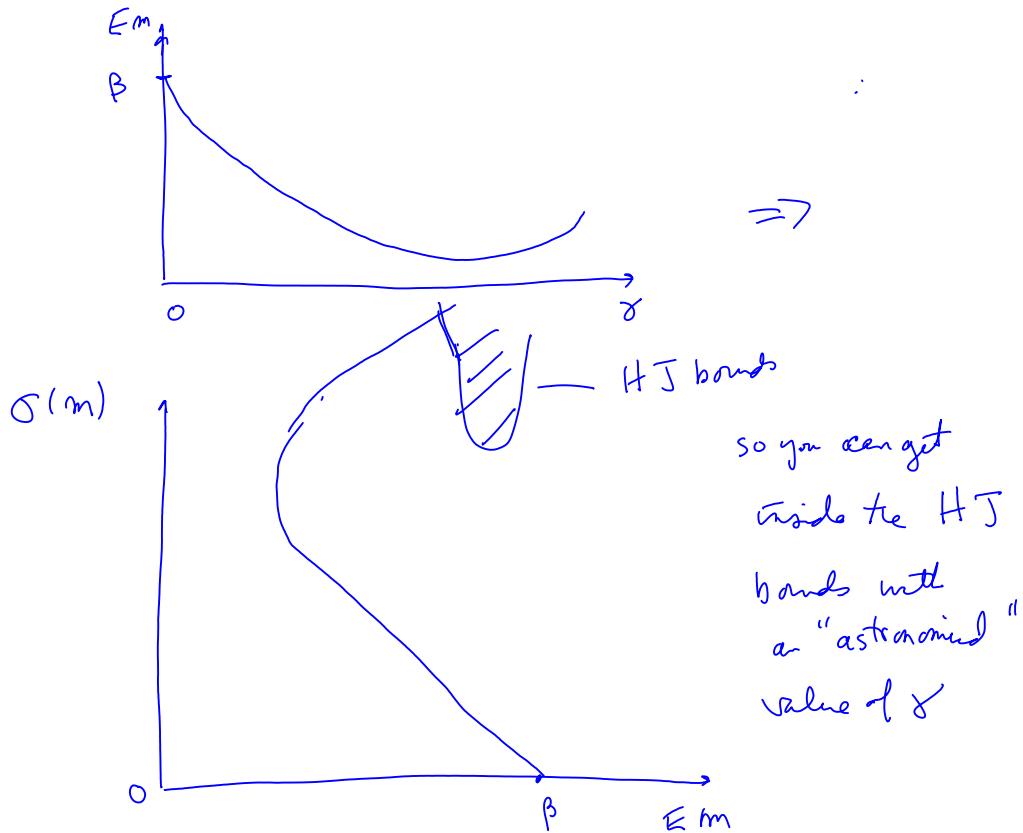
$$E_m = \beta \exp \left[\gamma \left(-\mu_c + \frac{\sigma_e^2}{2} \right) \right]$$

The $\exp(-\gamma \mu_c)$ term decreases as $\gamma \uparrow$
n. $\propto e^{-\gamma \mu_c} = \dots$

The $\exp\left(\delta^2 \frac{\sigma_e^2}{2}\right)$ term increases as $\delta \uparrow$

M_c & σ_e^2 empirically take values with aggregate consumption data for U.S. that make

E_m fall at first, then rise as $\delta \uparrow$



Remark: risk free rate puzzle is the tendency of E_m to fall (R_f to rise) as $\delta \uparrow$ for "low" values of δ

Remark: the equity premium puzzle is the high measured "market price of risk"

$$\frac{\sigma(m)}{E_m}$$

$$c_t = \log C_t$$

$$c_{t+1} - c_t = u_c + \sigma_\varepsilon \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, 1)$$

geometric random walk with drift

we established that with

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$$

$$(*) \quad E[m] = \beta \exp \left(\gamma (-M_c + \frac{\gamma \sigma_\varepsilon^2}{2}) \right)$$

\uparrow discounting \uparrow intertemporal smoothing \uparrow precautionary saving because of risk

$$\frac{\sigma(m)}{E(m)} = \left\{ \exp \left(\gamma^2 \sigma_\varepsilon^2 \right) - 1 \right\}^{\frac{1}{2}}$$

\uparrow risk aversion

Problem: the same parameter γ pushes up $\sigma(m)$ while pushing down $E[m]$ (because of preference for intertemporal smoothing)

Focus on

$$E[m] = \beta \exp \left(-\gamma M_c + \gamma^2 \frac{\sigma_\varepsilon^2}{2} \right)$$

\uparrow intertemporal substitution

$$\frac{\sigma(m)}{E[m]} = \sqrt{\exp(\gamma^2 \sigma_\varepsilon^2) - 1}$$

$$\overline{E(m)} = \overline{U(u)} + \sigma^2$$

risk aversion (across states)

Idea of Epstein-Zin-Weil: separate these two - set σ in 1 in $I \in S$, but σ large for risk aversion.

Solution: Epstein-Zin-Weil-Tallarini

Change utility function to

$$U_t = c_t - \beta \theta E_t \left[\exp \left(U_{t+1} / -\theta \right) \right]$$

where $\theta \equiv \frac{-1}{(1-\beta)(1-\gamma)}$

$$\gamma' = 1 + \frac{1}{(1-\beta)\theta} \quad (> 1 \text{ when } \theta < 0)$$

Then

$$E[m] = \beta \exp \left[-\mu_c + \frac{\sigma_\varepsilon^2}{2} (2\hat{\gamma}' - 1) \right]$$

$$\frac{\sigma(m)}{E(m)} = \left\{ \exp \left(\sigma_\varepsilon^2 \hat{\gamma}' - 1 \right) \right\}^{\frac{1}{2}}$$

\downarrow no σ here \downarrow this effect pushes $E[m]$
up slightly

$\leftarrow \hat{\gamma}' \text{ here}$

Reinterpretation of Tallarini.

Replace random walk model with "pessimistic" model

$$c_{t+1} - c_t = \mu_c + \sigma_\varepsilon (\varepsilon_{t+1} + w(\theta)) \quad w(\theta) = -\frac{\sigma_\varepsilon}{\theta}$$

$(-\beta) \theta$

$$= \hat{\mu}_c + \sigma_{\varepsilon} \varepsilon_{t+1}$$

$$\hat{\mu}_c = \bar{\mu} - \sigma_{\varepsilon}^2 (\hat{\gamma} - 1)$$

Plug in the mean $\hat{\mu}_c$ in formula \leftrightarrow , putting $\gamma=1$
(low risk aversion)

$$\begin{aligned} E_m &= \beta \exp \left[-\mu + \sigma_{\varepsilon}^2 (\hat{\gamma} - 1) + \frac{\sigma_{\varepsilon}^2}{2} \right] \\ &= \beta \exp \left[-\mu + -\frac{\sigma_{\varepsilon}^2 (2\hat{\gamma} - 1)}{2} \right] \end{aligned}$$

which is Tallaini's formula

give the two interpretations of Tallaini's.

- (1) separate intertemporal substn from risk-aversion
- (2) Risk aversion vs. model uncertainty

More implications of $E_t m_{t+1} R_{t+1} = 1$

Let $R_f^{-1} = E_m$ be the risk-free rate's reciprocal.

An excess return on asset i is defined as

$$\xi_{t+1}^i = R_{t+1}^i - R_f$$

Evidently, because $E_t m_{t+1} R_f = 1$, and $E_t m_{t+1} R_{t+1}^i = 1$

it follows that

$$(*) \quad E_t m_{t+1} \xi_{t+1}^i = 0 \quad \forall i$$

Def'n of (conditional) covariance:

$$\text{cov}_t(m_{t+1}, \xi_{t+1}^i) = E_t m_{t+1} \xi_{t+1}^i - E_t m_{t+1} E_t \xi_{t+1}^i$$

\Rightarrow

$$E_t m_{t+1} \xi_{t+1}^i = \text{cov}_t(m_{t+1}, \xi_{t+1}^i) + E_t m_{t+1} E_t \xi_{t+1}^i$$

||
0 by (*)

$$(*) \Rightarrow E_t m_{t+1} E_t \xi_{t+1}^i = -\text{cov}(m_{t+1}, \xi_{t+1}^i)$$

Now remember fact about (population) least squares regression

$$\xi_{t+1}^i = \underbrace{\frac{\text{cov}(m_{t+1}, \xi_{t+1}^i)}{\text{var}(m_{t+1})}}_{\approx} m_{t+1} + \varepsilon_{t+1}$$

$\varepsilon_{t+1} \perp m_{t+1}$

β regression coefficient

↑
regress ξ_{t+1}^i against m_{t+1}

∴

$$\text{Var}(\xi_{t+1}^i) = \frac{|\text{Cor}(m_{t+1}, \xi_{t+1}^i)|^2}{\text{Var}(m_{t+1})^2} \text{Var} m_{t+1} + \text{Var}(\varepsilon_{t+1}) > 0$$

⇒

$$\text{Var}(\xi_{t+1}^i) \geq \frac{|\text{Cor}(m_{t+1}, \xi_{t+1}^i)|^2}{\text{Var}(m_{t+1})}$$

(+)

$$| \geq \frac{|\text{Cor}(m_{t+1}, \xi_{t+1}^i)|^2}{\text{Var}(m_{t+1}) \text{Var}(\xi_{t+1}^i)} \quad "1 \geq R^2"$$

Now use (**) and (+)

$$\left| E_t m_{t+1} E_t \xi_{t+1}^i \right|^2 = |\text{Cor}(m_{t+1}, \xi_{t+1}^i)|^2 \quad (\text{this is } (**)\text{ squared}) \\ \leq \text{Var}(m_{t+1}) \cdot \text{Var}(\xi_{t+1}^i)$$

⇒

$$E_t m_{t+1} E_t \xi_{t+1}^i \leq \sigma_t(m_{t+1}) \cdot \sigma_t(\xi_{t+1}^i) \quad (\sigma_t \text{ is std}_t)$$

$$(++) \quad \frac{E_t \xi_{t+1}^i}{\sigma_t(\xi_{t+1}^i)} \leq \frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})} = \text{"market price of risk"}$$

"Sharpe ratio" ↑ "maximum Sharpe ratio"

Remark The HJ bounds are bounds on the RHS of (++)

Remark: Write (4*) as

$$E_t m_{t+1} E_t \xi_{t+1}^i = - \text{cov}_x(m_{t+1}, \xi_{t+1}^i)$$

on

$$E_t m_{t+1} E_t \xi_{t+1}^i = - \frac{\text{cov}_x(m_{t+1}, \xi_{t+1}^i)}{\text{var}(m_{t+1})} \cdot \text{var } m_{t+1}$$

$$E_t \xi_{t+1}^i = - \frac{\text{cov}(m_{t+1}, \xi_{t+1}^i)}{\text{var } m_{t+1}} \cdot \underbrace{\text{var } m_{t+1} \cdot R_{f+1}}_{\text{"undiversifiable risk.}}$$

$\hat{\beta} = \text{regression coefficient}$
of excess return on SDF.

Recall consumption based model of m_{t+1} :

$$m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}, \quad u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$$

$$c_t = \log C_t, \quad C_t = \exp(c_t), \quad \beta = \exp(-\gamma)$$

$$\Rightarrow c_{t+1} - c_t = M_c + \sigma_\varepsilon \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, 1)$$

$$\Rightarrow m_{t+1} = \beta \exp(-\gamma(c_{t+1} - c_t))$$

$$= \beta \exp(-\gamma(M_c + \sigma_\varepsilon \varepsilon_{t+1}))$$

$$(*) \quad \log m_{t+1} = -\gamma - \gamma M_c - \gamma \sigma_\varepsilon \varepsilon_{t+1}$$

\uparrow discounting \uparrow growth \uparrow risk $\gamma \sigma_\varepsilon$ is "risk exposure"

$$\text{Remark: } E_t m_{t+1} = \exp(-\gamma) \exp\left(-\gamma M_c + \frac{1}{2} \gamma^2 \sigma_\varepsilon^2\right)$$

$$= \exp(-r_t) \quad \text{where } r_t = \text{one period risk free rate (net)}$$

$$\text{Thus } r_t = \gamma + \gamma M_c - \frac{1}{2} \gamma^2 \sigma_\varepsilon^2$$

\uparrow correction for uncertainty

Thus, we can write

$$\log m_{t+1} = -r_t - \frac{1}{2} \gamma^2 \sigma_\varepsilon^2 - \gamma \sigma_\varepsilon \varepsilon_{t+1}$$

\uparrow short rate \uparrow "correction" \uparrow risk exposure

Because of failure of this model with reasonable γ to attain $\{T\}$ bonds, we want to

(1) generalize the model by allowing more sources of

macroeconomic risk ε_{t+1} to be priced

(2) divides ε_{t+1} from consumption alone.

"Bond price model"

$$z_{t+1} = \mu + \Phi z_t + C \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, I)$$

$$r_t = \delta_0 + \delta_z' z_t, \quad \text{"short rate"}$$

$$\log m_{t+1} = -r_t - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} \quad (\text{same } \varepsilon_{t+1} \text{ as above})$$

$$\Lambda_t = \Lambda_0 + \Lambda_z z_t$$

\Rightarrow

$$\begin{aligned} \log m_{t+1} = & -(\delta_0 + \delta_z' z_t) - \frac{1}{2} (\Lambda_0 + \Lambda_z z_t)' (\Lambda_0 + \Lambda_z z_t) \\ & - (\Lambda_0 + \Lambda_z z_t)' \varepsilon_{t+1} \end{aligned}$$

$$m_{t+1} = \exp(\text{" })$$

Object to estimate: μ, Φ, C ← (can estimate
VAR for z_t)

$$\delta_0, \delta_z'$$

$$\Lambda_0, \Lambda_z$$

estimate from moment conditions

$$E_t [m_{t+1} R_{t+1}^i] = 1 \quad \text{for a number of returns } R_{t+1}^i \text{ and dates } i=1, \dots, I, ; t=1, \dots, T$$

The system we'll estimate

$$r_t, \quad 1, \quad \dots, \quad -1, \quad \dots, \quad \dots, \quad \dots, \quad \dots, \quad \dots, \quad \dots$$

$$(2) E_t \left[\exp \left[-(\delta_0 + \delta_z z_t) - \frac{1}{2} (\lambda_0 + \lambda_z z_t)' (\lambda_0 + \lambda_z z_t) \right. \right. \\ \left. \left. - (\lambda_0 + \lambda_z z_t)' (C^{-1} (z_{t+1} - \Phi z_t)) \right] R_{t+1}^i \right] = 1$$

\uparrow
 ε_{t+1}

$$\Leftrightarrow z_{t+1} = u + \Phi z_t + C \varepsilon_{t+1}, \quad \varepsilon_{t+1} \perp z_t$$

Remark: This is an "affine model of the yield curve".

Let $y_t(\gamma)$ be the yield on a zero coupon bond of maturity γ at date t , $\gamma = 1, 2, \dots$

Then

$$y_t(\gamma) = a(\gamma) + b(\gamma) z_t$$

$$\text{where } a(1) = \delta_0, \quad b(1) = \delta_z$$

and $a(\gamma), b(\gamma)$ can be computed recursively
as functions of the δ 's & λ 's.

This is called the "affine" term structure model.

Remark: Objects in HJ bounds:

$$E_t m_{t+1} = \exp \left(-(\delta_0 + \delta_z' z_t) \right)$$

$$\text{std}_x m_{t+1} = \exp \left(-(\delta_0 + \delta_z' z_t) \right) \cdot \sqrt{e^{(\lambda_0 + \lambda_z z_t)' (\lambda_0 + \lambda_z z_t)} - 1}$$

Portfolio choice

Monday, April 26, 2010
9:12 AM

Basic portfolio choice model based on "fundamentals"

$$\begin{aligned} & \max_{c_0, c_1(\omega), b} u(c_0) + \beta \int u(c_1(\omega)) f(\omega) d\omega \\ \text{s.t. } & c_0 + p \cdot b \stackrel{n \times 1}{=} y_0 \\ & c_1(\omega) \leq b \cdot x(\omega) + y_1(\omega) \end{aligned}$$

Lagrangian:

$$\begin{aligned}
 & u(c_0) + \int u(c_1(\omega)) f(\omega) d\omega \\
 & + \theta [y_0 - c_0 - p \cdot b] \\
 & + \int \lambda(\omega) [b \cdot x(\omega) - c_1(\omega)] d\omega
 \end{aligned}$$

$$F_{DNC}: \text{ a) } p = \int \beta \frac{u'(c_1(\omega))}{u(c_0)} x(\omega) f(\omega) d\omega$$

$$c_1(\omega) = b \cdot x(\omega) + y_1(\omega)$$

Substitute for $c_i(\omega)$ into (8) to get

$$(+)^{m \times 1} p = \beta \int \frac{u'(b \cdot x(\omega) + y_1(\omega))}{u'(y_0 - p \cdot b)} x(\omega) f(\omega) d\omega$$

(+) are m equations in n unknowns. (b)

More superficial "Wall Street" formulation

$$\text{vector of random returns } R \sim N(\mu, \Sigma)$$

$m \times 1$ $m \times 1$ $m \times n$

portfolios b

$$\text{end of period wealth: } W = R \cdot b \quad \bullet \text{ inner product}$$

mean variance preferences : rank portfolios by

$$E[W] - \lambda \text{var}[W] \quad , \lambda > 0 \text{ measures} \\ = E[R \cdot b] - \lambda \text{var}(R \cdot b) \quad \text{"risk tolerance"}$$

$$= E[R \cdot b] - \lambda [b' (\text{cov}(R, R)) b]$$

$$= \mu \cdot b - \lambda [b' \Sigma b]$$

Problem: $\max_b \mu' b - \lambda [b' \Sigma b]$

FONC: $\mu - \lambda \Sigma b = 0$

$$\Rightarrow \boxed{\hat{b} = \lambda^{-1} \Sigma^{-1} \mu} \quad \text{optimal portfolio}$$

$$\text{var}(R \cdot b) = \lambda^2 \mu' \Sigma^{-1} \mu = \eta$$

$$\lambda^{-2} = \frac{\eta}{\mu' \Sigma^{-1} \mu}$$

$$\lambda^{-1} = \frac{\sqrt{\eta}}{\sqrt{\mu' \Sigma^{-1} \mu}}$$

mean $E R_b = \sqrt{\frac{\eta}{\mu' \Sigma^{-1} \mu}} \cdot \mu' \Sigma^{-1} \mu = \sqrt{\eta} \cdot \sqrt{\mu' \Sigma^{-1} \mu}$

Asset pricing, 2

Wednesday, April 07, 2010
10:32 AM

$$\begin{aligned} \mathbb{E}[R_{t+1}] &= E[m_{t+1} R_{t+1}], \quad \text{cov}(R_{t+1}, m_{t+1}) = E[m_{t+1} R_{t+1}] - E[m_{t+1}] E[R_{t+1}] \\ &\Downarrow \quad \Rightarrow \quad \mathbb{E}[m_{t+1} R_{t+1}] = \mathbb{E}[m_{t+1}] \mathbb{E}[R_{t+1}] + \text{cov}(R_{t+1}, m_{t+1}) \\ \mathbb{E}[R_{t+1}] &= \mathbb{E}[m_{t+1}] \mathbb{E}[R_{t+1}] + \text{cov}(R_{t+1}, m_{t+1}) \end{aligned}$$

$$\mathbb{E}[m_{t+1}] = R_f^{-1} \quad \text{risk free rate}$$

$$(\mathbb{E}[m_{t+1}])^{-1} = \mathbb{E}[R_{t+1}] + \text{cov}(R_{t+1}, m_{t+1})(\mathbb{E}[m_{t+1}])^{-1}$$

or

$$(*) \quad \mathbb{E}[R_{t+1}] - R_f^{-1} = -\text{cov}(R_{t+1}, m_{t+1}) R_f^{-1}$$

(*) applies to any asset with return R_t^i , and also
conditioned on any information at t \Rightarrow replace \mathbb{E} and
 cov with \mathbb{E}_t , cov_t .

$$\begin{aligned} (*) \quad \mathbb{E}_t[R_{t+1}^i] - R_f^{-1} &= -\text{cov}_t(R_{t+1}^i, m_{t+1}) \cdot R_f^{-1} \\ &= -\frac{\text{cov}_t(R_{t+1}^i, m_{t+1})}{\text{var}_t m_{t+1}} \cdot (\text{var}_{m_{t+1}}) R_f^{-1} \\ &\quad \underbrace{\qquad \qquad \qquad}_{\text{beta risk}} \quad \underbrace{\qquad \qquad \qquad}_{\text{risk}} \\ \text{note: } \hat{\beta}_i &= -\frac{\text{cov}_t(R_{t+1}^i, m_{t+1})}{\text{var}_t m_{t+1}} \end{aligned}$$

$\hat{\beta}_i$ is regression coefficient
(not to be confused with
our direct β below)

is regression coefficient of R_{t+1}^i on m_{t+1}

examples:

example:

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \quad [\beta \text{ here, not } \gamma \text{ above}]$$

\Rightarrow

$$\begin{aligned} \ln m_{t+1} &= \ln \beta - \gamma (\ln c_{t+1} - \ln c_t) \\ &= \ln \beta - \gamma (M + \sigma \varepsilon_{t+1}) \quad (\ln c_{t+1} - \ln c_t = M + \sigma \varepsilon_{t+1}) \\ &\qquad \qquad \qquad \varepsilon_{t+1} \sim N(0, 1) \quad \uparrow \\ &\qquad \qquad \qquad \text{"consumption is a geometric random walk with drift } \mu \text{"} \end{aligned}$$

$$(*) \quad m_{t+1} = \exp [\ln \beta - \gamma M - \gamma \sigma \varepsilon_{t+1}]$$

apply log normal formula to compute

$$\text{Var}_t m_{t+1} =$$

$$\begin{aligned} E_t m_{t+1} &= \\ \ln m_{t+1} &= \ln \beta - \gamma (M + \sigma \varepsilon_{t+1}) \\ (+) \quad &\sim N \left(\underbrace{\ln \beta - \gamma M}_{\mu_m}, \underbrace{\sigma^2 \sigma^2}_{\sigma_m^2} \right) \end{aligned}$$

log normal formulas:

$$\ln m \sim N(\mu_m, \sigma_m^2) \Rightarrow$$

$$\begin{aligned} E(m) &= \exp \left(\mu_m + \frac{1}{2} \sigma_m^2 \right) \\ \text{Var}(m) &= (e^{\sigma_m^2} - 1) e^{2\mu_m + \sigma_m^2} \quad \beta = e^{-f} \end{aligned}$$

$$\text{std}(m) = \left(e^{\mu_m + \frac{\sigma_m^2}{2}} \right) \sqrt{e^{\sigma_m^2} - 1} \quad \ln \beta = -f$$

so (+) \Rightarrow

$$\begin{aligned} E(m) &= \exp \left(\mu_m + \frac{1}{2} \sigma_m^2 \right) \quad \mu_m = -f - \gamma M \\ &= \exp \left(-f - \gamma M + \frac{1}{2} \gamma^2 \sigma^2 \right) \quad \sigma_m = \gamma \sigma \sigma \\ \sigma_m &\equiv \text{std}(m) = \left(e^{(-f - \gamma M) + \frac{1}{2} \gamma^2 \sigma^2} \right) \left(\sqrt{e^{\gamma^2 \sigma^2} - 1} \right) \end{aligned}$$

$\frac{\sigma_m}{E_m}$ is Hansen's market price of risk.

Stop reading here

Use these formulas to "fill in" the

$$\text{Price of Risk: } - \frac{\text{Cov}(R_{t+1}^i, m_{t+1})}{\text{Var}_t m_{t+1}}$$

amount of risk: $\text{Var}_t m_{t+1} R_f | t+1$

Link to continuous time formulation:

$$m_t = e^{-\gamma t} \cdot c_t^{-\delta}$$

$$\frac{dm}{dt} = -\kappa dt - \eta dB_x ,$$

η = price of risk

$$\gamma = \frac{u - r}{\sigma}$$

$$\text{eg } \eta = \sigma^5 \quad \text{and} \quad M - r = \sigma^{5^2}$$

$$\frac{\sigma(m)}{E[m]} = \eta \quad = \text{price of risk in the Lucas model} -$$

semitight different notion of prior risk in the

$$E_t R_{t+1}^i - R_{t+1} = \beta \cdot (\lambda_0 + \lambda_1 X_t)$$

✓
pin
of risk

$$\frac{\text{Var } m_{t+1}}{E_t m_{t+1}}$$

11 814, L

"risk

risk is the std. is being avoid.

Asset pricing, 3

Monday, March 01, 2010
9:15 AM

Extract implications of

$$I = \beta R \frac{u'(c_{t+1})}{u'(c_t)}$$

$$R = (1+r), \quad \beta = \frac{1}{1+r}$$

in non stochastic case

[later we'll consider implications of

$$I = \beta R E_t \frac{u'(c_{t+1})}{u'(c_t)}$$

when R is risk-free but
 c_{t+1} is stochastic.]

Form of utility: power or constant relative risk aversion

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1$$

$$= \log_e c \quad \text{if } \gamma = 1$$

$$u'(c) = c^{-\gamma} \quad . \quad \Rightarrow$$

$$I = \beta R \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

$$R = \beta^{-1} \left(\frac{c_{t+1}}{c_t} \right)^{\gamma}$$

C

Suppose

$$c_{t+1} = \delta c_t$$

where

$$\delta = 1 + g \quad , \quad g \text{ is net growth rate of consumption}$$

$$\log \delta \approx g$$

$$\log \beta^{-1} \approx \gamma$$

$$\log R \approx r$$

$$R = \beta^{-1} \delta^\gamma$$

$$\log R = -\log \beta + \gamma \log \delta$$

or

$$r \approx \gamma + \gamma g$$

• higher g raises r

• higher γ raises r

↑
distaste for unsuitability

need high r to reconcile the consumer to the growth.

Stochastic counterpart with log normal consumption growth

$$\text{Fact: } \log x_{...} \sim N(\mu, \sigma^2) \Rightarrow x_{...} = \exp(\mu + \sigma \varepsilon)$$

$$\varepsilon \sim N(0,1)$$

Fact: $\log \gamma_{t+1} \sim N(\mu, \sigma^2) \Rightarrow \gamma_{t+1} = \exp(\mu + \sigma \varepsilon)$

$$\log E \gamma_{t+1} = \mu + \frac{\sigma^2}{2} \quad E \gamma_{t+1} = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

L.S. $\log \gamma = \sigma \varepsilon_{t+1} - \frac{\sigma^2}{2}$

$$\sigma_\gamma = \exp\left(\sigma \varepsilon_{t+1} - \frac{\sigma^2}{2}\right)$$

where $\varepsilon_{t+1} \sim N(0, 1)$

$$E \log \gamma = -\frac{\sigma^2}{2}$$

$$\text{Var} \log \gamma = \sigma_\gamma^2 \quad (\text{coefficient on } \varepsilon_{t+1}^2)$$

$$\Rightarrow \log E \gamma = 0.$$

=

application

$$\frac{c_{t+1}}{c_t} = \delta \exp\left\{\sigma_c \varepsilon_{t+1} - \frac{\sigma_c^2}{2}\right\}, \quad \varepsilon_{t+1} \sim N(0, 1)$$

$$\log \frac{c_{t+1}}{c_t} = \log \delta + \sigma_c \varepsilon_{t+1} - \frac{\sigma_c^2}{2}$$

$$\sim N\left(\log \delta - \frac{\sigma_c^2}{2}, \sigma_c^2\right)$$

\Rightarrow

$$E\left(\frac{c_{t+1}}{c_t}\right) = \exp\left(\log \delta - \frac{\sigma_c^2}{2} + \frac{\sigma_c^2}{2}\right)$$

$$= \exp \log \delta = \delta$$

Now evaluate

✓

$$\begin{aligned}
 \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} &= \delta^{-\gamma} \exp \left(-\gamma \left(\sigma_c \varepsilon_{t+1} - \frac{\sigma_c^3}{2} \right) \right) \\
 &= \exp \left[\log \delta^{-\gamma} - \gamma \left(\sigma_c \varepsilon_{t+1} - \frac{\sigma_c^3}{2} \right) \right] \\
 &= \exp \left[\log \delta^{-\gamma} + \gamma \frac{\sigma_c^3}{2} - \gamma \sigma_c \varepsilon_{t+1} \right]
 \end{aligned}$$

$$\Rightarrow \log \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \sim \mathcal{W} \left(\log \bar{\delta}^{-\gamma} + \gamma \frac{\sigma_c^2}{2}, \gamma^2 \sigma_c^2 \right)$$

$$\begin{aligned}
 \Rightarrow F &= \left(\frac{C_{ext}}{C_0} \right)^{-\gamma} = \exp \left(\log \delta^{-\gamma} + \gamma \frac{\sigma_c^2}{2} + \gamma^2 \frac{\sigma_c^2}{2} \right) \\
 &\geq \exp \left(\log \delta^{-\gamma} + \frac{\sigma_c^2}{2} \gamma (1+\gamma) \right) \\
 &= \delta^{-\gamma} \exp \frac{(1+\gamma) \gamma \sigma_c^2}{2}
 \end{aligned}$$

Apply this to

$$l = \beta R F_t \frac{w'(c_{t+1})}{w'(c_t)}$$

$$\log(1+\varsigma) = -\log \beta + \gamma \log \delta - (1+\gamma)\gamma \frac{\delta_c^2}{2}$$

$$r \approx g + \gamma_g - (1+\gamma) \gamma \frac{\sigma_c^2}{2}, \quad \gamma = 1+g$$

\uparrow discount \downarrow growth \curvearrowleft precautionary savings

Asset pricing 4

Wednesday, April 07, 2010
3:27 PM

Expected utility two states

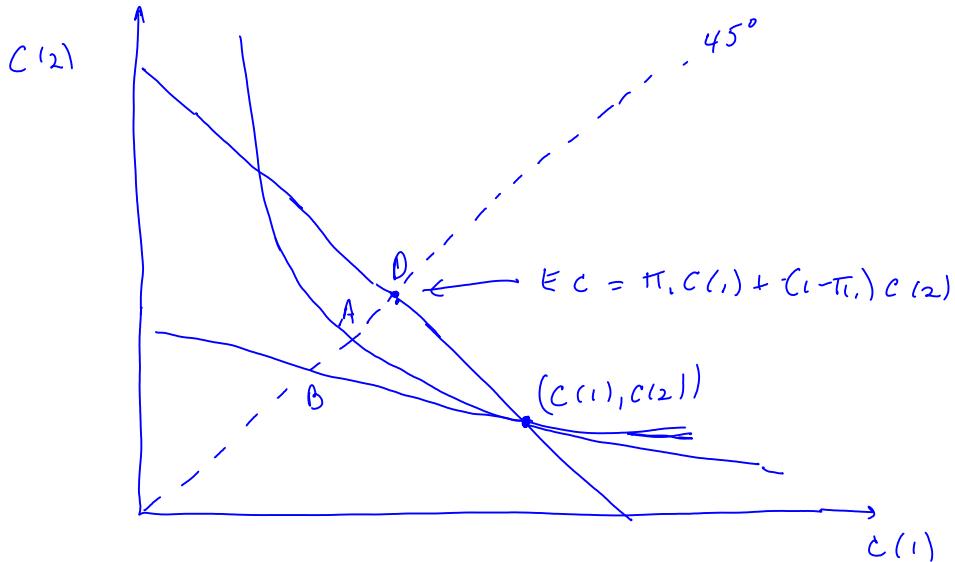
Background model is

$$C_0 + \pi_i u(C_{(1)}) + (1-\pi_i) u(C_{(2)})$$

$\uparrow_{\text{state}} \quad C_{(i)}$
 $\uparrow_{\text{state}} \quad \bar{z} = 1, 2$

Simplify & draw indifference curves for utility function

$$U = \pi_i u(c_{(1)}) + (1-\pi_i) u(c_{(2)})$$



D: (\bar{c}, \bar{c}) where $\bar{c} = \pi_i c_{(1)} + (1-\pi_i) c_{(2)}$

A: (\hat{c}, \hat{c}) where $u(\hat{c}) = \pi_i u(c_{(1)}) + (1-\pi_i) u(c_{(2)})$

B

$$U = \pi_1 u(c_{(1)}) + \pi_2 u(c_{(2)})$$

$$dU = \pi_1 u'(c_{(1)})dc_{(1)} + (1-\pi_1)u'(c_{(2)})dc_{(2)}$$

= 0 \Rightarrow

$$\frac{dc_{(2)}}{dc_{(1)}} = \frac{-\pi_1}{(1-\pi_1)} \frac{u'(c_{(1)})}{u'(c_{(2)})} =$$

Budget constraint:

$$I_1 = p_1 c_{(1)} + p_2 c_{(2)} (+ c_0)$$

$$p_1 \sim \frac{\text{time 0 good}}{\text{unit of } c \text{ in state 1 at time 1}}, p_2 = \frac{\text{time 0 good}}$$

$$dI = p_1 dc_{(1)} + p_2 dc_{(2)}$$

$$\frac{dc_{(2)}}{dc_{(1)}} = - \frac{p_1}{p_2}$$

$$\therefore \text{will equal } \frac{p_1}{p_2} = \frac{+\pi_1 u'(c_{(1)})}{(1-\pi_1)u'(c_{(2)})}$$

$$\text{eg } \frac{p_1}{p_2} = \frac{\pi_1}{1-\pi_1} \cdot \frac{\frac{1}{c_{(1)}}}{\frac{1}{c_{(2)}}} = \frac{c_{(2)}}{c_{(1)}}$$

$$\text{with } \pi_1 = .5, c_{(2)} < c_{(1)}, \frac{p_1}{p_2} < 1$$

Point B:

$$(p_1 + p_2) \hat{c} = p_1 c(1) + p_2 c(2)$$

$$\left(\frac{p_1}{p_2} + 1\right) \hat{c} = \frac{p_1}{p_2} c(1) + c(2)$$

$$\hat{c} = \frac{p_1}{p_1 + p_2} c(1) + \frac{p_2}{p_1 + p_2} c(2)$$

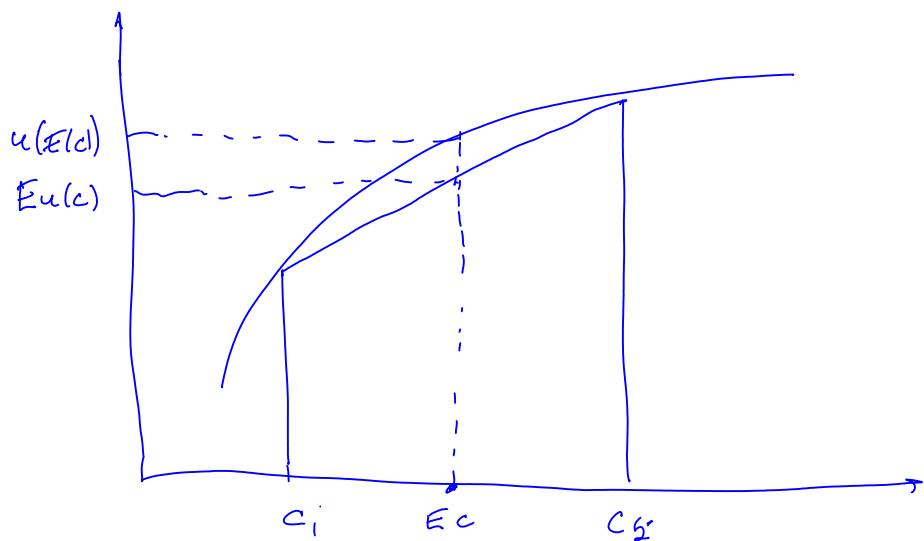
risk neutral
 probabilities

$$\hat{c} = \frac{\pi_1 u'(c_1)}{\pi_1 u'(c_{11}) + (1-\pi_1) u'(c_{12})} c(1) + \frac{(1-\pi_1) u'(c_2)}{\pi_1 u'(c_{21}) + (1-\pi_1) u'(c_{22})} c(2)$$

B: \hat{c} = expected consumption at risk neutral probabilities -

$(p_1 + p_2)$ = price of a sure claim on consⁿ at $t=1$

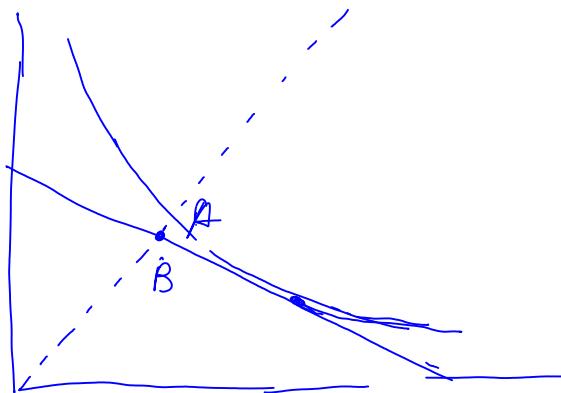
$\frac{1}{p_1 + p_2}$ = gross risk free interest rate.



concavity of $u \Rightarrow Eu(c) < u(Ec)$

Asset pricing:
price to demand.

\hat{B} is asset price
is certainty equivalent



let α be asset price:

$$\alpha = \frac{\pi_1}{\pi_1 + \pi_2} c(1) + \frac{\pi_2}{\pi_1 + \pi_2} c(2)$$

risk neutral
substitute

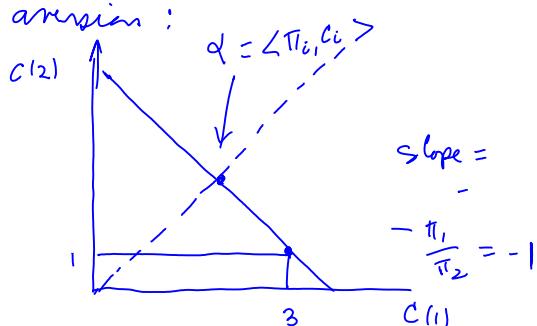
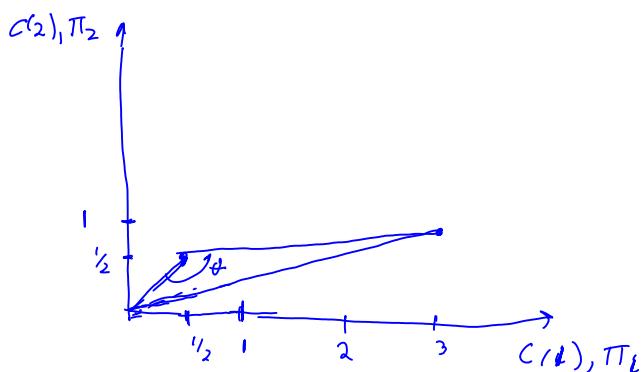
$$\alpha = \frac{\pi_1 u'(c_1)}{\pi_1 u'(c_1) + (1-\pi_1) u'(c_2)} c(1) + \frac{(1-\pi_1) u'(c_2)}{\pi_1 u'(c_1) + (1-\pi_1) u'(c_2)} c(2)$$

$$\alpha = \langle m(i), c(i) \rangle$$

$m(i)$ is a risk-neutral probability

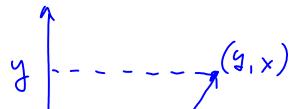
risk-neutral pricing - no risk aversion:

$$\alpha = \langle \pi_i, c_i \rangle$$

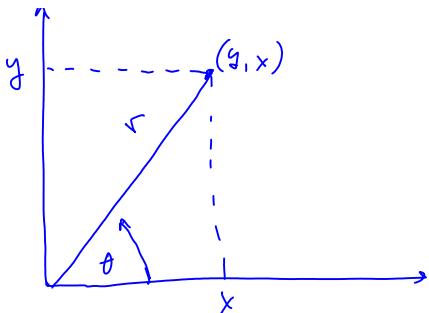


$$\alpha = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$$

$$\cos \theta = \langle \pi_i, c_i \rangle$$



$$\cos \theta = \langle \pi_i, c_i \rangle$$



$$\cos \theta = \frac{x}{r}, \quad r = \sqrt{x^2 + y^2}$$

Re do



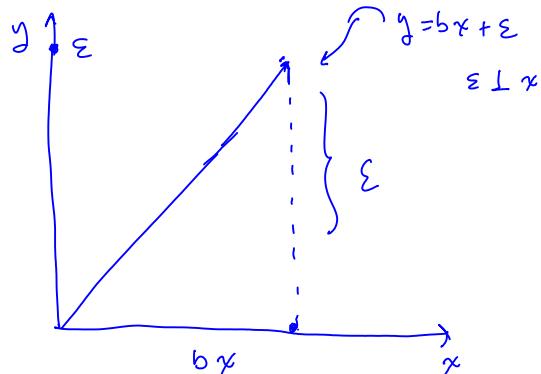
To specify graphically regression coefficient, correlation coefficient, covariance

$$b = \frac{\langle x, y \rangle}{\langle x, x \rangle}$$

$$\cos \theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$g = \frac{\langle x, y \rangle}{\langle x, x \rangle \langle y, y \rangle}$$

Get Luenberger



$$y = b x + \epsilon$$

$$\langle y, x \rangle = b \langle x, x \rangle + \langle \epsilon, x \rangle$$

$$b = \frac{\langle y, x \rangle}{\langle x, x \rangle}$$

↑ regression
as subspace

Start here:

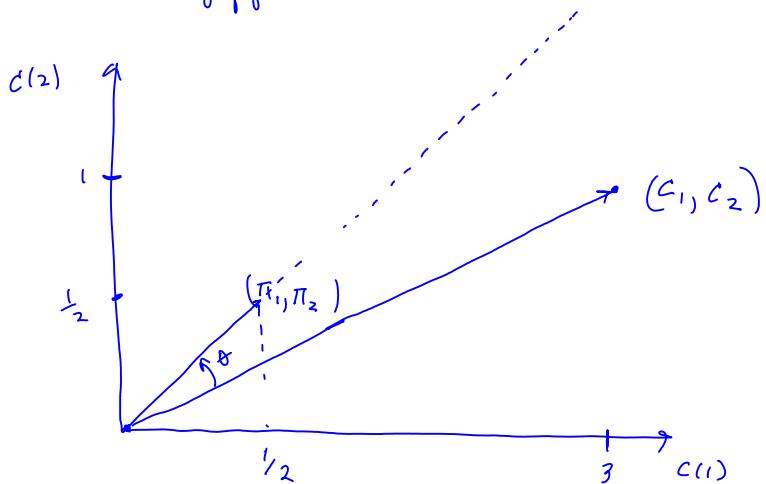
$$\text{norm: } |\langle x, x \rangle|^{1/2} \quad \text{normed linear space}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \langle x, x \rangle = (\sum x_i^2)$$

$$\text{vector 1: } c = (c_1, c_2)$$

$$\pi = (\pi_1, \pi_2)$$

expected value of payoff: $E c \cdot = \sum \pi_i c_i = \langle \pi, c \rangle$



$$\langle c, \pi \rangle = \sum c_i \pi_i$$

geometric interpretation of Cauchy-Schwarz inequality: $|\cos \theta| \leq 1$

$$\cos \theta = \frac{\langle \pi, c \rangle}{\sqrt{\langle \pi, \pi \rangle} \sqrt{\langle c, c \rangle}}$$

check out ↗

market price of risk:

$$\frac{\langle m, m \rangle^{1/2}}{\langle \pi, m \rangle}$$

be careful

def'n of inner product

$$\langle c, m \rangle = \sum \pi_i c_i m_i$$

$$\overbrace{\quad \quad \quad}^I \sum_{i=1}^I u(c_i) \pi_i$$

$$h^{-1} \sum_{i=1}^I \pi_i h(u(c_i))$$

extra adjustment for
e.g. risk

$$\text{e.g. } -\theta \log \sum_{i=1}^I \pi_i \exp \left(-\frac{u(c_i)}{\theta} \right) \quad \theta > 0$$

$$\cdot \quad \sum_{i=1}^I \dots \quad \cdot$$

$$\text{eq. } -\theta \log \sum_{i=1}^I \pi_i \exp \left(-\frac{u(c_i)}{\theta} \right) \quad \theta > 0$$

$$U = h^{-1} \sum_{i=1}^I \pi_i h(u(c_i))$$

$$f(U) = \sum_{i=1}^I \pi_i h(u(c_i))$$

$$h'(u) dU = \sum_i \pi_i h'(u(c_i)) u'(c_i) dc_i$$

Lect 2 ..

Sunday, January 24, 2010
8:42 PM

A preference specification.

$$U = \ln c - Bh$$

$$wh \leq c$$

h - hours
 c - consumption.

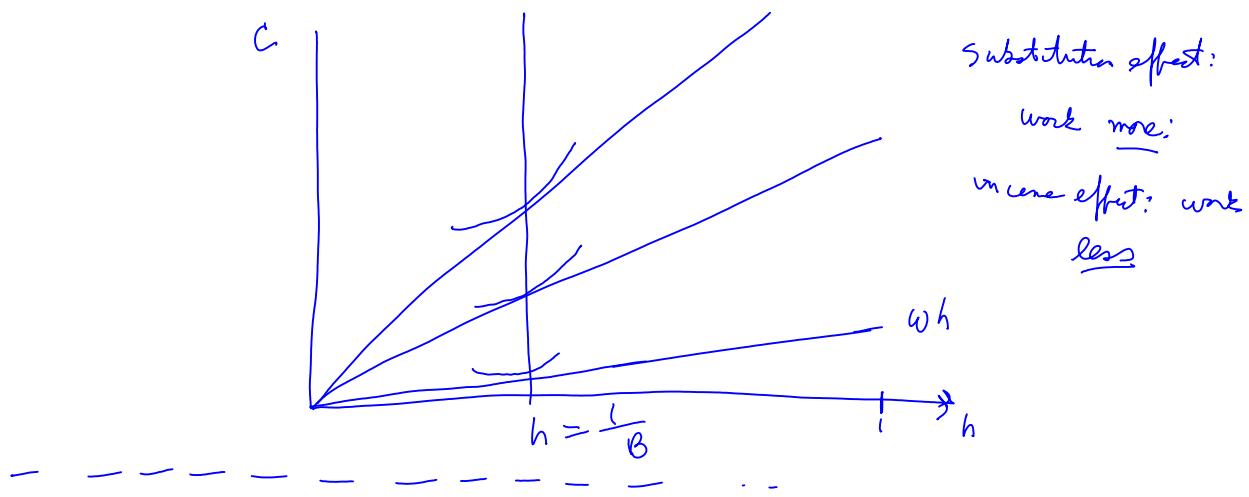
$$\max_{c,h} \ln c - Bh + \lambda [wh - c]$$

$$c: \frac{1}{c} - \lambda = 0$$

$$h: -B + \lambda w = 0 \Rightarrow h = \frac{w}{B}$$

$$c = \frac{w}{B}$$

Remark: income & substitution effects cancel:



Experiment 1: govt levies a tax γ_h on labor income

& uses λ to finance G - govt expenditures.

HH problem becomes:

$$\max_{c,h} \{ \ln c - Bh + \lambda [w(1-\gamma_h)h - c] \}$$

FONC:

$$c: \frac{1}{c} - \lambda = 0$$

$$h: -B + \lambda \omega(1-\tau_h) = 0$$

$$\Rightarrow B = \frac{1}{c} \omega(1-\tau_h)$$

$$\text{But } c = \omega(1-\tau_h)h$$

$$\Rightarrow B = \frac{1}{h} \quad \text{or} \quad h = \frac{1}{B}$$

Here the effect of the tax & spend policy is to leave h unaffected. government punches crowd out private consumer — c goes down by the increase in B .

Experiment 2: government imposes labor-tax rate τ_h

but rebates the proceeds lemp sum to the household.

HH optimum problem \Rightarrow

$$B = \frac{1}{c} \omega(1-\tau_h)$$

just as before. But now c is given by

$$c = \omega(1-\tau_h)h + T$$

where T is the rebate.

The rebate is treated as exogenous (beyond

influence) by the household.

But $T = w\tau_n h$ after all

$$\therefore C = w(1-\tau_n)h + w\tau_n h \\ = wh$$

$$\therefore B = \frac{1}{C} w(1-\tau_n) = \frac{1}{wh} w(1-\tau_n)$$
$$\Rightarrow h = \frac{(1-\tau_n)}{B}$$

So now, taxes discourage work.

Here we have isolated the substitution effect
and neutralized the income effect.

Remark: Describe Prescott's analysis.

Lect 4

Sunday, February 08, 2009
8:18 PM

Models of indivisible labor

1. employed lotteries model.

Hansen - Rogerson & Prescott

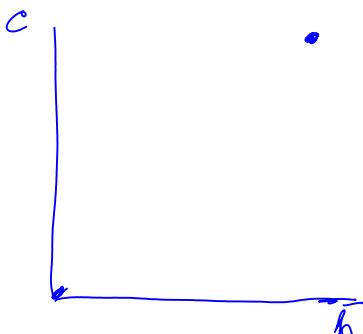
Preferences

$$u(c) = v(h)$$

$$\text{s.t. } c \leq w h \quad \text{but now}$$

$$h \in \{0, 1\}$$

↑ ↑
don't work work.



Two points are in the choice

$$\text{set: } (c, h) = (w, 1) \text{ or}$$

$$(0, 0)$$

choose to work if

$$u(w) - v(1) > u(0) - v(0)$$

reservation wage, if it exists, satisfies

$$u(\bar{w}) - v(1) = u(0) - v(0)$$

work if $w \geq \bar{w}$.

Example:

$$\ln c - Bh \quad , B > 0 \quad (B = v(1))$$

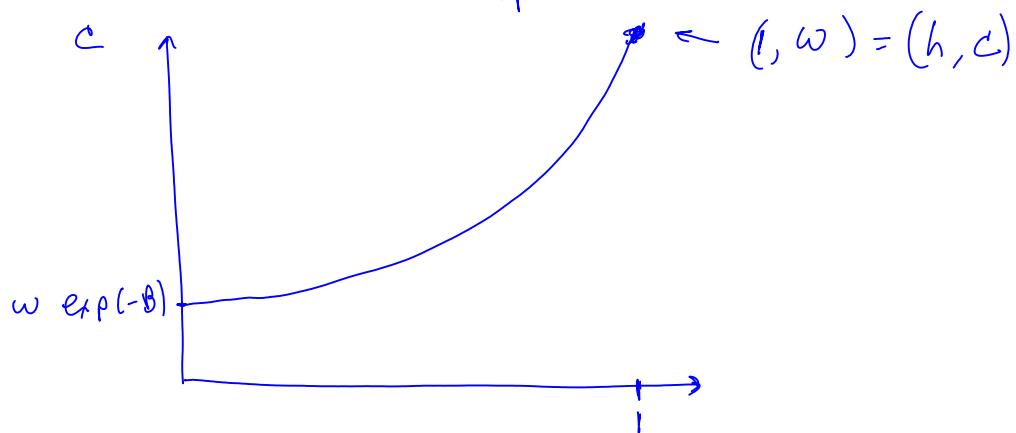
solve for indifference curve through $(c, h) = (\omega, 1)$

$$U = \ln c - B h$$

Solve for \tilde{c} the sets

$$\ln \tilde{c} - B h^{>0} = \ln \tilde{c} = \ln \omega - B$$

$$\Rightarrow \tilde{c} = \omega \exp(-B) > 0$$



Note: By himself, our worker would always work for these preferences, $\ln c - B h$.

Employment lotteries:

There is now a continuum of ex ante identical workers indexed by $j \in [0, 1]$, each of whom has identical preferences given by

$$u(c^j) - r(h^j), \quad h^j \in \{0, 1\}$$

\nearrow \uparrow

A benevolent planner chooses $\{c^j\}, \{h^j\}$ & $\bar{z} \in [0, 1]$
 to maximize

$$\int_0^1 [u(c^j) - v(h^j)] dz = \int_0^1 u(c^j) dz - v(1) \int_0^1 h^j dz$$

$$\text{s.t. } \int_0^1 c^j dz \leq w \int_0^1 h^j dz$$

let $\Phi = \int_0^1 h^j dz$ = fraction of j 's working
 Key step

Write the planning problem:

$$L = \int_0^1 u(c^j) dz - v(1) \Phi + \lambda \left[w \Phi - \int_0^1 c^j dz \right]$$

FONC:

$$c_j: u'(c^j) - \lambda = 0$$

$$\Phi: -v(1) + \lambda w = 0$$

$$\Rightarrow u'(c^j) = \lambda \quad \text{independent of } j \Rightarrow c^j = c \quad \forall j$$

$$w \Phi = c = \int_0^1 c^j dz = c \int_0^1 dz$$

$$N(l) = \lambda \omega$$

$$N(l) = u'(c)\omega$$

$$B = u'(\omega\phi)\omega$$

$$\text{or } u'(\omega\phi) = \frac{B}{\omega}$$

$$\text{in log case: } \frac{1}{\phi\omega} = \frac{B}{\omega} \Rightarrow \phi = \frac{1}{B}$$

now ϕ is fraction of people working.

This is how some people "calibrate" B .

ex post utility of people who work:

$$u(\bar{c}) - N(l) \quad \bar{c} = c^j + j.$$

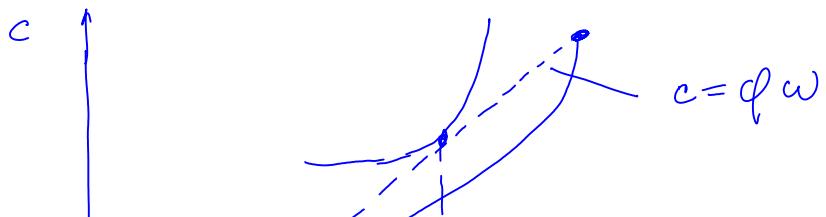
ex post utility of people who don't work

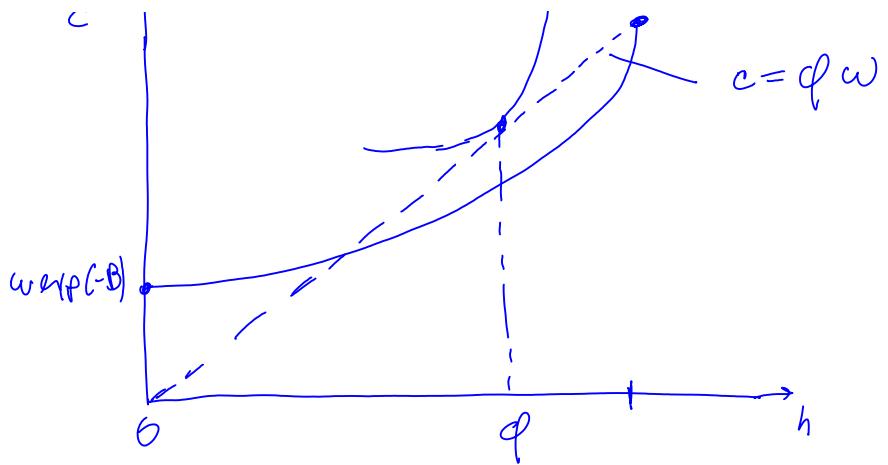
$$u(\bar{c})$$

ex ante utility is

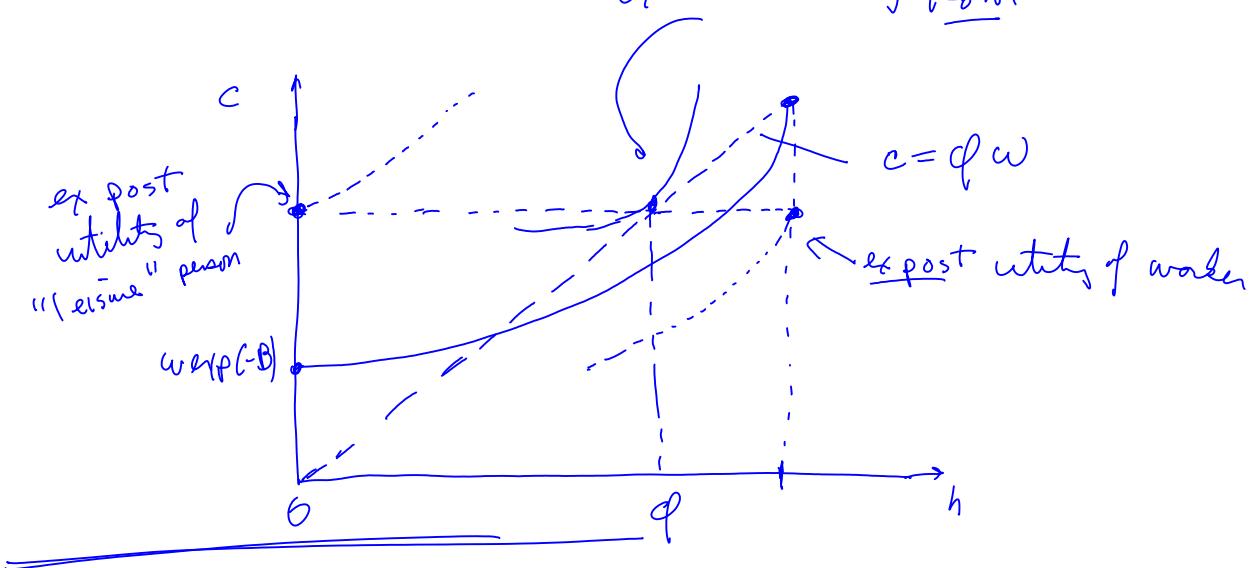
$$(1-\phi)u(\bar{c}) + \phi(u(\bar{c}) - B) = u(\bar{c}) - \phi B$$

Planner runs a lottery (ϕ) to assign people to work, gives everyone c .





ex ante utility of both.



Other way to calibrate β : - fraction of life time worked

No lotteries Instead - time averaging,

Consumer can own faces indivisibility.

inter-temporal problem

$$\max_{(c_t, h_t)} \int_0^1 e^{-\delta t} [\ln c_t - \beta h_t] dt, \quad h_t \in \{0, 1\}$$

$$\text{s.t. } \int_0^1 e^{-rt} [w h_t - c_t] dt$$

Focus on case where $\delta = r$

Let $\int_0^1 e^{-rt} h_t dt = H_t$ = present value of time worked

$$L = \int_0^1 e^{-rt} \ln c_t dt - B H_t + \lambda \left[w H_t - \int_0^1 e^{-rt} c_t dt \right]$$

FONC:

$$c_t: e^{-rt} \frac{1}{c_t} - \lambda e^{-rt} = 0 \\ \Rightarrow \frac{1}{c_t} = \lambda + t$$

\Rightarrow constant consumption through time

$$H_t: -B + \lambda w = 0$$

$$w = \frac{B}{\lambda} = B \bar{c} \quad \text{or} \quad \bar{c} = \frac{w}{B}$$

$$w \cdot H = \bar{c} \int_0^1 e^{-rt} dt$$

$$= \bar{c} \left[-\frac{1}{r} e^{-rt} \right]_0^1$$

$$= \bar{c} \left[\frac{1 - e^{-r}}{r} \right]$$

$$\Rightarrow w H = \underline{\underline{\frac{w}{r} \left[\frac{1 - e^{-r}}{r} \right]}}$$

$$\omega H = \frac{\omega}{\beta} \left[\frac{1-e^{-r}}{r} \right]$$

↑

Wage • PV of work
PV of cash

$$H = \frac{1}{\beta} \left[\frac{1-e^{-r}}{r} \right]$$

$$\lim_{r \rightarrow 0} \frac{1-e^{-r}}{r} = \lim_{r \rightarrow 0} \frac{e^{-r}}{1} = 1$$

agrees with earlier formula.

how about β ? - fraction of life time worked.

Lab 5

Saturday, February 14, 2009
4:37 PM

Consider a preference specification

$$u(c, 1-h) = \frac{c^{1-m}}{1-m} + \frac{(1-h)^{1-m}}{1-m}$$

c = Consumption

h = taxes

Remark $m \rightarrow 1 \Rightarrow \ln c + \ln(1-h)$

$$u_c = c^{-m}$$

$c = wh$ (constraint)

$$u_{1-h} = (1-h)^{-m}$$

$$\text{Problem: } \max_{c,h} \left\{ \frac{c^{1-m}}{1-m} + \frac{(1-h)^{1-m}}{1-m} \right. \\ \left. + \lambda \{ wh - c \} \right\}$$

$$\text{FONC: } c: c^{-m} - \lambda = 0$$

assume interior sol'n

$$h: - (1-h)^{-m} + \lambda w = 0$$

$$\Rightarrow - (1-h)^{-m} + c^{-m} w = 0$$

$$\Rightarrow w = \frac{(1-h)^{-m}}{c^{-m}} \quad \text{or}$$

$$\boxed{u_e = (1-h)^{-m}} = u_{1-h}$$

$$w = \frac{u_e}{u_c} = \frac{(1-h)^{-m}}{c^{-m}}$$

magical condition

+ to solve:

$$c = wh \quad \text{and} \quad w = \frac{(1-h)^{-m}}{c^{-m}}$$

from $c \neq h$,

$$(1-h)^{-m} = w c^{-m}$$

raise both sides to $-\frac{1}{m}$

$$(1-h) = w^{-\frac{1}{m}} c$$

$$\dots \quad 1 \dots \quad \dots \quad -\frac{1}{m} \quad \dots$$

$$\Rightarrow h = 1 - \omega^{-\frac{1}{m}} c \quad (+)$$

$$\text{but } c = \omega h \Rightarrow$$

$$c = \omega [1 - \omega^{-\frac{1}{m}} c]$$

$$\Rightarrow [1 + \omega^{1-\frac{1}{m}}] c = \omega$$

$$(*) \quad c = \frac{\omega}{1 + \omega^{1-\frac{1}{m}}}$$

$$(+ \Rightarrow h = 1 - \omega^{-\frac{1}{m}} \left(\frac{\omega}{1 + \omega^{1-\frac{1}{m}}} \right)^m$$

$$(\star) \quad h = 1 - \frac{\omega^{1-\frac{1}{m}}}{1 + \omega^{1-\frac{1}{m}}}$$

Note: (*) is upward sloping if $\alpha < m < 1$.

downward sloping if $m > 1$

constant if $m = 1$

To plot this & differentiate: use cobb douglas pref.m in
matlab program

Alternative "business cycle" type of specification

$$u(c, 1-n) = \frac{c^{1-\gamma}}{1-\gamma} - B n$$

$$\max \frac{c^{1-\gamma}}{1-\gamma} - B n \quad , \text{s.t.} \quad m \geq 0 \quad : M \\ 1-n \geq 0 \quad : \Theta$$

$$I = \underline{c^{1-\gamma}} - B n + \mu n + \theta (1-n) + \lambda [w_n - c]$$

$$L = \frac{c^{1-\gamma}}{1-\gamma} - Bn + \mu n + \theta(1-n) + \lambda[\omega n - c]$$

FONC:

$$c: c^{-\gamma} - \lambda = 0$$

$$n: -B + c^{-\gamma}\omega + \mu - \theta = 0$$

$$\mu n = 0, \quad \theta(1-n) = 0$$

$$c = \omega n \Rightarrow$$

$$-B + (\omega n)^{-\gamma}\omega = 0 \quad \text{at interior soln.}$$

$$\omega^{1-\gamma} \cdot n^{-\gamma} = B$$

$$\begin{aligned} \frac{\omega^{1-\gamma}}{B} &= n^\gamma \\ n &= \left(\frac{\omega}{B}\right)^{\frac{1}{1-\gamma}} \end{aligned} \quad (*)$$

Special case

$$\text{Special case: } \gamma = 1$$

$$L = \ln c - Bn + \lambda[\omega n - c] + \mu n + \theta(1-n)$$

$$\text{FONC: } c: \frac{1}{c} - \lambda = 0$$

$$n: -B + \frac{1}{c}\omega + \mu - \theta = 0, \quad \mu n = 0, \quad \theta(1-n) = 0$$

$$-B + \frac{\omega}{\mu n} + \mu - \theta = 0$$

$$\frac{1}{n} + \mu - \theta = B, \quad B = \frac{1}{n}, \quad \theta = 0, \mu = 0$$

at interior soln

If $B^{-1} > 1$, $n = 1$, i.e. if $B < 1$, $n = 1$

$$\mu = 0 \quad (\text{by logic})$$

$$\theta > 0 \text{ if } B < 1$$

$$-B + \frac{1}{n} - \theta = 0 \Rightarrow \boxed{\theta = 1-B} \quad \text{if } n = 1.$$

Redo with preferences

I ignore this section b/c "too ugly" because of polynomial.

$$u(c, 1-h) = \frac{c^{1-\varphi}}{1-\varphi} + \beta \frac{(1-h)^{1-\varphi}}{1-\varphi}$$

$$\max h = \frac{c^{1-\varphi}}{1-\varphi} + \beta \frac{(1-h)^{1-\varphi}}{1-\varphi}$$

$$+ \lambda [wh - c]$$

FONC:

$$c: c^{-\varphi} - \lambda = 0$$

$$h: -\beta (1-h)^{-\varphi} + \lambda w = 0$$

$$\Rightarrow \beta (1-h)^{-\varphi} = \omega c^{-\varphi}, \quad c = wh$$

$$\beta (1-h)^{-\varphi} = \omega (wh)^{-\varphi} = \omega^{1-\varphi} \cdot h^{-\varphi}$$

$$(1-h)^{-\varphi} = \frac{\omega}{\beta} (wh)^{-\varphi}$$

$$1-h = \left(\frac{\omega}{\beta}\right)^{\frac{1}{\varphi}} (wh)^{\frac{\varphi}{\varphi}}$$

a nasty polynomial

Remark: Box-Cox transformation:

$$y(\lambda) = \begin{cases} \frac{y^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln y & \text{if } \lambda = 0 \end{cases}$$

$$f(\lambda) = \begin{cases} \lambda & \text{if } \lambda \neq 0 \\ \log y & \text{if } \lambda = 0 \end{cases}$$

l' hospital's rule:

$$\lim_{\lambda \rightarrow 0} \frac{y^{\lambda-1}}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\lambda y^{\lambda-1}}{1} = \lim_{\lambda \rightarrow 0} \lambda y^{\lambda-1} = \log y$$

Permanent income, 4

Tuesday, February 17, 2009
4:28 PM

A person's income stream is random. In particular, at time 0, nature flips a "coin" that determines whether the consumer is "lucky" or not. The consumer's income stream is

$$y_t = \begin{cases} 0 & \text{for all } t \geq 0 \text{ if unlucky} \\ \bar{y} & \text{for all } t \geq 0 \text{ if lucky} \end{cases}$$

The probability of the consumer being "lucky" is $p \in (0, 1)$.

Consumer preferences - when outcomes are contingent on coin flip.

let $c_t(\omega) = \text{cons in "state" } \omega$

$\omega = 0$ if unlucky

$\omega = 1$ if lucky

$u' > 0, u'' < 0$

$$U = p \sum_{t=0}^{\infty} \beta^t u(c_t(1)) + (1-p) \sum_{t=0}^{\infty} \beta^t u(c_t(0))$$

\uparrow lucky \uparrow unlucky

Let $q_{bt}(1)$ = price of 1 unit of consumption
in date t , contingent on being lucky

$q_{bt}(0)$ = price of 1 unit of consⁿ; contingent
on being unlucky.

budget constraint:

$$\sum_{t=0}^{\infty} q_{bt}(1) c_t(1) + \sum_{t=1}^{\infty} q_{bt}(0) c_t(0)$$

"state contingent" prices of future consumption.

"coin flip" is realized just before time 0.

$$\leq \sum_{t=0}^{\infty} q_{bt}(1) \cdot y + \sum_{t=0}^{\infty} q_{bt}(0) \cdot 0$$

↑ unlucky

So Lagrangian is:

$$L = p \sum_{t=0}^{\infty} \beta^t u(c_t(1)) + (1-p) \sum_{t=0}^{\infty} \beta^t u(c_t(0))$$

$$+ \lambda \left[\sum_{t=0}^{\infty} q_{bt}(1) y - \sum_{t=0}^{\infty} q_{bt}(1) c_t(1) - \sum_{t=0}^{\infty} q_{bt}(0) c_t(0) \right]$$

FONC:

$$c_t(0) : (1-p)\beta^t u'(c_t(0)) - \lambda q_{bt}(0) = 0$$

$$\sim 1.1 \cdot 0.8^t u'(c_t(1)) - \lambda \cdot 1.1 = 0$$

$$c_k(1): \quad p \beta^t u'(c_k(0)) - q_t(1) = 0$$

take ratio:

$$\frac{(1-p) u'(c_k(0))}{p u'(c_k(1))} = \frac{q_t(0)}{q_t(1)}$$

ratio of probabilities • marginal utility = ratio of state contingent prices.

options

Saturday, February 21, 2009
10:35 AM

The screenshot shows a Mozilla Firefox window displaying the Wikipedia page for "Truncated normal distribution". The URL in the address bar is http://en.wikipedia.org/wiki/Truncated_normal_distribution. The page content includes sections on Definition, Moments, and See also, with mathematical formulas for the truncated normal distribution's density and variance. A blue box highlights a section about one-sided truncation, which includes the formula $E(X|a < X < b) = \mu + \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} \sigma$ and $Var(X|a < X < b) = \sigma^2 \left[1 + \frac{\frac{a-\mu}{\sigma} \phi(\frac{a-\mu}{\sigma}) - \frac{b-\mu}{\sigma} \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \left(\frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} \right)^2 \right]$. Handwritten notes in blue ink provide additional formulas: $\phi(x) = \text{normal density}$, $\Phi(x) = \text{normal c.d.f.}$, and $\Phi(x) = \int_{-\infty}^x \phi(s) ds$.

Option pricing:

Let p be the price of a stock and

assume that p is a random variable with probability density $f(p)$

A call option is the right to purchase

the stock tomorrow at strike price \bar{p} .

The value tomorrow of the call option is

$0 \text{ if } p < \bar{p}$

and $p - \bar{p} \text{ if } p > \bar{p}$

$$\sigma = \max(p - \bar{p}, 0)$$

The expected value of the option is then

$$\int_{\bar{p}}^{\infty} (p - \bar{p}) f(p) dp \\ = E(p | p \geq \bar{p})$$

Suppose p has a Gaussian or normal distribution

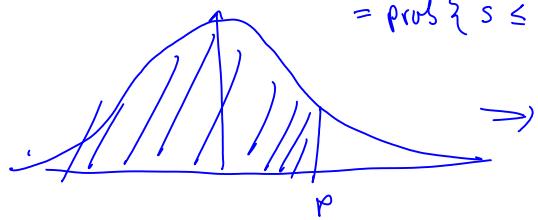
$$f(p) = \phi(p) \propto \exp\left(-\frac{1}{2} \frac{(p-\mu)^2}{\sigma^2}\right)$$

↑ ↑ ↑
p mean standard deviation
proportional to

$\phi(p)$ is the probability density

$$\Phi(p) = \int_{-\infty}^p \phi(s) ds = \text{cumulative density function (CDF)}$$

$$= \text{prob}\{s \leq p\}$$



Formula for the mean of a truncated normal :

$$E(p | p \geq \bar{p}) = \mu + \sigma \lambda(\alpha)$$

$$\alpha = \frac{(\bar{p} - \mu)}{\sigma}$$

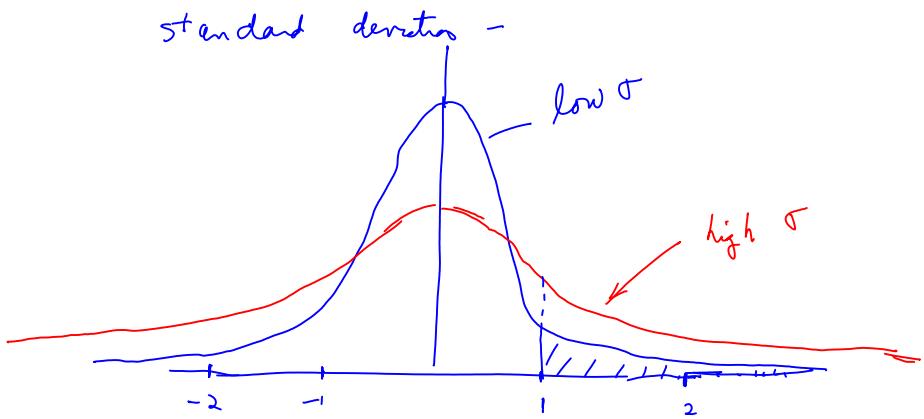
$$\lambda(\alpha) = \frac{\phi(\alpha)}{1 - \Phi(\alpha)}$$

density CDF

Example : $\mu = 1$

$$\bar{p} = 2$$

σ	$E[p p \geq \bar{p}]$
1	1.8
2	2.6
3	3.4
4	4.2
5	5.8



Finally, value of option today is

$$A = \beta \int_{\bar{p}}^{\infty} (p - \bar{p}) f(p) dp$$

Note:

$$\int_{\bar{p}}^{\infty} p \left[\frac{f(p)}{\int_{\bar{p}}^{\infty} f(s) ds} \right] dp = E[p | p \geq \bar{p}]$$

↑
(given by formula above)

density of $p | p \geq \bar{p}$ - it integrates to 1 ($\int_C f(p) dp = 1$)

$$A = \beta \left[\int_{\bar{p}}^{\infty} p f(p) dp - \bar{p} \int_{\bar{p}}^{\infty} f(p) dp \right]$$

$$= \beta \left[\int_{-\infty}^{\infty} p \left(\frac{f(p)}{\int_{\bar{p}}^{\infty} f(s) ds} \right) - \bar{p} \int_{-\infty}^{\infty} f(p) dp \right]$$

$$A = \beta \left[\int_{\bar{p}}^{\infty} p f(p) dp - \bar{p} \int_{\bar{p}}^{\infty} f(p) dp \right]$$

$$= \beta \left[\int_{\bar{p}}^{\infty} p \left(\frac{f(p)}{\int_{\bar{p}}^{\infty} f(s) ds} \right) \underbrace{\int_{\bar{p}}^{\infty} f(s) ds}_{1 - \Phi(\bar{p})} - \bar{p} \underbrace{\int_{\bar{p}}^{\infty} f(p) dp}_{1 - \Phi(\bar{p})} \right]$$

$$= \beta [E_p | p \geq \bar{p}] \cdot (1 - \Phi(\bar{p})) - \bar{p} (1 - \Phi(\bar{p}))$$

$$\boxed{A = \beta [E(p - \bar{p}) | p \geq \bar{p}] (1 - \Phi(\bar{p}))}$$

option price formula

A put option: the right to sell tomorrow at

price \underline{p} . Value of a put — today

$$\beta \int_{-\infty}^{\bar{p}} (\underline{p} - p) f(p) dp$$

↑ ↗
 you can buy it at this price
 you can sell it at this price

Sell it at
this price

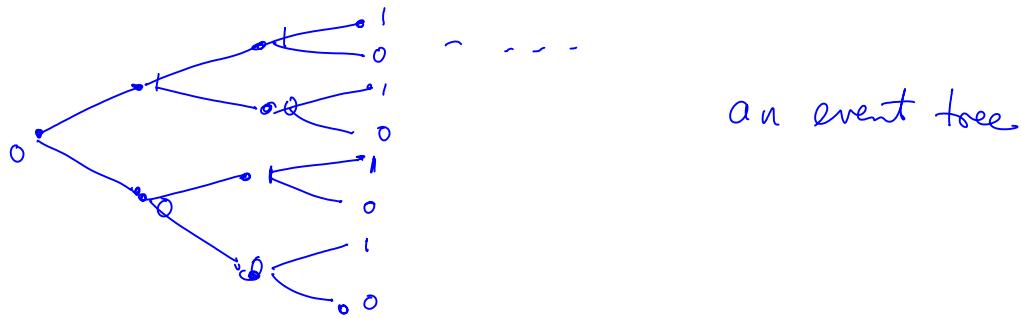
Complete markets

Saturday, March 21, 2009
6:02 PM

Asset pricing in complete markets

A state $s_t \in S$ for $t = 0, 1, 2, \dots, +\infty$

e.g. $S = \{0, 1\}$



Let $s^t = (s_t, s_{t+1}, \dots, s_1, s_0)$ be a history up to t .

s^∞ is an infinite history

note $s^t \in S \times S \times \dots \times S = S^t$

$s^\infty \in S^\infty$

Remark histories are huge objects

let $c_t(s^t)$ denote the amount a person consumes at date t contingent on history s^t .

note in general $c_t(\bar{s}^t) \neq c_t(\tilde{s}^t)$

for different histories $\bar{s}^t \neq \tilde{s}^t$

e.g. $\bar{s}_t = (0, 0, 0, 1, 1, 0)$

$$\tilde{s}_t = (1, 1, 1, 1, 1, 1) \quad (\text{different luck})$$

Let $\pi_t(s^t)$ be the probability of history s^t .

$$\sum_{s^t} \pi_t(s^t) = 1 ; \quad \pi_t(s^t) \geq 0 \quad \forall t, \forall s^t$$

probabilities are non-negative and sum to 1

To make sure that the probabilities for different t 's are consistent, we can just post a single joint probability for s^∞

$$\pi_\infty(s^\infty) \quad \text{where} \quad \pi_\infty(s^\infty) \geq 0 \quad \forall s^\infty$$

$$\text{and} \quad \sum_{s^\infty} \pi_\infty(s^\infty) = 1$$

sum over all infinite histories.

Then $\forall \bar{s}^t \in S^t$

$$\pi_{\#}(\bar{s}^t) = \sum_{s^\infty : s^t = \bar{s}^t} \pi_\infty(s^\infty)$$

\uparrow all ∞ histories such that partial history $s^t = \bar{s}^t$.

Preferences of a household over risky consumption streams:

$$\sum_{t=0}^{\infty} \sum_{s^t} \underbrace{\beta^t u(c_t(s^t))}_{\uparrow \quad \uparrow \quad \uparrow \quad \dots} \underbrace{\pi_t(s^t)}_{\uparrow \quad \dots}$$

time ↑ ↑ ↑
 all histories utility at history st ↓
 probability
of history st
 "expected utility"

Complete markets: at time 0, the household can buy or sell one unit of consumption at t contingent on history s^t at price $q_{ft}(s^t)$. (lots of prices!) Let the household have initial wealth W_0 at $t=0$. The household choose $\{c_t(s^t)\}_{t=0, s^t \in S^t}^\infty$ to maximize

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

subject to

$$(*) \quad \sum_{t=0}^{\infty} \sum_{s^t} q_{ft}(s^t) c_t(s^t) \leq W_0$$

Assume $u' > 0, u'' < 0, u'(0) = +\infty$

Form

$$\begin{aligned}
 L = & \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t) \\
 & + \lambda \left[W_0 - \sum_{t=0}^{\infty} \sum_{s^t} q_{ft}(s^t) c_t(s^t) \right]
 \end{aligned}$$

FONC:

$$c_t(s^t) : \beta^t u'(c_t(s^t)) \pi_t(s^t)$$

$$-\lambda g_t(s^t) = 0$$

\Rightarrow

$$(*) \quad \lambda g_t(s^t) = \beta^t u'(c_t(s^t)) \pi_t(s^t)$$

(*) and the budget constraint (**) are

to be solved for $c_t(s^t)$ as a function of
 $\pi_t(s^t)$, $g_t(s^t)$, and W_0 .

Arrow & Debreu & Hicks idea \uparrow

Lucas-Breeden asset pricing model
 (an extension of the midterms question!)

Consider a "representative consumer" with
 endowment in the form of income

$y_t(s^t)$ in history s^t . His initial
 wealth is evidently

$$W_0 = \sum_{t=0}^{\infty} \sum_{s^t} g_t(s^t) y_t(s^t).$$

Lucas-Breeden idea: find prices $g_t(s^t)$
 that cause the household to want not to
 trade - & to consume $y_t(s^t)$.

These prices evidently solve $y_t(s^t) = c_t(s^t)$

These prices evidently solve $y_t(s^t) = c_t(s^t)$

$$\lambda q_t(s^t) = \beta^t u'(y_t(s^t)) \pi_t(s^t)$$

we can set $\lambda =$ to any positive number, e.g. 1

(Multiplying all prices by the same positive scalar leaves budget sets and \therefore optimal consumption unchanged).

Thus, the Lucas-Breeden theory of history-contingent prices is

$$q_t(s^t) = \lambda^{-1} \beta^t u'(y_t(s^t)) \pi_t(s^t)$$

\uparrow
any positive
 $\#$

$q_t(s^t)$ are sometimes called state-price deflators

$$q_t(s^t) = \lambda^{-1} \beta^t u'(y_t(s^t)) \pi_t(s^t)$$

\nearrow \uparrow \nearrow
dis conty for rentless
risk aversion amount of good available
aft time.

for probabilities of s^t

Remark: The time t wealth of an agent endowed

with a stream $y_t(s^t)$ is

$$W_n = \sum_{s^t}^{\infty} q_t(s^t) y_t(s^t)$$

$t=0$

Suppose there are I households $i=1, \dots, I$
with preferences ordered by

$$\sum_k \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \quad u' > 0, u'' < 0$$

note: u and $\pi_t(s^t)$ are not indexed by
 i here - in principle they could be

but here, household $i=1, \dots, I$ have the
same preference ordering over
consumption plans $c_t^i(s^t)$

Then the prices evidently satisfy

$$\lambda^i q_t(s^t) = \beta^t u'(c_t^i(s^t)) \pi_t(s^t)$$

for some λ^i - Lagrange multipliers

$$i=1, \dots, I$$

where λ^i is chosen to make sure that

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) c_t^i(s^t) = W_i$$

Simple example: $I=2$, $s_t \in \{0, 1\}$

$$y_t^1(s^t) = \begin{cases} 1 & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 0 \end{cases}, t \geq 0$$

$$y_t^2(s^t) = \begin{cases} 0 & \text{if } s_t = 1 \\ 1 & \text{if } s_t = 0 \end{cases}, t \geq 0$$

$$\text{Nota: } u^1(s^t) + u^2(s^t) = 1 \quad \forall t \quad \forall s^t$$

$$\text{Note: } y_t^1(s^t) + y_t^2(s^t) = 1 \quad \forall t \quad \forall s^t$$

$$\text{Guess: } c_t^1(s^t) = \bar{c}_1 \quad \forall t \quad \forall s^t$$

$$c_t^2(s^t) = \bar{c}_2 \quad \forall t, \forall s^t$$

\Rightarrow

$$\lambda' q_t(s^t) = \beta^t u'(\bar{c}') \pi_t(s^t)$$

set $\lambda' = u'(\bar{c}')$ (a decision that scales prices)

$$q_t(s^t) = \beta^t \pi_t(s^t)$$

↑
discount factor ↘ probability

To compute

$$W^i = \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) y_t^i(s^t)$$

$$= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t y_t^i(s^t) \pi_t(s^t)$$

Suppose now that

$$\pi_t(s^t) = \pi(s_t) \pi(s_{t-1}) \cdots \pi(s_0)$$

independent, identically distributed s_t 's.

$$\text{and } \pi(1) = .5$$

$$\pi(0) = .5$$

then

$$W^i = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t y_t^i(s^t) \pi_t(s^t)$$

$$= \sum_{t=0}^{\infty} \beta^t \cdot 5 = \frac{.5}{1-\beta} \quad i=1, 2.$$

PV of expenditures:

$$\begin{aligned} & \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \bar{c}_i \pi_t(s^t) \\ &= \bar{c}_i \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) = \bar{c}_i \sum_{t=0}^{\infty} \beta^t = \frac{\bar{c}_i}{1-\beta} \end{aligned}$$

$$\text{to solve } \frac{\bar{c}_i}{1-\beta} = \frac{.5}{1-\beta} \Rightarrow \bar{c}_i = .5, i=1, 2$$

Note: here we have used different notation than in the growth model with infinitely lived agents.

Overlapping generations

Sunday, March 29, 2009
5:35 PM

The overlapping generations model:

Use to study

growth

- government debt & taxes
- social security schemes

Components: time: discrete $t=1, 2, \dots, +\infty$.

Demography: at each $t \geq 1$, there are born N

(no population growth) two period lived workers.

At $t=1$, there are alive N old people who want to consume but do not work. At each

$t \geq 1$, young agents work but consume nothing.

Old agents consume, but do no work. (This is a simplification designed to ease our computational burden while leaving in some essential forces.)

Technology: Aggregate:

$$Y_t = F(K_t, L_t) \quad (\text{constant returns to scale})$$

$$Y_t = C_t + I_t$$

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Technology: Aggregate:

$$Y_t = F(K_t, L_t) \quad (\text{constant returns to scale})$$

$$Y_t = C_t + I_t$$

$$K_{t+1} = (1 - \delta) K_t + I_t, \quad \delta \in (0, 1)$$

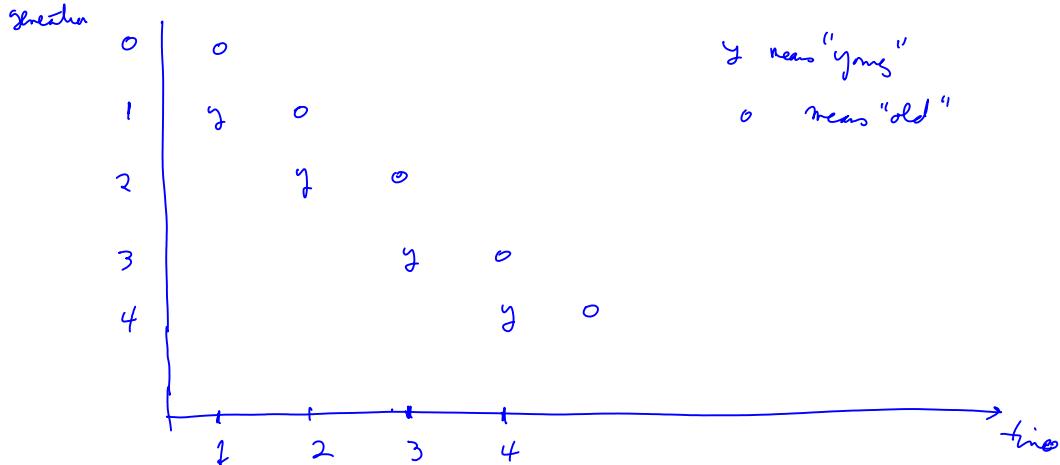
$$C_t = N \cdot c_{y_t} + N c_{o_t} \quad \underline{\text{feasibility}}$$

$$C_t = N \cdot c_{yt} + N c_{ot} \quad \underline{\text{feasibility}}$$

c_{yt} = census of a young person at t

c_{ot} = census of an old person at t .

Demography:



Demography - you are either young, or old, or not alive.

c_{ot} = census of an old person at t

c_{yt} = " " at young person at t

So, the consumption profile for a person of generation born in t

is (c_{yt}, c_{ot+1})

\uparrow
 young at t \uparrow
 old at $t+1$

"Command" economy (Solow)

$L_k = N$ (all young work)

$I_t = s F(K_t, L_t)$ s = saving rate commanded

\Rightarrow

$$\frac{K_{t+1}}{N} = \frac{K_t}{N} + s F\left(\frac{K_t}{N}, 1\right) - \delta \frac{K_t}{N}$$

or

$$k_{t+1} = (1-\delta)k_t + s f(k_t) \quad (\text{Solow model})$$

$x_{t+1} = v - w_t + u - (r_k) \quad (\text{lower bound})$

Now we'll consider a market economy

Agents → have preferences over work, consumption.

labor market (L_t, w_t)

goods market $(Y_t, C_{0t}, C_{yt}, I_t)$

capital market: r_t - net interest rate from t to $t+1$

Firms: rent labor and capital at competitive prices.

w_t - wage rate

g_t - rental rate for capital

Assume $Y_t = F(K_t, L_t) = K_t^\beta L_t^{1-\beta}$, $\beta \in (0, 1)$

β is share of capital

$(1-\beta)$ is share of labor

$$g_t = \frac{\partial F(K_t, L_t)}{\partial K_t} = \beta K_t^{\beta-1} L_t^{1-\beta}$$

$$= \beta k_t^{\beta-1}$$

so $g_t = \beta k_t^{\beta-1}$

$$w_t = \frac{\partial F(K_t, L_t)}{\partial L_t} = (1-\beta) K_t^\beta L_t^{-\beta}$$

$$= (1-\beta) \left(\frac{K_t}{L_t} \right)^\beta$$

$$= (1-\beta) k_t^\beta$$

$$\left[w_t = (1-\beta) k_t^\beta \right]$$

$$\begin{aligned}
 Y_t &= \underbrace{(\beta K_t^{\beta-1} L_t^{1-\beta})}_{\beta_t} K_t + \underbrace{((1-\beta) K_t^\beta L_t^{1-\beta})}_{w_t} L_t \\
 &= Y_t - \beta K_t^\beta L_t^{1-\beta} - (1-\beta) K_t^\beta L_t^{1-\beta} \\
 &= Y_t - \beta Y_t - (1-\beta) Y_t = 0
 \end{aligned}$$

Factor payments exhaust the product - zero profits

Simplifying assumptions:

- Agents don't value leisure
- Agents are only endowed with useful labor when young
- Agents only value consumption when old

e.g. utility fn is $u(c_{ot})$

Timing: young workers

- supply $L_t = N$,
- earn $w_t \cdot L_t$
- consume $C_{yt} = 0$ and deposit $a_{t+1} = w_t$ per worker
- deposit $N \cdot a_{t+1}$ with an "intermediary"

↘ saving at t
 ↗ earning at t
 $a_{t+1} = w_t$

old workers:

- withdraw $(1+r_{t+1}) N a_{t+1}$, ($r_{t+1} = g_{t+1} - \delta$)

[Note: it will turn out that

$$(1+r_{t+1}) N a_{t+1} = (1-\delta) K_{t+1} + \beta_{t+1} K_{t+1}$$

- consume $C_{ot+1} = (1+r_{t+1}) a_{t+1}$ ($= w_t \cdot (1+r_{t+1})$)

t...: 1 intermediate:

Financial intermediary:

at t gets deposits $N a_{t+1}$

- Uses the deposits to purchase $K_{t+1} = N a_{t+1}$

- At $t+1$ - rents K_{t+1} to firms and

- receives $\gamma_{t+1} K_{t+1}$ renting capl \leftarrow rents $\underbrace{K_{t+1}}$
- Pays out $(1-\delta) K_{t+1} + \gamma_{t+1} K_{t+1} \equiv (1+r_{t+1}) N a_{t+1}$
to depositors. so $[(1-\delta) + \gamma_{t+1}] K_{t+1} = (1+r_{t+1}) K_{t+1}$

Note: $(1-\delta) K_{t+1}$ of capital savings $\$$ is sold by the
time $(t+1)$ old to the time $(t+1)$ young; through
intermediary

Fn on specification:

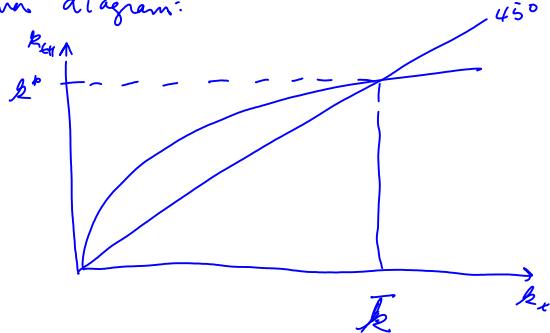
young at t earn

$$w_t N = N a_{t+1} = K_{t+1}$$

$$K_{t+1} = w_t N = (1-\beta) k_t^\beta N = K_{t+1} = N k_{t+1}$$

$$\Rightarrow k_{t+1} = (1-\beta) k_t^\beta.$$

\Rightarrow transition diagram:



$$\bar{k} = (1-\beta) \bar{k}^\beta$$

$$\bar{k}^{1-\beta} = (1-\beta)$$

$$(1-\beta) \log \bar{k} = \log (1-\beta)$$

$$v_k = \bar{v} + \beta v_{k+1}$$

$$(1-\delta) \log \bar{k} = \log(1-\beta)$$

$$\log \bar{k} = \frac{\log(1-\beta)}{1-\beta}$$

$$\text{or } \bar{k} = (1-\beta) \bar{k}^{\frac{1}{1-\beta}}$$

steady state factor payments:

$$\bar{g} = \beta \left[(1-\beta) \frac{1}{1-\beta} \right]^{\beta-1}$$

$$\bar{s} = \frac{\beta}{1-\beta}$$

$$\bar{s} = \frac{\beta}{1-\beta}$$

$$\bar{\omega} = (1-\beta) \bar{k}^\beta = (1-\beta)(1-\beta)^{\frac{\beta}{1-\beta}}$$

$$= (1-\beta)^{\frac{1-\beta}{1-\beta}} (1-\beta)^{\frac{\beta}{1-\beta}}$$

$$\bar{\omega} = (1-\beta)^{\frac{1}{1-\beta}}$$

$$\bar{\omega} = (1-\beta)^{\frac{1}{1-\beta}}$$

Please recall that $r_{t+1} = s_{t+1} - \delta$

$$\bar{r} = \frac{\beta}{1-\beta} - \delta \quad \text{may be bigger or less than 0 depending on } \beta \text{ & } \delta.$$

Budget constraint in the above problem:

$$c_{yt} + a_{t+1} = \omega_t$$

$$c_{yt+1} = (1+r_{t+1}) a_{t+1}$$

$$\text{we imposed: } c_{yt} = 0 \Rightarrow$$

$$c_{yt+1} = (1+r_{t+1}) a_{t+1} = (1+r_{t+1}) \omega_t$$

=

$$a_{t+1} = k_{t+1} \quad \text{all saving is in form of capital.}$$

return on capital:

$$(1-\delta) k_{t+1} + \underbrace{r_{t+1} k_{t+1}}_{\text{return on capital}}$$

amount left over after production

$$(1+r_{t+1}) = (1-\delta) + r_{t+1}$$

$$(1+r_{t+1}) = (1-\delta) + g_{t+1}$$

$$\Rightarrow r_{t+1} = g_{t+1} - \delta$$

Feasibility: $k_{t+1} = (1-\delta)k_t + g_t$

$$c_{0t} + c_{yt} + g_t = f(k_t)$$

$$\text{or } \quad \text{or } \quad \text{or } \quad + (k_{t+1} - (1-\delta)k_t) = k_t^\beta$$

* $c_{0t} + (k_{t+1} - (1-\delta)k_t) = k_t^\beta$ feasibility

$$c_{0t} = k_t(1+\varsigma_t) = k_t(1-\delta + \beta k_t^{\beta-1}) \text{ budget constraint}$$

$$(w_{t-1} \xrightarrow{\text{carried over from last period}}) \quad (w_{t-1} = (1-\beta)k_{t-1}^\beta)$$

$$\Rightarrow k_t(1-\delta + \beta k_t^{\beta-1}) + (k_{t+1} - (1-\delta)k_t) = k_t^\beta$$

$$\beta k_t^\beta + k_{t+1} = k_t^\beta$$

$$\boxed{k_{t+1} = (1-\beta)k_t^\beta}$$

✓ *Fit together.*

Now do a social security scheme:

Tax each young agent e

Transfer each old agent e

Budget constraints::

$$c_{yt} + a_{t+1} = w_t - e$$

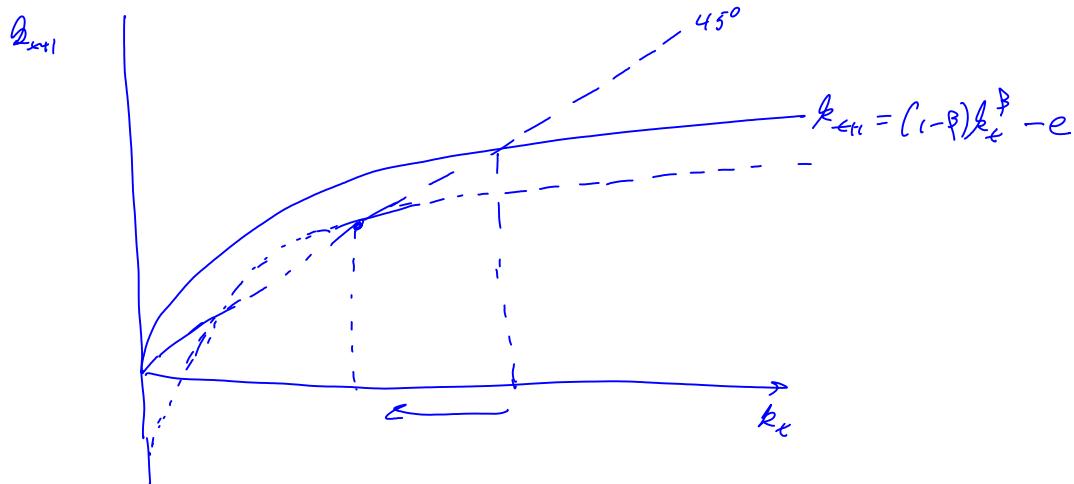
$$c_{0t+1} = a_{t+1}(1+r_{t+1}) + e.$$

Same preferences as before:

Same preferences as before:

$$a_{t+1} = k_{t+1} = (1-\beta)k_t^\beta - e$$

"
Savings



Who gains, who loses? It depends.

- initial time old gain always - because they get transfer of e when old.
- young to generations born at $t \geq 0$ lose e at time t and gain e at time $t+1$. The present value of the "gain" at t is:

$$-e + \frac{e}{1+r_{t+1}}$$

gain if $\frac{1}{1+r_{t+1}} > 1$ or $r_{t+1} < 0$

otherwise, a loss.

Key equation:

$$(w_t - e) = \underbrace{(1-\beta)k_t^\beta}_{\text{wage}} - e = k_{t+1}$$

↑
falls with
 k

- wage fall
- e rises (taxes ↑) \Rightarrow less capital accumulation

- Wage fall
- e rises (firms \uparrow) \Rightarrow less capital accumulation

$$c_{0,t} = e + (\tau_t + 1) k_t \quad \text{at } t=0, k_0 \text{ unchanged}$$

$\Rightarrow c_{0,t} \uparrow \text{ at } t=0.$

but

$$c_{0,1} = e + (\tau_1 + 1) \left[\underbrace{(1-\beta) k_0^\beta - e}_{\text{in}} \right]$$

$$= (\tau_1 + 1) (1-\beta) k_0^\beta - \tau_1 e$$

note key role of sign of τ_1

Summary of dynamic equations:

$$k_{t+1} = (1-\beta) k_t^\beta - e = w_t - e$$

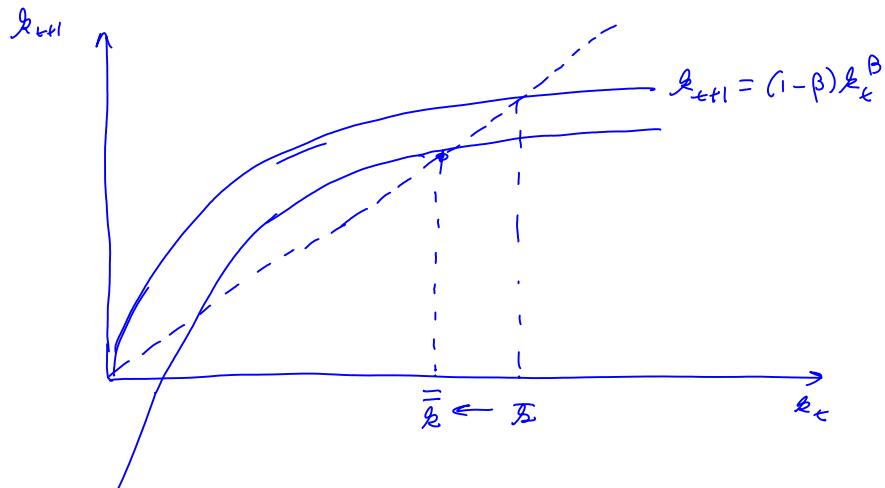
$$\rho_t = \beta k_t^{\beta-1}$$

$$w_t = (1-\beta) k_t^\beta$$

$$\tau_t = \rho_t - \delta$$

$$c_{0,t+1} = (1 + \tau_{t+1}) k_{t+1} + e = (1 + \tau_{t+1}) [w_t - e] + e$$

start from steady state with $e=0$ and
trace what happens to $c_{0,t+1}$ over time.



Starting from \bar{z}_k , $z_{k+1} \downarrow \bar{z} < \bar{z}$

what happens to $c_{0,t}$ compared to the no transfer $e=0$ case? Which generations gain? Which, if any, lose?

The answers depend on the value of $\bar{r} = \bar{\gamma} - \delta$ in the initial steady state, which in turn depends on (β, δ) , as well as on the size of the transfer.

The following propositions are true:

Proposition 1: Suppose $\bar{r} < 0$ in the $e=0$ steady state. Then there exists a positive social security transfer $e > 0$ that makes every generation better off, i.e. $c_{0,t}$ increases for all $t \geq 0$. Further, by driving \bar{z} downward, the social security system increases the eventual steady state \bar{r} .

Proposition 2: Suppose that (β, δ) are such that $r > 0$ in an initial steady state with $e=0$. Then a social security transfer $e > 0$ makes the initial old generation better off by increasing $c_{0,0}$ ($c_{0,t}$ at $t = 1$ has not yet been born) and subsequent

$t = 0$), but makes all subsequent generations worse off.

Remark: The little matlab program olg.m can be used to illustrate these propositions.

Remark: A steady state equilibrium in which $\bar{r} < 0$ is said to be dynamically inefficient.

One with $\bar{r} > 0$ is said to be dynamically efficient.

Summary lecture

Wednesday, April 29, 2009
9:29 AM

Last lecture in class we showed

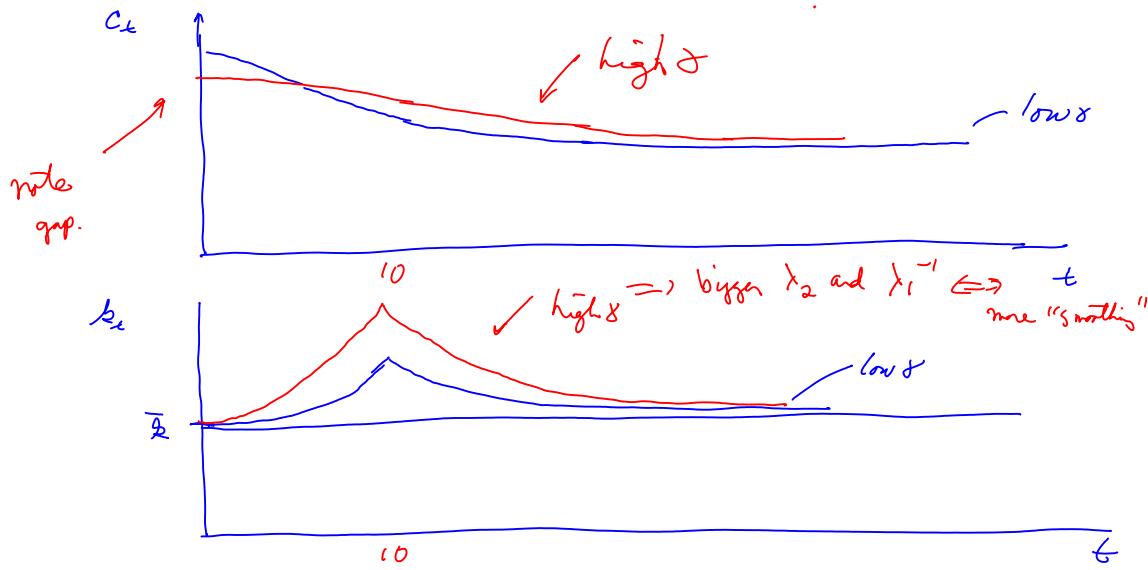
$$k_{t+1} = \lambda_2 k_t - \lambda_1^{-1} \phi^{-1} \sum_{j=0}^{\infty} (\lambda_1)^{-j} [A_0 + A_1 z_{t+j} + A_2 z_{t+j+1}]$$

λ_2 and λ_1 are functions of a number of parameters including γ .

λ_1^{-1} & λ_2 increase when γ increases.

Two economies - everything the same except γ

high γ & low γ economy



response to a γ increase \uparrow in γ at $t=10$
for a high & low γ economy

A preference for more smoothness in $c_t \rightarrow$ higher γ — surfaces in higher value of λ from last lecture.

Asset pricing:

q_t^0 lets us price anything - e.g. $\{d_e\}_{t=0}^\infty$

$$a_0 = \sum_{t=0}^{\infty} q_t^0 d_t$$

for much of course, we exogenously set q_t^0 , e.g.

$q_t = \beta^t$. Later we came up with a

$$\text{theory } q_t = \beta^t u'(c_t) \frac{1}{1 + \gamma_{ct}}$$

→
"consumption theory of asset pricing"

To add risk - $s_e \in S$, s^t , $T_k(s^t) \Rightarrow q_t^0(s^t)$

$$a_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$$

$$\text{e.g. } q_t^0(s^t) = \beta^t u'(c_e(s^t)) T_k(s^t)$$

• term structure of interest rates a special case

Asset pricing examples from early in course:

$$p_t = \beta p_{t+1} + d_{t+1}$$

$$d_t = G x_t$$

$$x_{t+1} = A x_t \Rightarrow$$

$$p_t = G (I - \beta A)^{-1} x_t + \text{bubble term}$$

Random version

$$x_{t+1} = A x_t + C \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, I)$$

$$p_t = \beta E_t p_{t+1} + d_{t+1}$$

Solve Bellman equation $p_t = H x_t = (\text{value function})$

$$H x_t = \beta E_t H x_{t+1} + G x_t \Rightarrow$$

$$H = G (I - \beta A)^{-1}$$

This is an example in which

$$g^0_t = \beta^t \pi_t(\varepsilon^t)$$

labor income as an asset:

Friedman model with stochastic labor income

$$x_{t+1} = A x_t + C \varepsilon_{t+1}$$

$$y_t = G x_t$$

$$F_{t+1} = \beta^{-1} [F_t + y_t - c_t]$$

F_t = financial assets

y_t = labor income

c_t = consumption

$$c_t = (1-\beta) [F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}]$$



"human wealth"

approximation is exact when

$$(a) \quad q_t^0 = \beta^t, \quad \text{and}$$

$$(b) \quad u(c_t) \text{ is quadratic} \Rightarrow u'(c_t) \text{ is linear in } c_t$$

we deduced

$$c_{t+1} - c_t = (1-\beta) \sum_{j=0}^{\infty} \beta^j (E_{t+j}, y_{t+j+1} - E_t y_{t+j})$$

innovation to present value of labor
income

\Rightarrow

$$c_{t+1} - c_t = (1-\beta) G(I - \beta A)^{-1} C \varepsilon_{t+1} \equiv M \varepsilon_{t+1} \equiv \varepsilon_{t+1}^c$$

consumption is a random walk

$$\varepsilon_{t+1}^c = M \varepsilon_t$$

$$E_t (c_{t+j} - c_t)^2 = \text{var}_t (\varepsilon_{t+j} + \varepsilon_{t+j-1} + \dots + \varepsilon_{t+1})$$

$$= j \sigma_e^2$$

\Rightarrow variance increases linearly with horizon j .

It put i subscripts on c_t^i, f_t^i, y_t^i

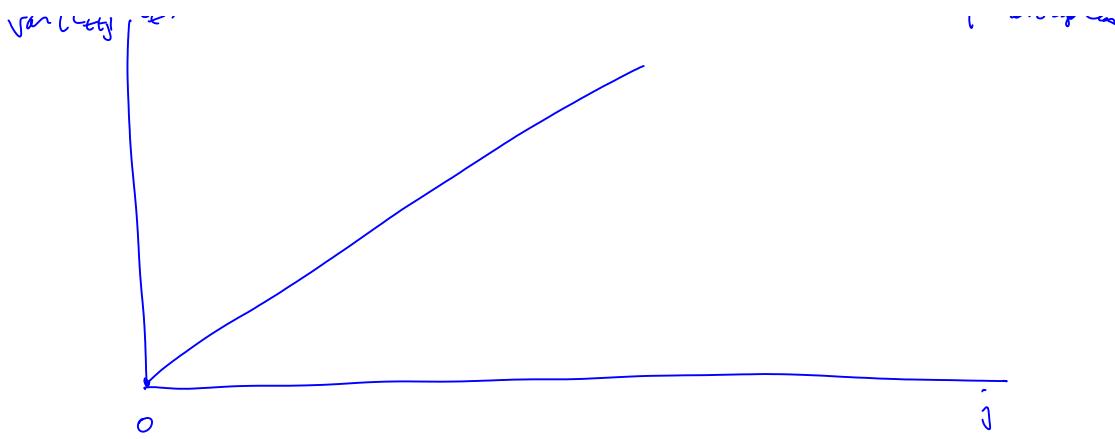
this becomes a theory of a spreading out

cross section distribution of consumption & wealth (R_t^i)

as j increases ; $\text{var}(c_{t+j}^i - c_t^i) = \text{cross section variance}$

$$\text{var}(c_{t+j}^i - c_t^i)$$

of consumption



Use as the basis of a theory to explain Doctor-Paxton

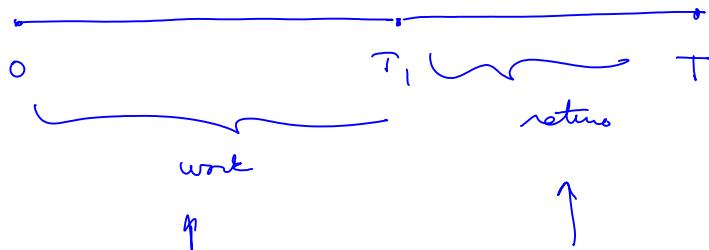
↳ Kaplan - Violante observations on

Consumption & wealth inequality

\bar{T}_t fit the data - change to a finite horizon model

$$E_t \sum_{t=0}^T \beta^t u(c_t)$$

$$F_{t+1} = f^{-1} [F_t + y_t - c_t]$$

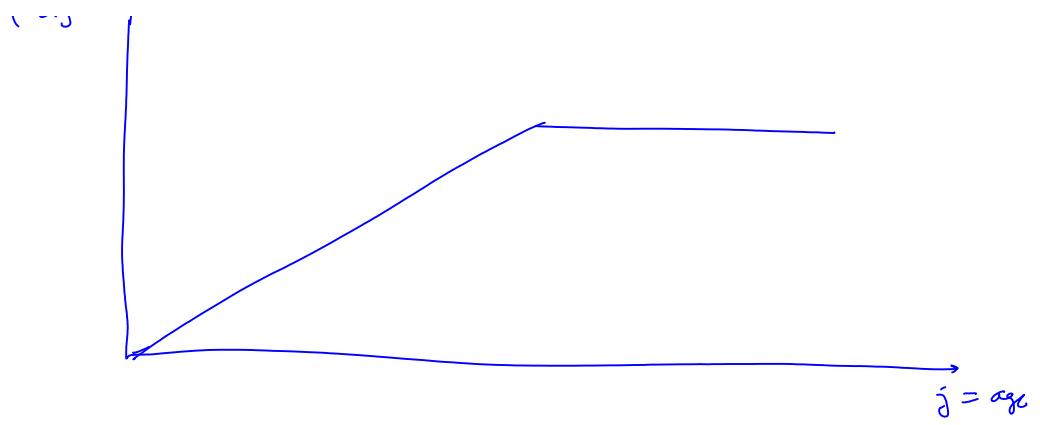


$$x_{t+1} = Ax_t + C\varepsilon_t$$

$$y_t = 0$$

$$y_t = Gx_t$$

$$\text{var}(c_{t+1}^i - c_t^j) \eta$$



Cross section variance of consumption π 's until date of
retirement.

Homework 1

Tuesday, February 10, 2009
11:10 AM

Homework 1

Instructions: Feel free to work in groups.

Please write up your own homework even if you have worked in a group.

Due: Feb 18

1. Consider a dividend stream described by the difference equation

$$d_{t+1} = \bar{d} + g_1 d_t \quad \text{for } t \geq 0$$

where \bar{d} is a constant, d_0 is given,
and $t=0, 1, 2, \dots$

At time t , a stock offering a perpetual claim to the stream d_t is valued at price p_t where p_t satisfies

$$p_t = \beta p_{t+1} + d_t \quad , \quad t \geq 0 .$$

- a. Find a formula for p_t expressed in terms of d_t, d_{t-1}, d_{t-2} and possibly calendar time t .

- b. Does your p_t formula display a "bubble"?
 Explain.
2. A worker's labor income is given by

$$y_{t+1} = \delta y_t, \quad \delta > 1$$

for $t = 0, 1, \dots, +\infty$

The worker's consumption c_t satisfies

$$c_{t+1} = (\beta R) c_t, \quad c_0 \text{ to be determined}$$

where $\beta < 1$, $R > 1$ and $(\beta R) < 1$.

Assume that $\beta \delta < 1$

The worker's consumption path must satisfy the intertemporal budget constraint

$$(*) \quad \sum_{t=0}^{\infty} R^{-t} (c_t - y_t) = 0.$$

a. Solve equation (*) for c_0 as a function of y_0 .

b. Now assume that

$$y_{t+1} = \delta_{11} y_t + \delta_{12} z_t$$

$$z_{t+1} = \delta_{21} y_t + \delta_{22} z_t$$

show how to solve for c in a function

Show how to solve for c_0 as a function of (y_0, z_0) .

3. The unemployment rate at month t , u_t , satisfies the difference equation.

$$(*) \quad u_{t+1} = \alpha (1 - u_t) + (1 - \varphi) u_t$$

where $\alpha = \text{rate of } \underline{\text{inflow}} \text{ into unemployment}$

$\varphi = \text{rate of } \underline{\text{exit}} \text{ from unemployment.}$

Definition: A steady state unemployment rate \bar{u} satisfies

$$(*) \text{ with } \bar{u} = u_{t+1} = u_t.$$

a. Compute a formula for the steady state unemployment rate \bar{u} as a function of α and φ .

b. In 2007 in the U.S., $\alpha = .0127$

and $\varphi = .2579$. Compute the implied steady state unemployment rate.

c. Suppose that in a recession year 20xx,

$\varphi = .15$ and $\alpha = .0175$. These are monthly rates.

Compute the steady state unemployment rate

in 20xx.

in year 2xxx.

d. Extra credit. Compute the average duration of a job in 2xxx. Compute the average duration of unemployment in 2xxx.

4. Answer question 7 on page 179 of Williamson.

After answering the question, say what would change if household preferences were to take the special form

$$u(c, l-h) = \ln c - \beta h , \quad \beta > 0 .$$

Homework 2

Wednesday, February 18, 2009
3:03 PM

Homework 2: due March 4.

1. An economy consists of two consumers, $i=1, 2$, each of whom orders consumption streams according to the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad \text{when } 0 < \beta < 1 \text{ and} \\ u' > 0 \text{ and } u'' < 0$$

each person has intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i .$$

Consumer 1 has endowment stream

$$y_t^1 = \{1, 1, 0, 1, 1, 0, 1, 1, 0, \dots\}$$

while consumer 2 has stream

$$y_t^2 = \{0, 0, 1, 0, 0, 1, 0, 0, 1, \dots\}$$

a. Find the optimal consumption paths

$$\{c_t^i\}_{t=0}^{\infty} \text{ for } i=1, 2.$$

b. Compute $c_t^1 + c_t^2$ for $t=0, 1, 2, \dots$

c. For $t = 0, 1, 2, \dots$, please compute

A'_{t+1} where

$$A'_{t+1} = \beta^{-1} [A'_t + y'_t - c'_t], \quad A'_0 = 0$$

d. For $t = 0, 1, 2, \dots$, please compute

A''_{t+1} where

$$A''_{t+1} = \beta^{-1} [A''_t + y''_t - c''_t].$$

2. Consider an economy consisting of two people, each of whom orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i), \quad i=1, 2, \quad u' > 0, \quad u'' < 0$$

$$0 < \beta < 1$$

Person 1 has income stream

$$y_t^1 = \delta^t, \quad t \geq 0, \quad \delta \in (0, 1).$$

Person 2 has income stream

$$y_t^2 = 1 - \delta^t, \quad t \geq 0, \quad \delta \in (0, 1).$$

Each person has the intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t [c_t^i - y_t^i] = 0$$

- a. Find the optimal consumption paths
 $\{c_k^i\}_{t=0}^{\infty}$ for $i=1, 2$.
- b. Compute $c_k^1 + c_k^2$ for $t=0, 1, 2, \dots$

c. For $t=0, 1, 2, \dots$, please compute

$$A_{t+1}^1 \text{ where}$$

$$A_{t+1}^1 = \beta^{-1} [A_t^1 + y_t^1 - c_t^1], \quad A_0^1 = 0.$$

d. For $t=0, 1, 2, \dots$, please compute

$$A_{t+1}^2 \text{ where}$$

$$A_{t+1}^2 = \beta^{-1} [A_t^2 + y_t^2 - c_t^2].$$

e. Given β with $0 < \beta < 1$, please find a value of δ that makes consumers 1 and 2 consume identical amounts for $t=0, 1, 2, \dots$.

e. Assume the same two consumers with the same income processes mentioned above. But now assume that each consumer has the sequence of budget constraints

$$A_{t+1}^i = \beta^{-1} [A_t^i + y_t^i - c_t^i],$$

where $A_0^i = 0$ and

where $A_0^i = 0$ and

$$(*) \quad A_{t+1}^i \geq 0 \quad \text{for } t=0, 1, \dots$$

Here (*) is a no-borrowing constraint.

Please find the optimal consumption path

$$\{c_t^i\}_{t=0}^{\infty} \quad \text{for } i=1, 2.$$

3. A stock price obeys the difference equation

$$p_t = \beta E_t p_{t+1} + d_t$$

where $\beta \in (0, 1)$, p_t is the stock price today

$E_t p_{t+1}$ is the best forecast of the price tomorrow,
and d_t is the dividend process; d_t obeys the
random difference equation

$$(*) \quad d_{t+1} = \bar{d} + \varphi_1 d_t + \varphi_2 d_{t-1} + c \varepsilon_{t+1},$$

d_0, d_{-1} given, and

where $\{\varepsilon_t\}_{t=0}^{\infty}$ is an independently and identically
distributed sequence of normal random variables

with mean 0 ($E_t \varepsilon_{t+1} = 0$) and

standard deviation 1 ($\sqrt{E_t (\varepsilon_{t+1} - 0)^2} = 1$).

a. Find a fundamental solution of (*) of the form

$p_t = f(d_t, d_{t+1})$, giving an explicit expression for the function f .

b. Find a solution of (*) that displays a bubble.

c. A government wants to minimize the following measure of tax distortions:

$$\sum_{t=0}^{\infty} \beta^t D(T_t)$$

where $D(T_t)$ is a measure of the costs of distortions at date t , T_t are total tax revenues, and $D' > 0$, $D'' > 0$.

The government has an exogenous stream of expenditures $\{G_t\}_{t=0}^{\infty}$ and faces the sequence of government budget constraints

$$B_{t+1} = R [B_t + G_t - T_t] , B_0 = 0$$

where B_{t+1} is government debt issued at t and due at $t+1$. The government can

borrow or lend. $R = 1 + r > 1$
 is the gross rate of return on government
 debt. Assume that $R = \beta^{-1}$.

Assume that $\lim_{T \rightarrow \infty} \beta^t B_{t+1} = 0$ as

a condition to rule out "Ponzi schemes"

a. Consider the government expenditure process

$$G_t = \begin{cases} 1 & t \text{ even} \\ 0 & t \text{ odd} \end{cases}$$

Find the optimal setting for taxes $\{T_t\}_{t=0}^{\infty}$ for
 $t = 0, 1, 2, \dots$

b. Consider the expenditure process

$$G_t = \begin{cases} 1 & t \neq 10j, j = 0, 1, 2, \dots; t = 0, 1, 2, \dots \\ 10 & t = 10j, j = 0, 1, 2, \dots \end{cases}$$

here a "war" happens in periods 0, 10, 20, ...
 and "peace" occurs the rest of the time.

Find the optimal level of $\{T_t\}_{t=0}^{\infty}$.

c. Does the analysis in this problem remind you
 of any other problem or model you have seen?

If so, explain.

Homework 3

Tuesday, March 24, 2009
2:51 PM

Homework 3 :

Due Monday April 6.

- Each period, a previously unemployed worker draws two offers to work forever from the c.d.f.

$$F(w) = \sqrt{\frac{w}{B}} , \quad 0 \leq w \leq B .$$

Successive draws within a period and across periods are identically and independently distributed.

The offers are to work forever. There is no quitting, no firing, and no recall of offers from past periods.

The worker wants to maximize

$$E_{-1} \sum_{t=0}^{\infty} \beta^t y_t , \quad 0 < \beta < 1$$

where $y_t = \begin{cases} w & \text{if employed at wage } w \\ c & \text{if unemployed} \end{cases}$

where w is the wage and c is unemployment compensation, and E_{-1} is an expected value (mean) before the two offers are drawn in period 0.

- Verify that $F(w) = \sqrt{\frac{w}{B}}$, $0 \leq w \leq B$ is a legitimate c.d.f. (cumulative distribution function).

c. d.f. (cumulative distribution function).

- b. Argue rigorously that the worker's optimal strategy
'has a reservation wage strategy':

accept \tilde{w} if $\tilde{w} > \bar{w}$

reject \tilde{w} if $\tilde{w} < \bar{w}$

where $\tilde{w} = \max(w_1, w_2) =$ maximum of two offers w_1, w_2
drawn this period. (i.e. find a Bellman equation
in an optimal value function).

- c. Find a formula for \bar{w} as a function of

(β, f, c) . Hint: you will have to compute a definite
integral.

- d. Find a formula for

$\psi \equiv$ Probability of leaving unemployment this
period

as a function of (β, f, c) .

- e. Give the value of \bar{w} when

$\beta = .9$, $B = 10$, $C = 0$. Find the value of ψ .

- f. Give the value of \bar{w} when $\beta = .9$, $B = 10$,

$c = 6$. Find the value of ψ .

2. Consider Solow's growth model in the
special case $\delta = 1$ and $f(k) = zk^\alpha$, $\alpha \in (0, 1)$.

- a. Find a difference equation expressing $\log k_{t+1}$ as a function of $\log k_t$.
- b. Find the steady state value of $\log k_t$.
- c. Argue that the steady state is "stable" in the sense that
- $$\lim_{t \rightarrow \infty} \log k_t = \log \bar{k}$$
- starting from any initial conditions $\log k_0 > -\infty$.
3. Please answer question 10 on page 226 of Williamson, 3rd edition.
4. Please answer question 11 on page 226 of Williamson, 3rd edition.

Note: the following problem is challenging.

5. Each period, a previously unemployed worker draws one and only one offer to work from a known c.d.f. There is no quitting, no firing, and no recall of offers

no quitting, no firing, and no recall of offers drawn in previous periods. The worker wants to maximise

$$E_{-1} \sum_{t=0}^{\infty} \beta^t y_t, \quad 0 < \beta < 1$$

$$y_t = \begin{cases} w & \text{if employed at } t \\ c & \text{if unemployed at } t \end{cases}$$

Here c is unemployment compensation and w is the wage.

For each t odd (i.e., $t=1, 3, 5, \dots$)

the worker draws one offer from a c.d.f.

$$F_o(w), \quad F_o(0) = 0, \quad F_o(B) = 1$$

For each t even (i.e., $t=0, 2, 4, \dots$)

the worker draws one offer from a c.d.f.

$$F_e(w), \quad F_e(0) = 0, \quad F_e(B) = 1.$$

Successive draws are independent, but not identically distributed.

a. Describe how to solve the worker's problem

using "dynamic programming."

b. Find the form of the worker's optimal

strategy. Is it of the "reservation wage" form? Explain.

Hint: Let $N_0(w)$ be the optimal value of the problem for a previously unemployed worker who has just drawn wage w in an odd period. Let ...

Practice problems 1

Friday, February 26, 2010
9:49 AM

Here are some practice problems. They are not assigned but are designed to help you learn.

1. A consumer has multi-period utility function

$$(1) \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

$$\text{where } u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}, \quad \gamma \geq 0$$

The consumer chooses $\{c_t, A_{t+1}\}_{t=0}^{\infty}$ to

maximize (1) subject to $c_t \geq 0$

$$A_{t+1} = R_t [A_t + y_t - c_t]$$

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) A_{T+1} = 0, \quad A_0 = 0$$

where R_t is the gross interest rate on financial assets between t and $t+1$,

$y_t \geq 0$ is the consumer's labor income at t ,

and A_t is the consumer's financial assets at the beginning at period t .

a. Verify that $u' > 0, u'' < 0$.

b. It just happens that $\{R_t\}_{t=0}^{\infty}$ is such that

The consumer want to set $c_t = y_t \quad \forall t \geq 0$

and $A_{t+1} = 0 \quad \forall t \geq 0$; where

$$y_t = \begin{cases} y_0 \delta^t, & t = 0, \dots, T \\ y_0 \delta^T \phi^{t-T}, & t = T+1, \dots, \infty \end{cases}$$

Find a formula for R_t at each $t \geq 0$.

Justify your formula.

Interpret your formula for R_t in terms

of (i) the consumer's impatience, and (ii)

the consumer's income growth.

2. A consumer has preferences ordered by

$$(1) \quad \sum_{t=0}^T \beta^t c_t, \quad 0 < \beta < 1.$$

She maximizes (1) subject to

$$A_{t+1} = R [A_t - c_t], \quad c_t \geq 0, \quad A_{T+1} \geq 0$$

where $A_0 > 0$ and $R > 0$ is a constant gross rate of interest on financial assets. The consumer can borrow or lend.

Find the consumer's optimal plan for $\{c_t, A_{t+1}\}_{t=0}^T$.

Long Hint: View this as an exercise in using

the Kuhn-Tucker conditions.

Form the Lagrangian

$$L = \sum_{t=0}^T \beta^t \left\{ c_t + \lambda_t [R(A_t - c_t) - A_{t+1}] \right\}$$

where $\lambda_t \geq 0$ is a Lagrange multiplier.

FONC:

$$c_t: \beta^t [1 - \lambda_t R] \leq 0, = 0 \text{ if } c_t > 0, \quad (\star) \\ \text{because } c_t \geq 0 \quad t=0, \dots, T$$

$$A_t: \lambda_t R - \beta^{-1} \lambda_{t-1} = 0, \quad t=1, \dots, T \quad (\star) \\ \text{because } A_t \text{ can be } + \text{ or } -$$

$$A_{T+1}: -\beta^T \lambda_T \leq 0, = 0 \text{ if } A_{T+1} > 0$$

$$\Rightarrow \beta^T \lambda_T A_{T+1} = 0$$

$$\Rightarrow A_{T+1} = 0 \text{ if } \lambda_T > 0.$$

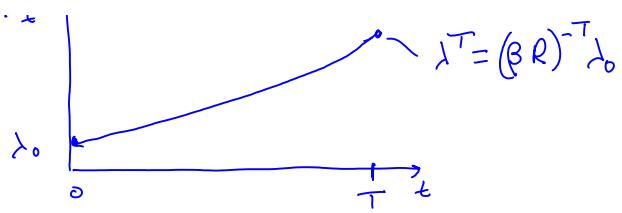
$$(\star) \Rightarrow \lambda_t = \beta R \lambda_{t+1}, \quad t=0, \dots, T-1$$

$$\text{or } \lambda_{t+1} = (\beta R)^{-1} \lambda_t \quad (\star\star)$$

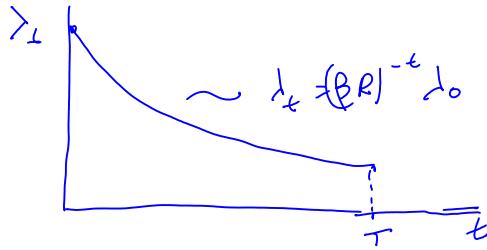
Three cases:

$$(i) \beta R < 1 \Rightarrow (\beta R)^{-1} > 1$$

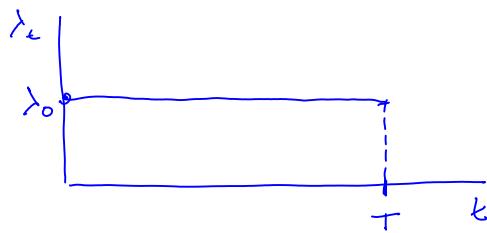
$$\lambda_{t+1} > \lambda_t$$



$$(ii) \quad \beta R > 1 \Rightarrow (\beta R)^{-t} < 1 \Rightarrow$$



$$(iii) \quad \beta R = 1 \Rightarrow (\beta R)^{-t} = 1 \Rightarrow \lambda_t = \lambda_0$$



Now return to (**) \Rightarrow

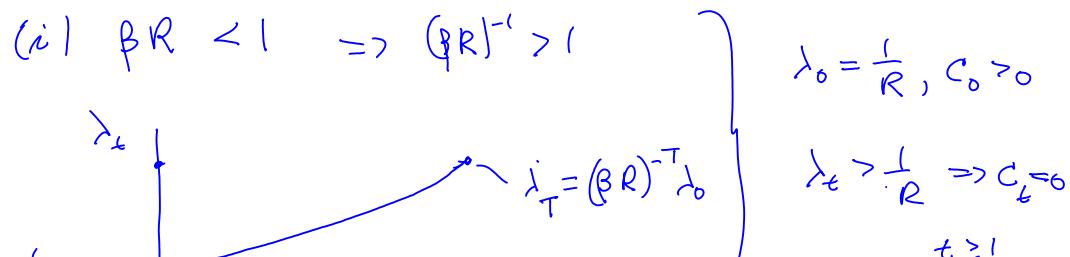
$$1 - \lambda_t R \leq 0, = 0 \text{ if } c_t > 0.$$

$$1 \leq \lambda_t R, = 0 \text{ if } c_t > 0$$

or

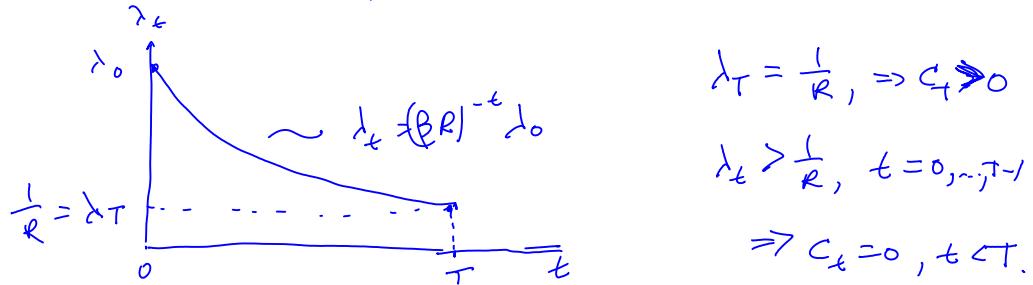
$$\frac{1}{R} \leq \lambda_t, = 0 \text{ if } c_t > 0.$$

Evidently, the figures for case (i) and (ii) are



$$\frac{1}{R} = \lambda_0 \quad \begin{array}{c} \nearrow \\ \text{---} \\ \searrow \end{array} \quad \begin{array}{c} \nearrow \\ \text{---} \\ \searrow \end{array} \quad \begin{array}{c} \nearrow \\ \text{---} \\ \searrow \end{array}$$

$$(ii) \quad \beta R > 1 \Rightarrow (\beta R)^{-1} < 1 \Rightarrow$$



So when $(\beta R) < 1$, the consumer sets $C_0 = A_0$,

$A_{t+1} = 0, t = 0, \dots, T$, and $C_t = 0$ for $t = 1, \dots, T$.

When $(\beta R) > 1$, the consumer sets $A_t = (\beta R)^t A_0$

and $c_T = A^T$, $c_t = 0, t = 0, 1, \dots, T-1$.

In case (ii), $\{c_t\}_{t=0}^T, \{A_{t+1}\}_{t=0}^{T-1}$ are indeterminate,

there being many solutions. Please find one
 of them and explain. (Hint: Impatience and
positive interest just offset each other in case (ii).)

Further explanation and intuition.

Let's interpret the condition

$$(\star) \quad \beta^t \cdot 1 - \beta^t \lambda_t R \leq 0, = 0 \text{ if } c_t > 0$$

$$\text{or} \quad \beta^t \leq \beta^t \lambda_t R, = 0 \text{ if } c_t > 0$$

First consider the time 0 utility functional

First consider the form of utility functional

$$U_0 = \sum_{t=0}^{\infty} \beta^t c_t$$

with

$$\frac{\partial U_0}{\partial c_t} = \beta^t .$$

so the term $\beta^t \cdot 1$ in (*) is the marginal utility, as measured by U_0 , of c_t .

Now, what about the term $\beta^t \lambda_t R$ in (*)?

well, if we differentiate the Lagrangian w.r.t. A_t we get $\beta^t \lambda_t R$. And at the optimum, differentiating the Lagrangian L w.r.t. A_t is "like" differentiating U_0 w.r.t. A_t (why?, because at the optimum,

$$\sum \beta^t \lambda_t [R(A_t + y_t - c_t) - A_{t+1}] = 0$$

so $L = U_0$ at the optimum.)

Thus, $\beta^t \lambda_t R$ is the marginal utility, in the sense of U_0 , of A_t .

The condition (*) says

$$\beta^t \cdot 1 < \beta^t \lambda_t R \Rightarrow c_t = 0$$

↑
marginal
utility of
consumption

↑
marginal
utility
of assets at t

i.e., only consume when marginal utility of consumption =
 marginal utility of assets; otherwise, just Save.

3. A consumer wants to maximize the expected value of

$$\sum_{t=0}^{\infty} \beta^t \left\{ -\frac{1}{2} (c_t - b)^2 \right\}, \quad 0 < \beta < 1$$

where b is a very big positive number representing
 "bliss" consumption,

subject to the budget constraints

$$F_{t+1} = \beta^{-1} [F_t + y_t - c_t], \quad F_0 = 0$$

where F_t is financial assets at t ,
 β^{-1} is the gross interest rate from t to $t+1$
 and y_t is labor income.

labor income obey

$$y_{t+1} = \bar{\delta} + \delta_1 y_t + \delta_2 y_{t-1} + \Gamma \varepsilon_{t+1}$$

where ε_{t+1} is an i.i.d. (independently and
 identically distributed) sequence of scalar
 random variables with

$$\varepsilon_{t+1} \sim \mathcal{N}(0, 1)$$

$$\varepsilon_{t+1} \sim \mathcal{N}(0, 1)$$

(normal with mean 0 and variance 1)

The consumer's optimal decision rule satisfies

$$(*) c_t = (1-\beta) \left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

here E_t is the expectation conditioned on information known

- a. Find $\alpha_0, \alpha_1, \alpha_2$ at t .
 an expression for the consumer's decision rule of the form

$$c_t = (1-\beta) \left[F_t + \alpha_0 + \alpha_1 y_t + \alpha_2 y_{t-1} \right],$$

please describe how to find formulas for $\alpha_0, \alpha_1, \alpha_2$.

- b. The growth rates of consumption for this consumer over the last 4 years were as follows:

<u>quarter</u>	<u>$c_t - c_{t-1}$</u>
2008 I	1000
2008 II	0
2008 III	0
2008 IV	<u>-4000</u>

What can you infer from these numbers

What can you infer from these numbers about the consumer's past and future labor income?

4. (Another exercise in Kuhn-Tucker)

A consumer lives for two periods and has multi-period utility function

$$\ln c_1 - \bar{\beta} h_1 + \beta [\ln c_2 - \bar{\beta} h_2]$$

$$0 < \beta < 1, \bar{\beta} > 0$$

$$c_1 \geq 0, c_2 \geq 0, h_1 \geq 0, h_2 \geq 0 \quad (***)$$

c_i is consumption in period i

h_i is labor supplied in period i

The consumer's intertemporal budget constraint is

$$c_1 + \beta c_2 = w_1 h_1 + \beta (w_2 h_2)$$

where here β^{-1} is the one period gross interest rate.

and $w_1 > 0$ is the first period real wage and $w_2 > 0$ is the second period real wage.

a. Suppose that $w_1 > w_2$. Find the

consumer's optimal plan for c_1, c_2, h_1, h_2 .

(please remember the inequalities in (**)).

b. Suppose that $w_2 > w_1$. Find the consumer's optimal plan for c_1, c_2, h_1, h_2 .

c. Explain why your answers in (a) and (b) differ.

Practice problems 2

Friday, February 26, 2010
8:31 PM

Here are some sample problems for practice.

1. "Equalizing differences."

A person who has just graduated from high school can enter the work force now (at time t)

and earn $w_{t+j}^h = \delta_h^j w_t$, $j=0, 1, \dots, T$; where $\delta_h > 0$. (He/she must retire after $t+T+1$.) The present

value of his life time earnings if he goes to work

right away is

$$PV_t^h = \sum_{j=0}^T \beta^j w_{t+j}^h . \quad \beta = \frac{1}{1+r}, r > 0$$

If the high school graduate instead enters college (then graduate school) and starts working

in period $t+k$, he/she earns

w_s^c in period $s=t+k, t+k+1, \dots, t+T$

where $w_{t+j}^c = \delta_c w_t^c$, $\delta_c > 0$

The present value of going to college during

$t, t+1, \dots, t+k-1$ (during which he/she learns nothing) is

$$PV_t^c = \sum_{j=k}^T \beta^j w_{t+j}^c$$

- Find a formula for PV_t^h as a function of $\beta, w_t^h, T, \delta_h$

$$\beta, w_t^h, T, \delta_h$$

- b. Find a formula for PV_t^c as a function of $\beta, w_t^c, T, \delta_c$, and k .
- c. Given $\beta, w_t^h, T, \delta_h, \delta_c, w_t^h$, find a value of w_t^c that equals the present values of going to college and not going to college.

2. Equalizing differences, again.

A truck driver works every period $t \geq 0$ at wage

$$w_{dt} = \delta_d^t w_{d,0}, \quad 0 < \delta_d < \frac{1}{\beta}, \quad \beta = \frac{1}{1+r}, \quad r > 0$$

A construction worker works only every even period at wage

$$w_{ct} = \begin{cases} \delta_c^t w_{c,0} & \text{if } t=0, 2, 4, \dots \\ 0 & \text{other wise} \end{cases}$$

Each worker lives forever.

- a. Compute the present value of the wages of a truck driver

$$PV_{d,0} = \sum_{t=0}^{\infty} \beta^t w_{dt}$$

- b. Compute the present value of the wages of a construction worker

$$PV_{c,0} = w_{c,0} + \beta^2 w_{c,2} + \beta^4 w_{c,4} + \dots$$

- c. Find a formula for $w_{c,o}$ that equates the present value of a construction worker to the present value of a truck driver.

3. Consider an economy consisting of two people, each of whom orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad , \quad i=1,2, \quad u' > 0, \quad u'' < 0$$

$0 < \beta < 1$

Assume that $u(c) = \ln c$

Person 1 has income stream

$$y_t = \delta^t, \quad t \geq 0, \quad \delta \in (0, 1).$$

Person 2 has income stream

$$y_t^2 = (-\delta^t), t \geq 0, \delta \in (0, 1).$$

Each consumer faces the sequence of budget constraints

$$A_{t+1}^{\quad i} = R_k \left[A_k^{\quad i} + y_k^{\quad i} - c_k^{\quad i} \right],$$

$$A_o^i = 0 \quad i=1,2$$

A_{t+1}^i , $i=1, 2, \dots, t \geq 0$, can be

either positive or negative but must obey

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) A_T = 0.$$

- a. Assume that $R_t = \beta^{-1}$. (There is an outside lender who supplies loans at this rate). Find optimal consumption, asset plans $(c_t^i, A_{t+1}^i)_{t=0}^\infty$ for $i=1,2$.

=

- From now on assume that the consumer cannot borrow so that $A_{t+1}^i \geq 0$ for all $t \geq 0$ for $i=1,2$.

- b. Now relax the assumption that $R_t = \beta^{-1}$.

Instead, find a sequence of gross interest rates $\{R_t\}_{t=1}^\infty$ that makes consumer 1 want to consume his endowment each period so that

$$c_t' = \delta^t, t \geq 0$$

Hint: get the FONC's for the consumer's problem, set $c_t' = \delta^t, t \geq 0$, solve, and solve for $\{R_t\}_{t=0}^\infty$

- c. At the $\{R_t\}_{t=0}^\infty$ that you found in part b, verify that consumer 2 will not want to lend.

- d. Argue that $\{R_t\}_{t=1}^\infty$ that you found in part (b)

a. Argue that $\{R_t\}_{t=1}^{\infty}$ (let you find in part (b)) is a competitive equilibrium $\{R_t\}_{t=1}^{\infty}$ for the closed economy in which $C_t^1 + C_t^2 = y_t^1 + y_t^2$ and in which there is no outside lender.

c. Now relax the assumption that $R_t = \beta^{-1}$.

Instead, find a sequence of gross interest

rates $\{R_t\}_{t=1}^{\infty}$ that makes consumer 2

want to consume his endowment each period so that

$$C_t^2 = 1 - \delta^t, t \geq 0$$

Is R_t higher or lower than what you found in part b? At this R_t , will consumer 1 want to borrow or lend?

Practice problems 3

Friday, February 26, 2010
11:11 AM

1. Consumer $i=1, 2$ orders consumption

streams $\{c_t^i\}_{t=0}^\infty$ according to

$$(1) \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{where } u' > 0, u'' < 0 \\ c_t \geq 0, \quad 0 < \beta < 1$$

The consumer has exogenous labor income y_t^i

and must pay a constant lump sum tax γ

(the same for both $i=1, 2$) each period.

The income of consumer 1 is

$$y_t^1 = \begin{cases} \bar{Y}_0, & 0 \leq t \leq T \\ \bar{Y}_1, & T+1 \leq t \end{cases}$$

where $\bar{Y}_1 > \bar{Y}_0$

The income of consumer 2 is

$$y_t^2 = \begin{cases} \bar{Y}_1, & 0 \leq t \leq T \\ \bar{Y}_0, & T+1 \leq t \end{cases}, \quad t=0, 1, 2, \dots$$

The tax rate $\gamma_t = \bar{\gamma}$ for all $t \geq 0$

where $0 < \bar{\gamma} < \bar{Y}_0 < \bar{Y}_1$. (Please notice this restriction on $\bar{\gamma}$.)

Consumer i faces a sequence of budget constraints

$$A_{t+1}^i = \beta^{-1} [A_t^i + y_t^i - \gamma_t - c_t^i]$$

$$c_t^i \geq 0, A_0^i = 0$$

$$A_{t+1}^i \geq 0, t=0, 1, 2, \dots$$

Notice that $A_{t+1}^i \geq 0$ is a no-borrowing constraint.

Notice that both consumers pay the same tax $\gamma_t = \bar{\gamma}$ each period.

a. Please give formulas for the optimal consumption paths of consumers $i=1, 2$ for each t ,

$$\{c_t^i\}_{t=0}^{\infty}, i=1, 2.$$

b. Now suppose that the tax for each person is changed to

$$\gamma_t = \begin{cases} 0, & 0 \leq t \leq T \\ \gamma^*, & T+1 \leq t \end{cases}$$

where $\gamma^* = \frac{\bar{\gamma}}{\beta^{T+1}}$ (H)

Find optimal levels of $\{c_t^i\}_{t=0}^{\infty}, i=1, 2$.

c. Compare your solutions from parts (a) and (b) and discuss why they differ (if they do differ).

d. Interpret condition (f).

2. A consumer wants to choose $\{c_t, A_{t+1}\}_{t=0}^{\infty}$

$$\sum_{t=0}^{\infty} \beta^t u(c_t), 0 < \beta < 1, u' > 0, u'' < 0$$

subject to the sequence of budget constraints

$$F_{t+1} = \beta^{-1} [F_t + g_t - c_t], F_0 = 0.$$

\uparrow financial assets \uparrow endowment \nwarrow consumption

$$R = \beta^{-1}$$

$$R = (1+r)$$

↑ net interest rate

$$F_0 = 0 \text{ is given.}$$

we also require

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) F_T = 0$$

the consumer can borrow and lend, so

F_{t+1} can be + or -.

There is no uncertainty, so the consumer has perfect foresight about $\{g_t\}_{t=0}^{\infty}$.

Consider the endowment path

$$y_t = \begin{cases} y_0 \delta^t, & t = 0, 1, \dots, T \\ y_0 \delta^T, & \text{for } t \geq T \end{cases}$$

where $1 < \delta < \frac{1}{\beta}$.

a. Graph the endowment path

b. Compute the optimal path for $\{c_t\}_{t=0}^{\infty}$ - i.e. give a formula for c_t .

c. Compute F_x for $t \geq T$.

Now assume that the consumer cannot borrow but can lend. Suppose $\delta < 1$.

d. Compute the optimal path for $\{c_t\}_{t=0}^{\infty}$.

e. Compute F_x for $t \geq T$.

Now assume that the consumer cannot borrow but can lend and that $\delta > 1$.

f. Compute the optimal path for $\{c_t\}_{t=0}^{\infty}$.

g. Compute F_x for $t \geq T$.

3. A consumer chooses $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad u' > 0, u'' < 0$$

subject to $\sum_{t=0}^{\infty} q_t^0 c_t \leq \sum_{t=0}^{\infty} q_t^0 y_t$

where $\{y_t\}_{t=0}^{\infty}$ is an exogenous endowment

and q_t^0 = the price of one unit of consumption at time t , measured in units of time 0 consumption,

so

$$q_0^0 = 1 \quad (\text{a normalization})$$

Assume that

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1$$

$$= \ln c \quad \text{if } \gamma = 1$$

$$(\text{Remember that } q_t^0 = \frac{1}{(1 + g_{0t})^t})$$

where g_{0t} is the yield to maturity on a t -period zero-coupon bond)

Consider a "Lucas model" where there is a representative consumer and prices $\{q_t^0\}_{t=0}^{\infty}$ must adjust to induce the consumer to set $c_t = y_t \quad \forall t \geq 0$.

a. Suppose $y_t = y_0 \delta^t, \quad 0 < \delta < \frac{1}{\beta}$.

Compute q_t^0 . Compute g_{0t} .

b. In the special case of $\gamma = 1$ (log utility),

b. In the special case of $\gamma = 1$ (log utility),
compute q_t^0 and f_{0t} .

c. When $\delta = 1$, compute q_t^0 .

d. Is the one-period interest rate f_{01}
"high" or "low" when $\delta > i$?

e. Is the one-period interest rate f_{01} "high" or "low"
when $\delta < i$?

Practice problems 4

Saturday, February 27, 2010
3:36 PM

1. A consumer receives an exogenous endowment $\{y_t\}_{t=0}^{\infty}$ that evolves according to the law of motion

$$y_{t+1} = y_t + \sigma_t \varepsilon_{t+1}$$

where $y_0 > 0$ is given, $\{\varepsilon_t\}_{t=0}^{\infty}$ is an independently and identically distributed normal random variable with mean 0 and variance 1, and

$$\sigma_t = \begin{cases} \bar{\sigma} > 0 & \text{for } t = 0, 1 \\ \sigma_t = 0 & \text{for } t \geq 2 \end{cases}$$

At time t , the consumer knows the history

$y^t = (y_t, y_{t+1}, \dots, y_0)$ but not the future values of y_s , $s > t$.

The consumer chooses $\{c_t, F_{t+1}\}_{t=0}^{\infty}$ subject to the sequence of budget constraints

$$F_{t+1} = \beta^{-1} [F_t + y_t - c_t]$$

$$F_0 \geq 0 \text{ given}$$

where $\beta^{-1} = R = (1+r)$ is the gross one-period interest rate on risk-free borrowing or lending. The consumer can borrow or lend, but his/her plan must satisfy

$$E_0 \beta^T u'(c_T) F_T = 0$$

(no Ponzi scheme)

The consumer wants to choose $\{c_t, F_{t+1}\}_{t=0}^\infty$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

mathematical expectation given time 0 information

$$\text{where } u(c_t) = \alpha_0 - \frac{\alpha_1}{2} c_t^2, \quad \alpha_0 > 0, \quad \alpha_1 > 0$$

(quadratic utility)

The consumer's first-order condition with respect to $c_{t+1}, F_{t+1} \Rightarrow$

$$u'(c_t) = E_t u'(c_{t+1})$$

which in turn implies, through quadratic utility,

$$E_t C_{t+1} = C_t$$

↑

expectation conditioned on time t "information"

This condition and the budget constraints leads the consumer to set C_t so that

$$C_t = (1-\beta) \left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

↑ ↑
financial wealth "human wealth"

a. Draw 5 "sample paths" for $\{y_t\}_{t=0}^5$

b. Find a "consumption function" of the form

$$C_t = \delta_0 + \delta_1 F_t + \phi_0 y_t + \phi_1 y_{t-1}$$

Please give formulas for $\delta_0, \delta_1, \phi_0, \phi_1$ in terms of β and the other fundamental parameters of the model.

c. Please draw 5 sample paths of
 $\{C_t\}_{t=0}^5$

Parts d. and e. are extra credit:

d. Find a formula for $\{F_t\}$ of the form

$$F_{t+1} = F_t + \alpha_t \varepsilon_{t+1}$$

and give a formula for α_t , $t \geq 0$

e. Find a formula for $\{c_t\}$ of the form

$$c_{t+1} = c_t + \eta_t \varepsilon_{t+1}$$

and give a formula for η_t , $t \geq 0$.

Note: we allow α_t and η_t to depend on t , so you have to find an α_t and η_t for each $t \geq 0$.

2. A consumer's endowment $\{y_t\}_{t=0}^{\infty}$ follows the law of motion

$$y_{t+1} = f(y_t + \sigma \varepsilon_{t+1}), \quad \sigma > 0$$

$$y_0 > 0 \text{ given}, \quad |f'| \leq 1$$

$$\varepsilon_{t+1} \sim N(0, 1) \quad (\text{Gaussian i.i.d.})$$

The consumer knows $y^t = y_t, y_{t-1}, \dots, y_0$ at time $t \geq 0$.

The consumer sets $\{c_t, F_{t+1}\}_{t=0}^{\infty}$ according to

$$c_t = (1-\beta) \left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right],$$

$$F_{t+1} = \beta^{-1} [F_t + y_t - c_t]$$

$$0 < \beta < 1$$

a. Find an expression for c_t of the form

$$c_t = \delta_0 + \delta_1 F_t + \phi_0 y_t + \phi_1 y_{t-1}$$

giving formulas for ϕ_0 and ϕ_1
as functions of β and σ and τ .

b. For $\beta = .95$, $\sigma = .05$, please fill
in the following table:

σ	ϕ_0	ϕ_1
-1	[]	[]
-.5	[]	[]
0	[]	[]
.5	[]	[]
1	[]	[]

that is, fill in the missing entries for ϕ_0 and ϕ_1 .

Practice problems 5a

Monday, April 12, 2010
9:05 AM

The first several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t \log c_t , \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\varphi}, \quad \varphi > 0$$

where c_t is consumption per worker.

The technology is .

$$y_t = f(k_t) = z k_t^\alpha, \quad 0 < \alpha < 1$$

$$z > 0$$

where y_t is output per unit labor and

k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta)k_t + x_t, \quad 0 < \delta < 1$$

Assume a competitive equilibrium with a price system

$\{q_t, r_t, w_t\}_{t=0}^{\infty}$ and a government policy $\{g_t, T_t\}_{t=0}^{\infty}$.

1. Assume that the government finances its expenditures by levying lump sum taxes. There are no

distortionary taxes. Assume that at time 0, the economy starts out with a capital per unit labor k_0 . This equals the steady state value appropriate for an economy in which g_t had been zero forever.

a. Find a formula for the steady state capital stock when $g_t = 0 \forall t$.

b. Compare the steady state capital labor ratio \bar{k} in the competitive equilibrium with the capital labor ratio \tilde{k} that maximizes steady state consumption per capita; i.e., \tilde{k} solves

$$\tilde{c} = \max_k f(k) - \delta k$$

Is \tilde{k} greater than or less than \bar{k} ? If they differ, why? Is \tilde{c} greater or less than $\bar{c} = f(\bar{k}) - \delta \bar{k}$? Explain why.

c. Now assume that at time 0, g_t suddenly jumps to the value $g = \frac{1}{2} \bar{c}$ where \bar{c} is the value of consumption per capita in the initial steady state in which g was zero forever. Starting from $k_0 = \bar{k}$ for the old $g=0$ steady state, find the time paths of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ associated with the new path

$g_t = g > 0$ for government expenditures per capita.

Also show the time path for $R_{t+1} = (1-\delta) + f'(k_{t+1})$.

Explain why the new time paths are as they are.

2. Trade and growth

Consider the problem of a planner in a small economy. When the economy is closed to trade, the planner chooses $\{c_t, k_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 , \quad \beta = \frac{1}{1+\gamma} \\ \gamma > 0$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0, 1)$$

where $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \quad , \quad \gamma > 0$

$$f(k_t) = z k_t^\alpha \quad , \quad 0 < \alpha < 1$$

Let \bar{k} be the steady state value of k_t under the optimal plan.

a. Find a formula for \bar{k} .

b. Assume that $k_0 < \bar{k}$. Describe time paths

for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $R_{t+1} = (1-\delta) + f'(k_{t+1})$.

c. What is the steady state value of R_{t+1} ?

d. Is R_{t+1} less or greater than its steady state

Value when $k_{t+1} < \bar{k}$?

Part II.

Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$ at which the planner can borrow or lend.

The planner can use the proceeds of borrowing to purchase goods on the international market. These goods can be used to augment capital or to consume.

Let \bar{k}_0 be the level of initial capital ($\bar{k}_0 < \bar{k}$) just before the country opens up to trade just before time 0. Let \bar{k}_0 be the same initial capital per capita $\bar{k}_0 < \bar{k}$ studied in parts a.-d.

At time $t=-1$, after \bar{k}_0 was set, trade opens up. At $t=-1$, the planner can issue IOU's or bonds in amount B_{-1} and use the proceeds to purchase capital, thereby setting

$$k_0 = \bar{k}_0 + B_{-1}$$

where B_{-1} is denominated in time -1 consumption goods.

The bonds are one-period in duration and bear the

constant world gross interest rate $R = \beta^{-1}$.

For $t \geq 0$, the planner faces the constraints

$$c_t + k_{t+1} + R B_{t-1} = f(k_t) + (1-\delta) k_t + B_t,$$

Here $R B_{t-1}$ is the interest and principal on bonds issued in $t-1$, and B_t is the amount of one-period bonds issued at t .

The planning problem is now to choose

$$\{c_t, k_t, B_t\}_{t=0}^{\infty} \text{ and } B_{-1}, \text{ subject}$$

to k_0^- given. (Note that k_0 is now a choice variable and that k_0^- is an initial condition.)

Solve the planning problem in this small open economy and compare the solution to the solution in the closed economy.

Hint: Form the Lagrangian

$$\begin{aligned} L = & \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) \right. \\ & + \lambda_t \left[f(k_t) + B_t - c_t - k_{t+1} + (1-\delta) k_t - R B_{t-1} \right] \left. \right\} \\ & + \phi [k_0^- + B_{-1} - k_0] \end{aligned}$$

where $\{\lambda_t\}_{t=0}^{\infty}$, ϕ are Lagrange multipliers.

FONC:

$$c_t: u'(c_t) - \lambda_t = 0, t=0, 1, \dots$$

$$\lambda_t: \lambda_t [f'(k_t) + (1-\delta)] - \beta^{-1} \lambda_{t-1} = 0, t=1, 2, \dots$$

$$k_0: \lambda_0 [f'(k_0) + (1-\delta)] - \phi = 0$$

$$\beta_t: \lambda_t - \lambda_{t+1} R \beta = 0, t=0, 1, \dots$$

$$\beta_{-1}: \phi - \lambda_0 R = 0$$

$$\Rightarrow \phi = \lambda_0 R$$

$$\lambda_{t+1} = \lambda_t, t \geq 0 \Rightarrow \lambda_t = \lambda \quad \forall t$$

$$\lambda_0 [f'(k_0) + (1-\delta)] = \lambda_0 R$$

$$\text{or} \\ f'(k_0) + (1-\delta) = (1+\rho)$$

$$\text{or} \\ f'(k_0) = \rho + \delta$$

$\Rightarrow k_0 = \bar{k}$ for optimal steady state in the

closed economy.

$$\text{also } \lambda_t = \lambda \quad \forall t \Rightarrow$$

$$f'(\bar{k}_t) + (1-\delta) = 1+\rho \rightarrow$$

$$f'(\bar{k}) = \rho + \delta \quad \forall t$$

$$\Rightarrow k_t = \bar{k} \quad \forall t.$$

Thus the soln is

$$\beta_{-1} = \bar{k} - k_0 \quad (k_0 = \bar{k})$$

$$\beta_t = \beta_{t-1} \quad \forall t \geq 0$$

roll over the debt
- - - - -

$$B = (\bar{k} - k_0^-)$$

Consumption C_t :

$$C_t + b_{t+1} + R B_{t+1} = f(k_t) + (1-\delta) k_t + \bar{B}_t$$

with $k_t > \bar{k}$ and $B_t = \bar{B} \Rightarrow$

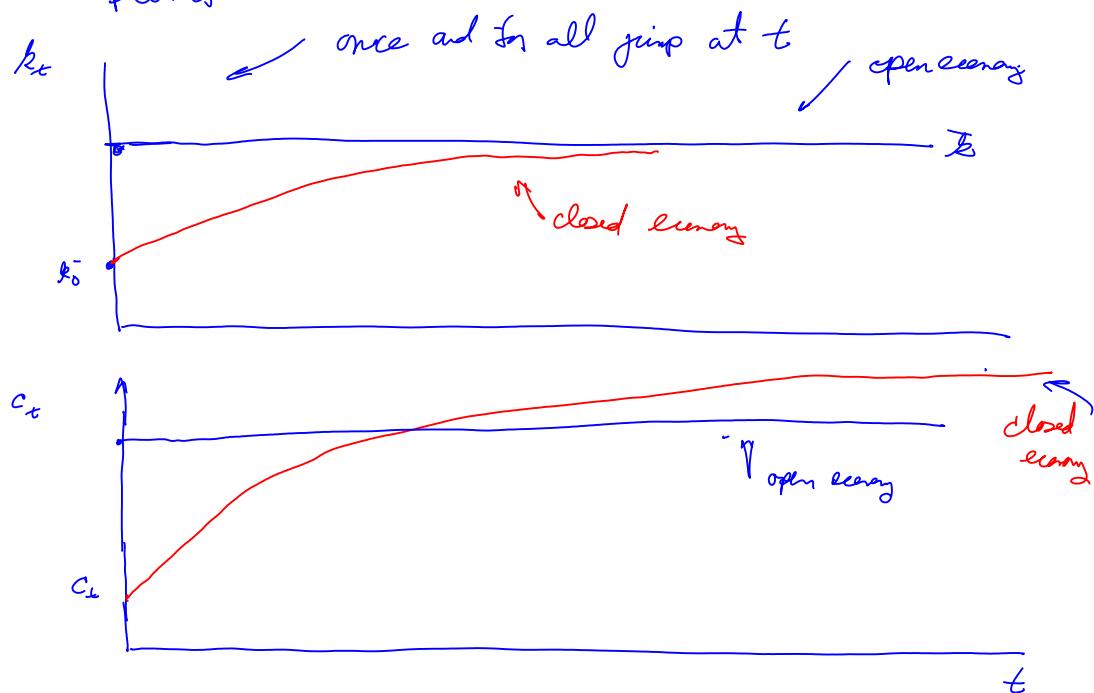
$$\bar{c} + \delta \bar{k} + (R-1) \bar{B} = f(\bar{k})$$

$$\bar{c} = f(\bar{k}) - \delta \bar{k} - (R-1) [\bar{B} - \bar{k}_0^-]$$

$$\bar{c} = f(\bar{k}) - \delta \bar{k} - g \underbrace{[\bar{k} - \bar{k}_0^-]}_{\text{initial borrowing}}$$

net interest on initial borrowing.

So the path is



difference in consumption asymptotically

Closed economy: $\bar{c} = f(\bar{k}) - \delta \bar{k}$

Open economy: $\bar{c} = f(k) - \delta k - g[\bar{k} - k_0]$

Question: Is welfare $\sum_{t=0}^{\infty} \beta^t u(c_t)$ higher
in the "open" or "closed" economy.

Hint: (The planner is free to set $B_t = 0$
 $\forall t \geq -1$ but chooses not to)

Practice problems 5

Sunday, April 11, 2010
3:03 PM

The next several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\gamma}, \quad \gamma > 0$$

where c_k is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma = 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

The technology is ..

$$y_t = f(k_t) = z k_t^\alpha, \quad , \quad 0 < \alpha < 1 \\ z > 0$$

where y_t is output per unit labor and

k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

χ_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta)k_t + \gamma_t \quad , \quad 0 < \delta < 1$$

The government finances its expenditures by levying some combination of a flat rate tax T_{ct} on the value of consumption goods purchased at t , a flat rate tax of T_{mt} on the value of labor earnings at t ,

a flat rate tax γ_{kt} on earnings from capital at t , and a lump sum tax of γ_{ct} in the consumption goods per worker at time t .

Let $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ be a pure system.

3. Consider an economy of the type described above in

which $g_t = \bar{g} > 0$ for all $t \geq 0$

and in which initially the government finances all expenditures by lump sum taxes.

a. Find a formula for the steady state capital labor ratio k^* for this economy. Find formulas for the steady state levels of c_t and

$$R_t = [(1-\delta) + (1-\gamma_{k+1}) f'(k_{t+1})].$$

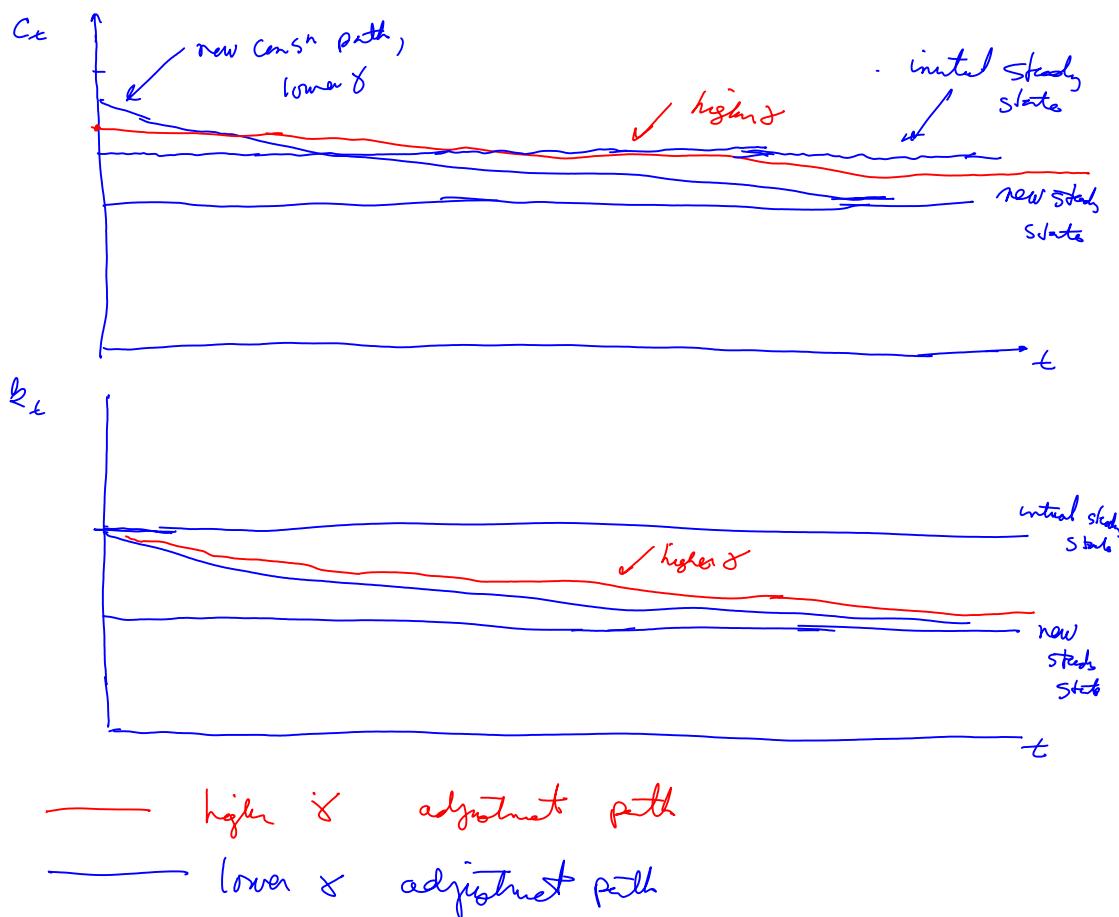
b. Now suppose that starting from $k_0 = \bar{k} =$ steady state flat you computed in part a, the government suddenly increases the tax on earnings on capital to a constant level $\gamma_k > 0$. The government adjusts lump sum taxes to keep the government budget balanced. Derive competitive equilibrium time paths for c_t , k_{t+1} , R_t and their relationship to corresponding values in the old steady state flat you derived in part a.

c. Describe how the shapes of the paths that you found in part b. depend on the curvature parameter γ in the utility function

$u(c) = \frac{c}{1-\gamma} c^{1-\gamma}$. Higher values of γ imply higher curvature and more deviation to consumption paths that fluctuate.

Higher values of γ imply that the consumer values smooth consumption paths even more.

Thus, here is what we expect:



This look this way because of the consumer's

This looks this way because of the consumer's attitude about consumption variability. The interest rate and k_t adjust to make this happen.

Remark: formula in section 11.7.3 of the typed chapter verifies this by giving a formula for λ as a function of γ where λ was the decay rate discussed in class.

d. Starting from the steady state \bar{k} that you computed in part a, now consider a situation in which the government announces at time 0 that starting in period 10 the tax on earnings from capital T_k will rise permanently to $\bar{T}_k > 0$. The government adjusts its lump sum taxes to balance its budget.

- i) find new steady values for \bar{k}_t , \bar{c}_t , \bar{R}_t .
- ii) describe the shapes of the transition paths from the initial steady state to the new one for k_t , R_t , c_t .
- (iii) decide how the shapes of these transition paths depend on the curvature parameter γ in the utility function $u(c)$.

Hint: when γ is bigger, consumers more strongly prefer smoother consumption paths.

4. Trade and growth, version II

Consider the problem of a planner in a small economy. When the economy is closed to trade, the planner chooses $\{c_t, k_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 , \quad \beta = \frac{1}{1+\gamma} \\ \gamma > 0$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0, 1)$$

where $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \quad , \quad \gamma > 0$

$$f(k_t) = z k_t^\alpha \quad , \quad 0 < \alpha < 1$$

Let \bar{k} be the steady state value of k_t under the optimal plan.

a. Find a formula for \bar{k} .

b. Assume that $k_0 > \bar{k}$. Describe time paths

for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $R_{t+1} = (1-\delta) + f'(k_t)$.

c. What is the steady state value of R_{t+1} ?

d. Is R_{t+1} less or greater than its steady state

value when $k_{t+1} > \bar{k}$?

Part II.

- - - - - + + - - . . . + : + + + + + +

e. Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$ at which the planner can borrow or lend. In particular, the planner is free to use the following plan. The planner can sell all of its capital k_0^- and simply consume the interest payments.

Let k_0^- be the level of initial capital ($k_0^- > \bar{k}$) just before the country opens up to trade just before time 0. Let \bar{k}_0^- be the same initial capital per capita $\bar{k}_0^- > \bar{k}$ induced in parts a.-d.

At time $t=-1$, after k_0^- was set, trade opens up. At $t=-1$, the planner sells k_0^- in exchange for IOU's from the rest of the world in the amount $A_{-1} = k_0^-$.

The bonds A_{-1} are one-period in duration and bear the

constant world gross interest rate $R = \beta^{-1}$.

After the sale of $k_0^- = A_{-1}$,

the planner has zero capital and so shuts down the technology. Instead, the planner uses the asset market to smooth consumption. The planner thus chooses (A_{t+1}, c_t)

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

to maximize

$$\beta = \frac{1}{1+r}$$

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad \beta = \frac{1}{1+\rho}$$

$$c_t + \beta A_{t+1} = A_t, \quad A_0 = R_0$$

$$A_{t+1} = \beta^{-1} [A_t - c_t]$$

Form the Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t [\beta^{-1}(A_t - c_t) - A_{t+1}] \right\}$$

FONC:

$$c_t: \quad u'(c_t) - \lambda_t \beta^{-1} = 0$$

$$A_t: \quad -\beta^{-1} \lambda_{t-1} + \beta^{-1} \lambda_t = 0$$

$$\lim_{T \rightarrow \infty} \beta^T \lambda_T A_{T+1} = 0$$

$$\Rightarrow \lambda_t = \lambda_0 \quad \forall t$$

$$u'(c_t) = u'(c_0) \quad \forall t$$

$$c_t = c_0 \quad \forall t.$$

then roll over assets & just consume interest:

$$A_{t+1} = \beta^{-1} [A_t - c_t] \Rightarrow$$

$$\bar{A} = \beta^{-1} [\bar{A} - c_0] \Rightarrow$$

$$(1-\beta^{-1}) \bar{A} = -\beta^{-1} c_0$$

$$\beta^{-1} c_0 = (\beta^{-1} - 1) \bar{A}$$

$$c_0 = (1-\beta) \bar{A}$$

$$1-\beta = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} = \beta \rho$$

$$c_0 = g \bar{f} \bar{A} \quad , \quad \bar{A} = A_0 = \mathcal{L}_0^{-1}$$

Thus, the optimal path for consumption is

$$c_t = g \bar{A} = g k_0^{-1}$$

f. Compute the time path of (c_t, k_t, R_t)

that you computed in parts a, b, c, d with "no trade" with the part e one "with trade" in which the government "shuts down the home technology" and lives entirely from returns on foreign assets.

Can you say which path the representative consumer would prefer?

* Hint: I think it could go either way for reasons that might be easier to appreciate after you look at part g.

g. Now return to the economy in part e with $k_0 > \bar{k}$ from part d. Assume that the planner is free to borrow or lend capital at the fixed gross interest rate of $\beta^{-1} = 1 + f$, as before.

But now assume the planner chooses the optimal amount of k_o^{-1} to sell off and is not necessarily sell off the entire k_o^{-1} and possibly continues to operate the technology.

i) Find the solution of the planning problem.

Hint: you might want to review your solution to problem 2 of lecture problems 4.

ii) Explain why it is optimal not to shut down the technology. Hint: Starting from having shut the technology down, think of putting a small amount ε of capital into the technology - this earns $\geq \varepsilon^\alpha$ and costs $\leq \varepsilon$ in terms of foregone interest. because $0 < \alpha < 1$ $\geq \varepsilon < \varepsilon^\alpha$ for small ε . - The technology is very productive for small ε - so use it.

"

3. Term structure of interest rates.

This problem assumes the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$
$$\beta = \frac{1}{1+r}, r > 0$$

where c_t is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma = 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

$$u(c) = \log c \quad \text{if } \delta = 1$$

The technology is .

$$y_t = f(k_t) = z k_t^\alpha, \quad 0 < \alpha < 1 \\ z > 0$$

where y_t is output per unit labor and

k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta)k_t + x_t, \quad 0 < \delta < 1$$

c_t = consumption per worker

The government finances its expenditures by levying some combination of a flat rate tax T_{ct} on the value of consumption goods purchased at t , a flat rate tax of T_{nt} on the value of labor earnings at t , a flat rate tax T_{kt} on earnings from capital at t , and a lump sum tax of T_{st} in the t consumption goods per worker at time t .

Let $\{q_t^0, r_t^0, w_t^0\}_{t=0}^\infty$ be a pure system.

Here the superscript 0 means the true or pure system.

a. Recall that we can represent

$$q_{bt}^0 = q_{b0}^0 m_{0,1} m_{1,2} \dots m_{t-1,t}$$

$$\text{where } m_{t-1,t} = \frac{q_{bt}^0}{r_t^0}$$

$$\text{where } m_{t-1,t} = \frac{g_t^0}{g_{t-1}^0}$$

$$\text{and } m_{t-1,t} = \exp(-r_{t-1,t}) \approx \frac{1}{1+r_{t-1,t}}$$

Further recall that the t period long yield is

satisfies

$$g_t^0 = \exp(-t r_{0,t})$$

or

$$r_{0,t} = \frac{1}{t} [r_{0,1} + r_{1,2} + \dots + r_{t-1,t}]$$

Now suppose that at $t=0$, $k_0 = \bar{k}$, where

\bar{k} is the steady state appropriate for an economy with constant $g_t = \bar{g} > 0$ and all expenditures financed by lump sum taxes. Find

g_t^0 for this economy.

b. Plot $r_{t-1,t}$ for this economy for $t=1, 2, \dots, 10$.

c. Plot $r_{0,t}$ for this economy for $t=1, \dots, 10$.

(this is what Bloomberg plots)

d. Now assume that at time 0, starting from

$k_0 = \bar{k}$ for the steady state you computed in part a,

the government unexpectedly and permanently raises the tax rate on income from capital $T_{kt} = T_b > 0$ to a

+ 1

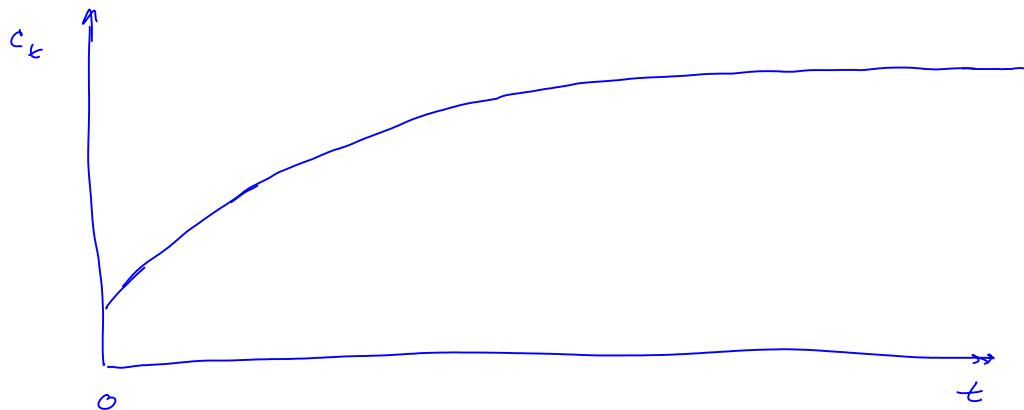
positive rate.

i) Plot $r_{t-1,t}$ for $t=1, \dots, 10$ for this economy. Explain how you got this outcome.

ii) Plot $r_{0,t}$ for $t=1, \dots, 10$ for this economy. Explain how you got this outcome.

4. The following problem assumes the same economic environment as the previous problem-i.e., the "growth model" with fiscal policy.

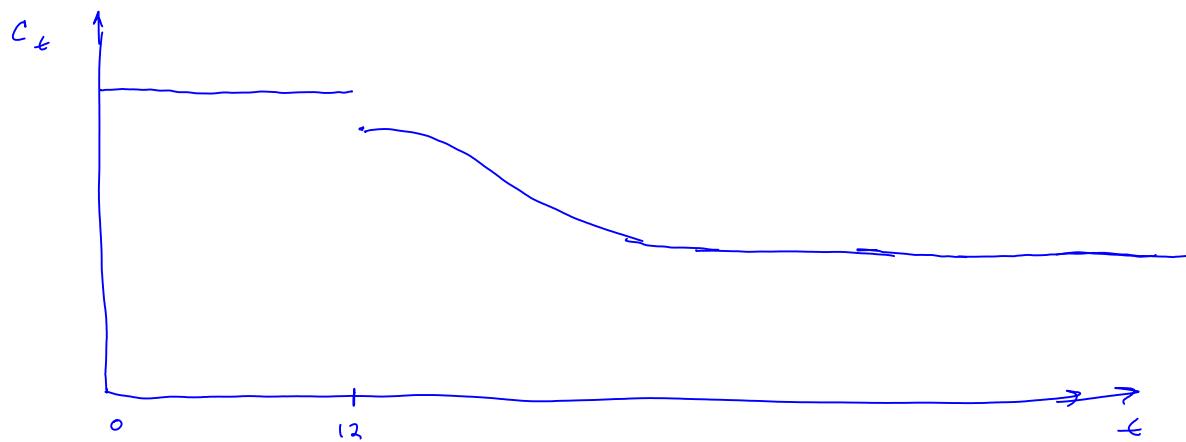
Suppose that you observe the following path for consumption per capita



Say what you can about likely behavior over time of k_t , R_t , $[(1-\delta) + (1-\gamma_k) f'(k_t)]$, g_t , and T_t . (You are free to make up any story you like that is consistent with the model.)

5. Assume the same economic environment as in the previous

two problems. Assume that someone has observed the following time path for c_t :



- a. Describe a consistent set of assumptions about fiscal policy that explains this time path for c_t . In doing so, please distinguish carefully between changes in taxes and expenditures that are foreseen versus unforeseen.
- b. Describe what is happening to k_t , R_t , and J_t over time. Make whatever assumptions you must to get a complete but consistent story — here "consistent" means "consistent with the economic environment we have assumed."

Practice problems 6

Sunday, April 11, 2010
3:19 PM

- Consider the optimal growth model with a representative consumer with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t, \quad 0 < \beta < 1,$$

with technology

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \quad \delta \in (0, 1)$$

$$c_t \geq 0, \quad k_0 > 0 \text{ given}$$

$$f' > 0, \quad f'' < 0, \quad \lim_{k \rightarrow 0} f'(k) = +\infty, \quad \lim_{k \rightarrow \infty} f'(k) = 0$$

Here c_t = consumption / person

k_t = capital / worker

Let \bar{k} solve $f'(\bar{k}) = g + \delta$.

- For k_0 given formulate and solve the optimal planning problem.
- For $k_0 > \bar{k}$, describe the optimal time path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.
- For $k_0 < \bar{k}$, describe the optimal time path of (c_t, k_{t+1}) .
- Let the saving rate s_t be defined as the s_t that satisfies

$$k_{t+1} = s_t f(k_t) + (1-\delta) k_t$$

(in the Solow model, we assumed that $s_t = s$, an exogenous constant.) Here s_t in general varies along a $\{c_t, k_{t+1}\}$. Say what you can about how s_t varies as a function of k_t .

Hint: Form the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ c_t + \lambda_t [f'(k_t) + (1-\delta) k_t - c_t - k_{t+1}] \right\}$$

FONC:

$$(1) \quad c_t: 1 - \lambda_t \leq 0, = 0 \text{ if } c_t > 0, t \geq 0$$

$$(2) \quad k_t: \lambda_t [f'(k_t) + (1-\delta)] - \beta^{-1} \lambda_{t-1} = 0, \quad t \geq 1$$

$$(1) \Rightarrow \lambda_t \geq 1, \quad \lambda_t = 1 \Rightarrow c_t > 0$$

$$\lambda_t > 1 \Rightarrow c_t = 0$$

$$(2) \Rightarrow \lambda_t = \left(\frac{\beta^{-1}}{f'(k_t) + (1-\delta)} \right) \lambda_{t-1}, \quad \beta = \frac{1}{1+\rho}$$

$$\Rightarrow \lambda_t = \frac{(1+\rho)}{f'(k_t) + (1-\delta)} \lambda_{t-1}$$

this, $\frac{\lambda_t}{\lambda_{t-1}} < 1$ when $f'(k_t) + (1-\delta) > (1+\rho)$
or

$$f'(k_t) > \rho + \delta$$

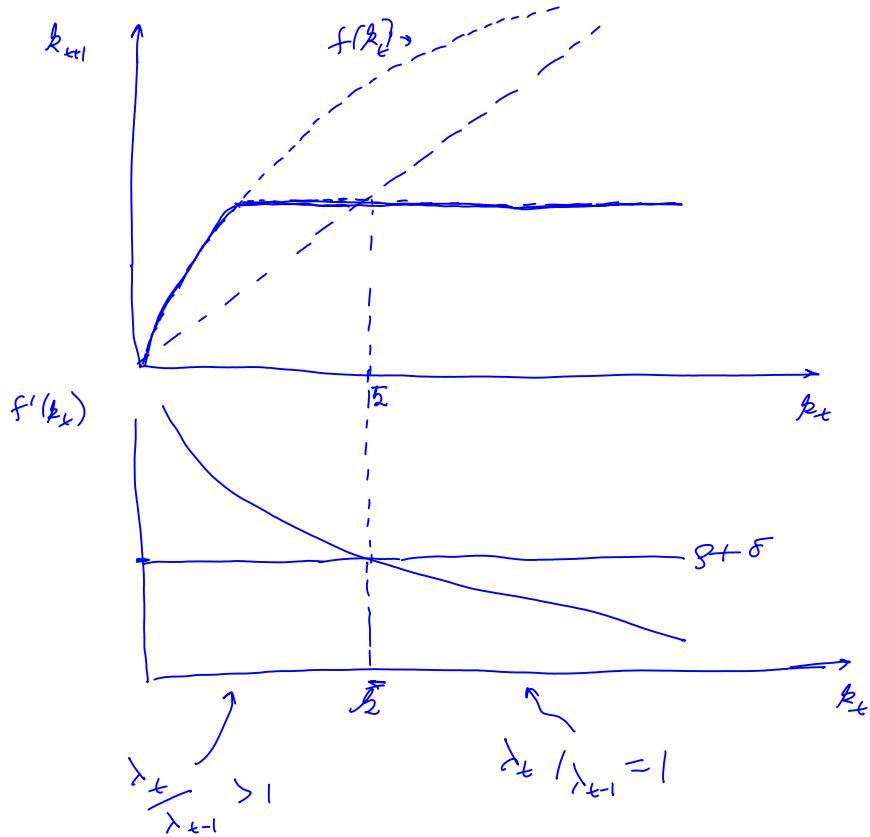
$$\frac{\lambda_t}{\lambda_{t-1}} = 1 \text{ when } f'(k_t) = \rho + \delta$$

$$\lambda_{t-1}$$

$$\frac{\lambda_t}{\lambda_{t-1}} > 1 \quad \text{when} \quad f'(k_t) < g + \delta$$

These outcomes indicate the following optimal path.

when $f'(k_t) > g + \delta$, $\lambda_t > \lambda_{t-1}$.



Optimal policy is :

if $k_0 > \bar{k}_t$,

set c_0 to solve

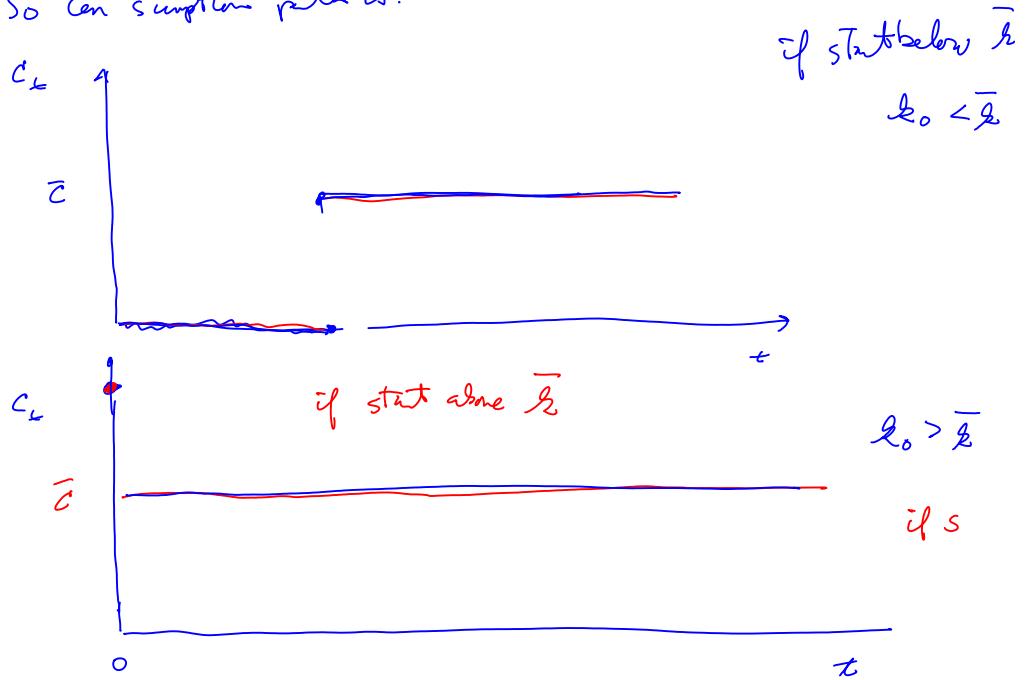
$$\bar{k}_t + c_0 = f(k_0) + (1-\delta)k_0$$

so that you jump to \bar{k}_t in one step.

if $k_0 < \bar{k}_t$,

save everything until you get to \bar{k} \Rightarrow
get to \bar{k} as fast as you can.

So consumption path is:



Note: with preference $\sum_{t=0}^{\infty} \beta^t c_t$ - consumer

is willing to substitute consumption across time easily —
leads to a very unsmooth consumption path
under the optimal plan.

Compare above optimal growth model with the Solow model. The Solow model has the same technology

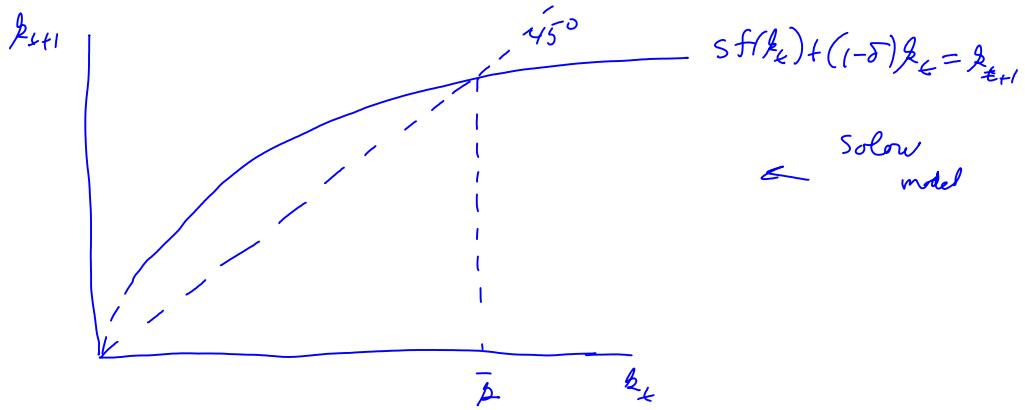
$$c_t + k_{t+1} = (1-\delta)k_t + f(k_t)$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + (f(\hat{k}_t) - c_t)$$

The Solow model assumes

$$\text{Saving} \equiv f(k_t) - c_t = s f(k_t)$$

where s is a constant saving rate.



Steady state :

$$(*) \quad s f(\bar{k}) = \delta \bar{k} \quad \text{at steady state of Solow model}$$

(*) is eqn that determines steady state of Solow model
as function of assumed constant saving rate s .

In the above optimal growth model with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t \quad (\text{i.e. } u(c_t) = c_t)$$

the saving rate is not constant. Instead of (*), the steady state capital stock is determined by
the usual equation

$$f'(\bar{k}) = g + \delta$$

The saving rate (s) adjusts to make this happen.

Of course, at the steady state of the optimal growth model

$$k_{t+1} = (1-\delta)k_t + (f(k_t) - c_t) \Rightarrow$$

$$\delta \bar{k} = f(\bar{k}) - \bar{c} \equiv \bar{s} f(\bar{k})$$

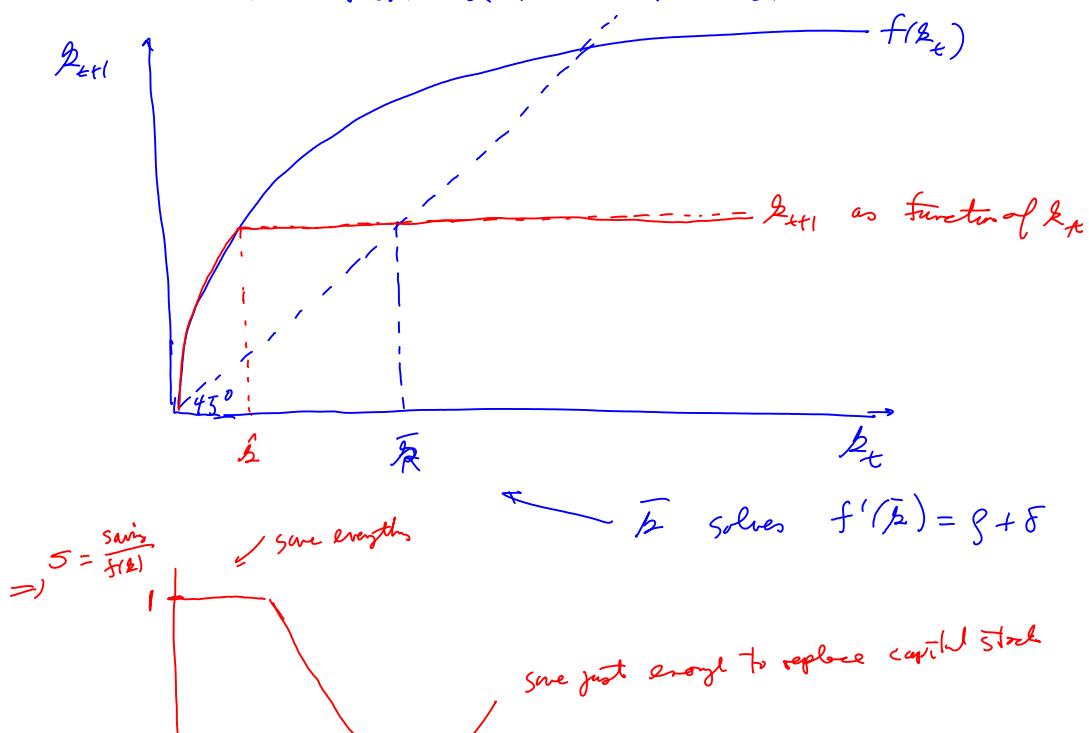
[where \bar{s} is saving rate at the steady state]

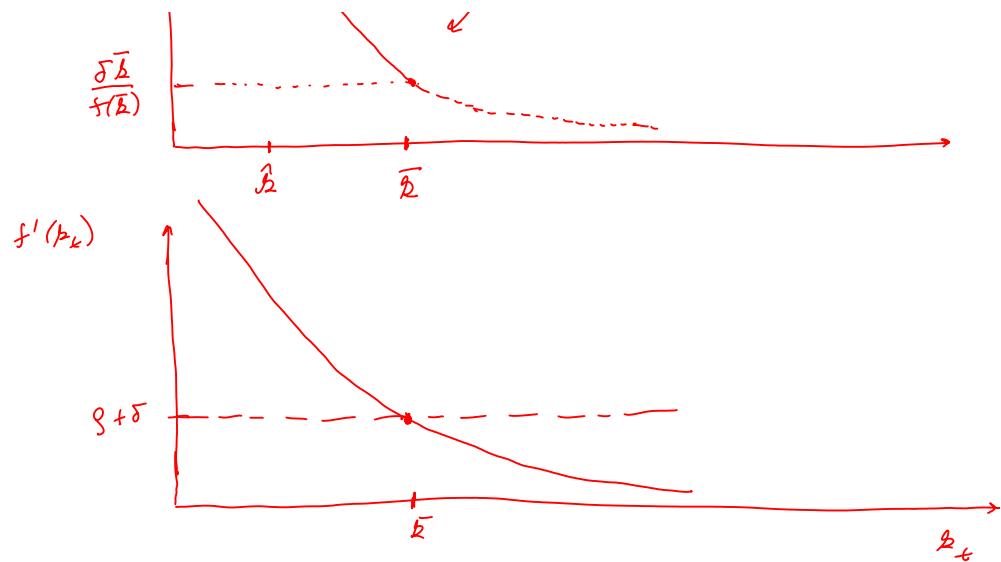
$$\bar{s} = \frac{\delta \bar{k}}{f(\bar{k})}$$

but here \bar{s} is an outcome - to be solved for as a function of \bar{k} - and not a determinant of \bar{k} as it is in the Solow model.

What about the saving rate $\frac{f(k) - c}{f(k)}$

away from the steady state? We can deduce the answer from our earlier picture:





Note how the rate of return on capital
co varies with the saving rate here. Compare
this with the constant saving rate assumed
in the Solow model.

Practice problems 7

Wednesday, April 28, 2010
2:35 PM

1. Trade and growth

Consider the problem of a planner in a small economy. When the economy is closed to trade, the planner chooses $\{c_t, k_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 , \quad \beta = \frac{1}{1+\gamma} \\ \gamma > 0$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0, 1)$$

$$\text{where } u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \quad , \quad \gamma > 0$$

$$f(k_t) = z k_t^\alpha \quad , \quad 0 < \alpha < 1$$

Let \bar{k} be the steady state value of k_t under the optimal plan.

a. Find a formula for \bar{k} .

b. Assume that $k_0 > \bar{k}$. Describe optimal time paths for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $R_{t+1} = (1-\delta) + f'(k_{t+1})$.

c. What is the steady state value of R_{t+1} ?

d. Is R_{t+1} less or greater than its steady state value when $k_{t+1} > \bar{k}$?

Part II.

Now assume that the economy is open to international trade

Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $\tilde{R} = \beta^{-1}$ at which the planner can borrow or lend.

The planner can use the proceeds of borrowing to purchase goods on the international market. These goods can be used to augment capital or to consume. Alternatively, the planner can sell capital to the rest of world at any time $t \geq 0$ and invest the proceeds abroad by lending at the risk-free gross interest rates $\beta^{-1} \equiv \tilde{R}$.

Let k_0^- be the level of initial capital ($k_0^- < \bar{k}$) just before the country opens up to trade just before time 0. Let k_0^+ be the same initial capital per capita $k_0 > \bar{k}$ studied in parts a.-d.

At time $t=-1$, after k_0^- was set, trade opens up. At $t=-1$, the planner can issue loans in amount B_{-1} and use the proceeds to purchase capital, thereby setting

$$k_0 = k_0^- + B_{-1}$$

where B_{-1} is denominated in time -1 consumption goods.

The bonds are one-period in duration and bear the

The bonds are one-period in duration and bear the constant world gross interest rate $\tilde{R} = \beta^{-1}$.

For $t \geq 0$, the planner faces the constraints

$$c_t + k_{t+1} + R B_{t-1} = f(k_t) + (1-\delta) k_t + B_t,$$

Here $R B_{t-1}$ is the interest and principal on bonds issued in $t-1$, and B_t is the amount of one-period bonds issued at t . If $B_t < 0$, the planner is lending to the rest of the world.

The planning problem is now to choose

$$\{c_t, k_t, B_t\}_{t=0}^{\infty} \text{ and } B_{-1}, \text{ subject}$$

to k_0 given. (Note that k_0 is now a choice variable and that k_0 is an initial condition.)

Solve the planning problem in this small open economy and compare the solution to the solution in the closed economy.

2. (A special permanent income model)

A consumer faces a time-invariant risk free gross-interest rate of $R = e^r$, where $r > 0$.

The consumer can borrow or lend at this rate up to a "no-Ponzi scheme" limit.

up to a "no-Ponzi scheme" limit.

The consumer chooses $\{c_t, b_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \quad U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

$$\beta = e^{-\rho} = \exp(-\rho), \quad \rho > 0$$

$$\text{s.t. } c_t + b_{t+1} \leq e^r b_t + y_t, \quad t \geq 0$$

$$b_0 = 0$$

$$y_{t+1} = y_t \exp(\sigma_\varepsilon \varepsilon_{t+1})$$

where $\varepsilon_{t+1} \sim N(0, 1)$. (The consumer knows y_t, b_t at time t when he sets c_t, b_{t+1}).

a. Find the first order necessary conditions for the firm's optimum problem and verify that they imply

$$1 = E_t e^r m_{t+1} = e^r E_t m_{t+1} \quad (\text{because } e^r \text{ is a constant})$$

$$\text{where } m_{t+1} \equiv \beta \frac{U'(c_{t+1})}{U'(c_t)}$$

b. Verify that

$$m_{t+1} = \beta \exp(-\gamma(c_{t+1} - c_t))$$

where $c_t = \log C_t$

c. The consumer is observed to set

$$* \left\{ \begin{array}{l} B_{t+1} = 0 \\ C_{t+1} = Y_{t+1} \end{array} \right. , \forall t \geq 0$$

Find a value for the constant net interest rate r for which policy $(*)$ is optimal.

Hint: where

$$C_{t+1} - C_t = M_c + \sigma_\varepsilon \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, 1)$$

we computed in class

$$\mathbb{E}_t m_{t+1} = \exp(-f - \gamma M_c + \frac{1}{2} \gamma^2 \sigma_\varepsilon^2)$$

If $C_t = Y_t$, then we must have

$$M_c = 0 \quad (\text{because } \log Y_{t+1} = \log Y_t + \sigma_\varepsilon \varepsilon_{t+1} \uparrow \text{no "drift"})$$

then to satisfy \checkmark drift M_c set to zero

$$\exp(-r) = \exp(-f + \frac{1}{2} \gamma^2 \sigma_\varepsilon^2)$$

or

$$\boxed{r = f - \frac{1}{2} \gamma^2 \sigma_\varepsilon^2}$$

$$r = g - \frac{1}{2} \gamma^2 \sigma_e^2$$

3. For a long time period, $t=1, \dots, T$, an economist has data on excess returns $\{g_{t+1}^i\}$ on assets $i=1, \dots, I$, where

$$g_{t+1}^i = R_{t+1}^i - R_{t+1}^f$$

where R_{t+1}^f is the risk free gross rate of return between t and $t+1$. Here T is a very large number.

The economist uses the long time series to estimate

$$\frac{\mathbb{E} g_{t+1}^i}{\sigma(g_{t+1}^i)} = \text{Sharpe ratio for asset } i$$

where he/she estimates $\mathbb{E} g_{t+1}^i$ as the sample average and $\sigma(g_{t+1}^i)$ as the sample standard deviation.

The economist has recorded the following outcomes:

$i :$	1	2	3	4	6	7	8	9
-------	---	---	---	---	---	---	---	---

Sharpe ratio :	-0.01	0.25	0.05	0.1	0.09	0.02	-0.03	0.24
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From these data, say what you can about

E_m and $\sigma(m)$, the mean and standard deviation of a stochastic discount factor that

- ..
-
- .. +

prices all assets.

Hint #1: "asset pricing is all about covariance"

Hint #2: " R^2 statistics in regressions are always less than 1".

4. (Hansen-Jagannathan bounds)

Let $R_{t+1} = \begin{pmatrix} R_{1,t+1} \\ R_{2,t+1} \\ \vdots \\ R_{n,t+1} \end{pmatrix}$ be an $m \times 1$ vector of gross returns on assets $i=1, \dots, n$

Let m_{t+1} be a stochastic discount factor.

The fundamental equation of asset pricing is

$$(1) \quad E_{t+1} m_{t+1} R_{t+1} = 1$$

a. Justify equation (1).

Hint: (1) is the first-order condition for the portfolio choice problem with Lagrangian

Lagrangian:

$$u(c_0) + \beta \int u(c_i(w)) f(w) dw \quad \left|_{P^{(m \times 1)}} \right.$$

$$+ \theta [y_0 - c_0 - p \cdot b] \quad \left(b \cdot (n \times 1) \right)$$

$$+ \int \lambda(\omega) [b \cdot x(\omega) - c_1(\omega)] d\omega$$

FONC: a) $\hat{p} = \int \beta \frac{u'(c_1(\omega))}{u(c_0)} x(\omega) f(\omega) d\omega$

$$c_1(\omega) = b \cdot x(\omega) + y_1(\omega)$$

Notre that (*) \Rightarrow

$$I = \int_{\Omega} m(\omega) R(\omega) f(\omega) d\omega$$

$$m(\omega) = \beta \frac{u'(c_1(\omega))}{u'(c_0)}, \quad R(\omega) = \frac{x(\omega)}{p}$$

$\underbrace{\phantom{\beta \frac{u'(c_1(\omega))}{u'(c_0)}}}_{\text{returns}}$

- b. Assume that we don't know the utility-functions or consumption levels of the households who hold assets but still want to extract implications of

$$E[m \cdot R] = I$$

for observations on a vector of returns R .

Think of m as a scalar random variable that we don't observe, but R as an $(n \times 1)$ vector of returns that we observe after returns have been realized (i.e., at date $t+1$).

Consider the population regression

$$m = a + R' b + e$$

1x1 1x1 1xn mx1 1x1

Show that the least squares orthogonality condition

$$E e \cdot R = 0$$

implies that

$$b = [\text{cov}(R, R)]^{-1} \text{cov}(R, m) .$$

(This is the population regression formula)

c. Show that

$$\text{cov}(R, m) = I - E_m E_R$$

and conclude that

$$(t) \quad b = [\text{cov}(R, R)]^{-1} (I - E_m E_R)$$

d. Use the population regression equation

$$m = a + R' b + e , \quad E e R = 0$$

to conclude that

$$(tt) \quad \sigma(m) \geq [b' \text{cov}(R, R) b]^{1/2}$$

e. Describe how (t) and (tt) together with data on estimated values of E_R , $\text{cov}(R, R)$ can be used to construct the Hansen-Tsakennatha.

can be used to construct the Hansen-Jagannathan
bounds on $(\bar{E}m, \Sigma(m))$. (Here, $\bar{E}R$ is
the mean return vector and $\text{cov}(R) = E(R - \bar{E}R)(R - \bar{E}R)'$
is the covariance matrix of returns.

Practice problems 3

Sunday, March 08, 2009
7:54 PM

1. A consumer-worker lives for two periods and has utility function

$$c_1 + \beta c_2 + \ln(1-h_1) + \beta \ln(1-h_2), \quad 0 < \beta < 1$$

where $c_t \geq 0$ is consumption in period $t=1, 2$;

$h_t \in [0, 1]$ is time spent working in period $t=1, 2$.

The worker's intertemporal budget constraint is

$$c_1 + \beta c_2 = w_1 + \beta w_2$$

where β^{-1} is the interest rate and $w_1 \geq 1, w_2 \geq 1$

are the real wages in period 1 and 2.

We require that $c_t \geq 0$ and $h_t \in [0, 1]$, $t=1, 2$

- a. Find the consumer's optimal level of c_1, c_2, h_1, h_2 as functions of w_1 and w_2 .

- b. Find the optimal level of c_1, c_2, h_1, h_2 when $w_1 = w_2 = 1$.

Practice problems 6

Tuesday, April 14, 2009
7:20 PM

1.

Jeff Sachs calculation (his article criticizing the Treasury plan)

Consider an asset that pays off

$X > 0$ with probability $\pi > 0$

and

$Y = 0$ with probability $(1-\pi)$

A risk neutral investor would be willing to

pay

$$\pi \cdot X + (1-\pi) Y = \pi X \quad \text{for the asset}$$

Now suppose that the government offers to

lend the buyer α . what he pays for the asset. The loan has interest rate $r=0$ and

is "non-recourse"- the borrower can walk away from the loan costlessly if the asset does not pay off. Here $\alpha \in (0,1)$

How much would a risk-neutral investor be willing to pay for the asset?

let $P =$ amount he is willing to pay

He borrows αP and puts up $(1-\alpha)P$ of his own money.

His expected payoff is

$$\pi(X - \alpha P) + (1-\pi) \max(0, Y - \alpha P)$$

\uparrow repay loan "o

payoff in
good state

$$= \pi(X - \alpha P)$$

In terms of his own money, the expected payoff
 $\pi(X - \alpha P)$ costs him $(1-\alpha)P$

so to equate the expected payoff to the cost, set

$$\pi(X - \alpha P) = (1-\alpha)P$$

$$\Rightarrow P = \frac{\pi}{\pi\alpha + (1-\alpha)} \cdot X$$

Note that $\lim_{\alpha \uparrow 1} P(\alpha) = X$ —

a. Use these formulas to replicate Jeff Sacks'

calculations.

b. Without the government-supplied leverage, the asset is worth

πX , while with the leverage it is worth

πX , while with the leverage it is worth

$$\frac{\pi}{\pi \alpha + (1-\alpha)} X .$$

The expected value of the loan to the government is

$$\underbrace{\pi \cdot \alpha p}_{\substack{\text{payoff in} \\ \text{good states}}} - \underbrace{\alpha p}_{\substack{\text{T_amt lent}}}$$

$$= -\alpha p (1-\pi) = -\frac{\alpha \pi (1-\pi)}{\pi \alpha + (1-\alpha)} X < 0$$

Note that the value paid by the investor + the value of the loan \Rightarrow the government is

$$\frac{\pi X}{\pi \alpha + (1-\alpha)} - \frac{\alpha \pi (1-\pi)}{\pi \alpha + (1-\alpha)} X$$

$$= \frac{\pi [1 - \alpha(1-\pi)]}{\pi \alpha + (1-\alpha)} X = \pi X = \text{price of asset without the government loan.}$$

\circlearrowleft
 $1 - \alpha(1-\pi)$

Remark: Sachs' point is that the Treasury plan

transfers $\frac{\alpha \pi (1-\pi)}{\pi \alpha + (1-\alpha)}$ from the Treasury to

the original owner of the asset that offers

payoff $(X, 0)$ in the good state, bad state.

—

2. Consider the overlapping generations model with capital discussed in class.

At each date $t \geq 1$ there are born N identical

young agents who work when young, consume when old. The consumption stream of a young agent born in t is (c_{yt}, c_{t+1})



The young person's utility function is $u(c_{t+1}) - \alpha$ / she doesn't value consumption when young.

Each young person is endowed with one unit of labor that he/she supplies inelastically.

The technology and financial structure are as described in class. In particular,

$$\text{for } t \geq 1, \quad y_t = k_t^\beta, \quad y_t = \text{output per young worker}$$

$$\text{wage} = w_t = (1-\beta)k_t^\beta \quad k_t = \text{capital per young worker}$$

$$\text{rental rate} \quad f_t = \beta k_t^{\beta-1} \quad N = \# \text{ of young workers,} \\ \text{on capital} \quad \text{constant over time}$$

$$\beta \in (0, 1)$$

$$\delta \in (0, 1) \quad \text{depreciation rate}$$

$$r_t = f_t - \delta \quad \text{return on savings from } t-1 \text{ to } t$$

$$\text{Per capita savings} = w_t$$

e = social security tax paid when young

= social security payment received when old

$k_{t+1} = a_{t+1}$ savings devoted at t

$k_{t+1} = w_t - e$

$$(*) \quad k_{t+1} = (1-\beta)k_t + e$$

consumption when old , $t \geq 0$ (includes initial old)

$$c_{0,t+1} = (r_{t+1} + 1)k_t + e \quad \text{at } t=1$$

a. Suppose that $e = 0$ for all $t \geq 0$.

Find formulas for the steady state values of

k_t , w_t , r_t , $c_{0,t+1}$.

b. Suppose that $\beta = 1/4$, $\delta = 1/3$. Find

'steady' state values of k , w , r , c_0 .

↑
consumption of old.

c. Now suppose that $\beta = 1/4$, $\delta = 1/3$, and

k_1 = steady state value computed in part b.

Starting at time $t=1$, suppose that we institute

a social security scheme with $e = .1$ —

this is a transfer to the old at t financed by an equal tax on the young. Find the

path of k_{t+1} , c_{0t} , r_t , w_t associated with this new policy for $t = 1, 2, \dots, 10$.

Which generations are better off with $e = .1$? How will $e = 0$? Which are worse off?

Hint: You will need to write a matlab or excel program to iterate on (*) starting from the given value of k_1 .

d. Now suppose that $\beta = 1/5$ and $\delta = 1/3$ and $e = 0$. Find steady state values of k , w , r , and c_0 .

e. Starting from the steady state k that you found in d as the initial capital owned by the initial old, on which they earn payoffs of r_t , k_{t+1} , suppose there is started a social security transfer system at time 1 with $e = .1$. Find values of c_{0t} , r_t , k_{t+1} for $t \geq 1, \dots, 10$.

Compared to the steady states you computed in part d, which generations are better/worse off?

Hint: You will have to write an excel or matlab program to iterate on (*).

The next several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\gamma}, \quad \gamma > 0$$

where c_t is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma = 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

The technology is .

$$y_t = f(k_t) = z k_t^\alpha, \quad 0 < \alpha < 1$$

where y_t is output per unit labor and

k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta)k_t + x_t, \quad 0 < \delta < 1$$

The government finances its expenditures by levying some combination of a flat rate tax τ_{ct} on the value of consumption goods purchased at t , a flat rate tax of T_{mt} on the value of labor earnings at t ,

a flat rate tax $\tau_{k,t}$ on earnings from capital at t , and a lump sum tax of $\tau_{ct,t}$ in the t consumption goods per worker at time t .

Let $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ be a price system.

3. Consider an economy of the type described above in which $g_t = \bar{g} > 0$ for all $t \geq 0$

and in which initially the government finances all expenditures by lump sum taxes.

- a. Find a formula for the steady state capital labor ratio k_t for this economy. Find formulas for the steady state levels of c_t and

$$R_t = [(1-\delta) + (1-\tau_{k,t+1}) f'(k_{t+1})].$$

- b. Now suppose that starting from $k_0 = \bar{k} =$ steady state level you computed in part a, the government suddenly increases the tax on earnings on capital to a constant level $\tau_k > 0$. The government adjusts lump sum taxes to keep the government budget balanced. Describe competitive equilibrium time paths for c_t, k_{t+1}, R_t and their relationship to corresponding values in the old steady state. Hint: use L.H.L: $c_t = k_t$.

Steady states that you derived in part a.

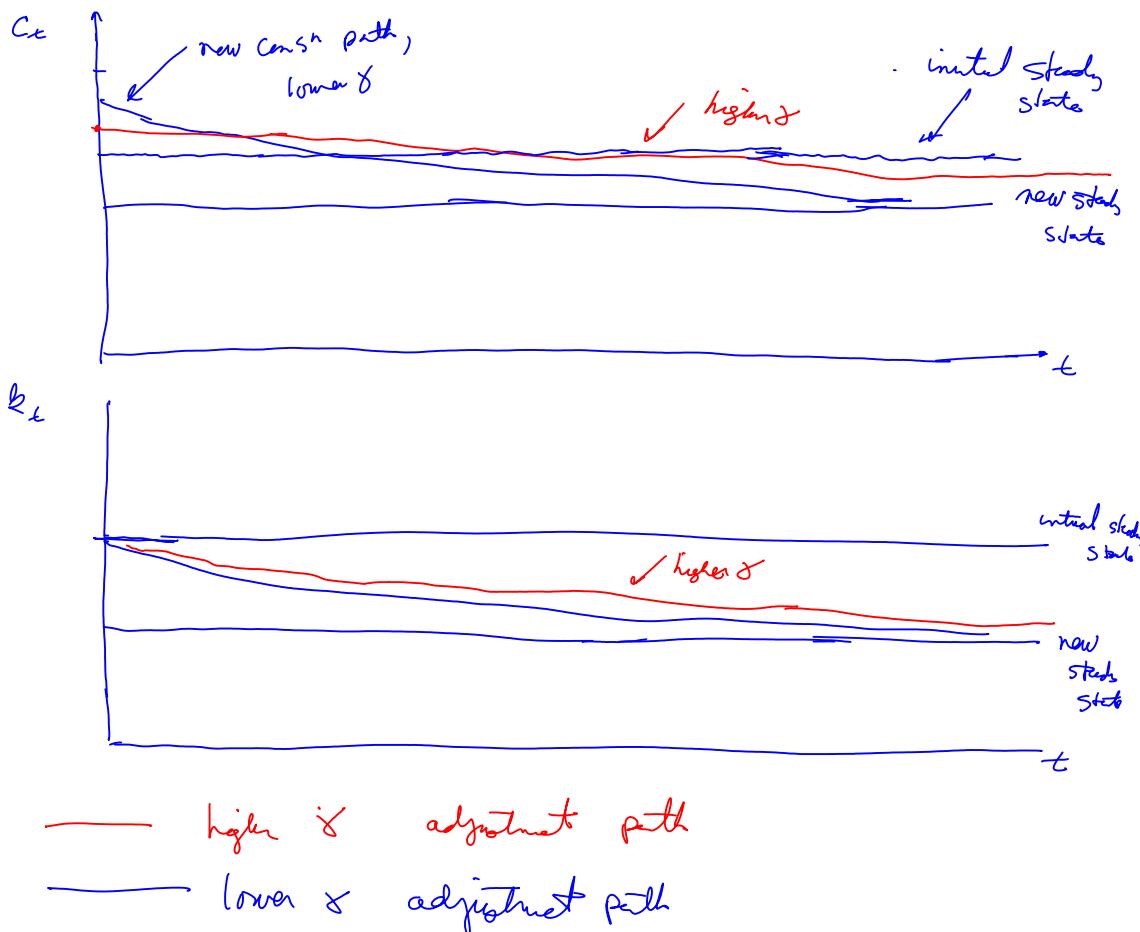
c. Describe how the shapes of the paths that you found in part b. depend on the curvature parameter γ in the utility function

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma}. \text{ Higher values of } \gamma \text{ imply}$$

high curvature and more aversion to consumption paths that fluctuate.

Higher values of γ imply that the consumer values smooth consumption paths even more.

Thus, here is what we expect:



This look this way because of the consumer's attitude about consumption variability. The interest rate and k_+ adjust to make this happen.

Remark: formula in section 11.7.3 of the typed chapter verifies this by giving a formula for

λ as a function of γ where

λ was the decay rate discussed in class.

d. Starting from the steady state \bar{k} that you computed in part a, now consider a situation in which the government announces at time 0 that starting in period 10 the tax on earnings from capital T_k will rise permanently to $\bar{T}_k > 0$. The government adjusts its lump sum taxes to balance its budget.

- i) find new steady values for k_+ , c_+ , R_+ .
- ii) Describe the shapes of the transition paths from the initial steady state to the new one for k_+ , R_+ , c_+ .
- iii) decide how the shapes of these transition paths depend on the curvature parameters

paths depend on the curvature parameter γ in the utility function $u(c)$.

Hint: when γ is bigger, consumers now strongly prefer smoother consumption paths.

4. Trade and growth, version II

Consider the problem of a planner in a small economy. When the economy is closed to trade, the planner chooses $\{c_t, k_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 , \quad \beta = \frac{1}{1+\delta} \\ \delta > 0$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0, 1)$$

$$\text{where } u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \quad , \quad \gamma > 0$$

$$f(k_t) = \gamma k_t^\alpha \quad , \quad 0 < \alpha < 1$$

let \bar{k} be the steady state value of k_t under the optimal plan.

a. Find a formula for \bar{k} .

b. Assume that $k_0 > \bar{k}$. Describe time paths

for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $R_{t+1} = (1-\delta) + f'(k_t)$.

c. What is the steady state value of R_{t+1} ?

d. Is R_{t+1} less or greater than its steady state

value when $k_{t+1} > \bar{k}$?

Part II.

e. Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$ at which the planner can borrow or lend. In particular, the planner is free to use the following plan. The planner can sell all of its capital \bar{k}_0 and simply consume the interest payments.

Let \bar{k}_0 be the level of initial capital ($\bar{k}_0 > \bar{k}$) just before the country opens up to trade just before time 0. Let \bar{k}_0' be the same initial capital per capita $\bar{k}_0' > \bar{k}$ studied in parts a.-d.

At time $t=-1$, after \bar{k}_0 was set, trade opens up. At $t=-1$, the planner sells \bar{k}_0 in exchange for IOU's from the rest of the world in the amount $A_{-1} = \bar{k}_0$.

The bonds A_{-1} are one-period in duration and bear the

constant world gross interest rate $R = \beta^{-1}$.

After the sale of $\bar{k}_0 = A_{-1}$,

the planner has zero capital and so shuts down the technology. Instead, the planner uses the asset market to

Smooth consumption. The planner thus chooses (A_{t+1}, c_t) to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad \beta = \frac{1}{1+r}$$

$$c_t + \beta A_{t+1} = A_t, \quad A_0 = \bar{A}_0$$

$$A_{t+1} = \beta^{-1} [A_t - c_t]$$

Form the Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t [\beta^{-1}(A_t - c_t) - A_{t+1}] \right\}$$

FONC:

$$c_t: u'(c_t) - \lambda_t \beta^{-1} = 0$$

$$A_t: -\beta^{-1} \lambda_{t-1} + \beta^{-1} \lambda_t = 0$$

$$\lim_{T \rightarrow \infty} \beta^T \lambda_T A_{T+1} = 0$$

$$\Rightarrow \lambda_t = \lambda_0 \quad \forall t$$

$$u'(c_t) = u'(c_0) \quad \forall t$$

$$c_t = c_0 \quad \forall t$$

Then roll over assets & just consume interest:

$$A_{t+1} = \beta^{-1} [A_t - c_t] \Rightarrow$$

$$\bar{A} = \beta^{-1} [\bar{A} - c_0] \Rightarrow$$

$$(1 - \beta^{-1}) \bar{A} = -\beta^{-1} c_0$$

m

$$\beta^{-1} c_0 = (\beta^{-1} - 1) \bar{A}$$

$$c_0 = (1 - \beta) \bar{A}$$

$$c_0 = g \beta \bar{A}, \quad \bar{A} = A_0 = k_0^{-1}$$

Thus, the optimal path for consumption is

$$c_t = g \bar{A} = g k_0^{-1}$$

f. Compute the time path of (c_t, k_t, R_t)

that you computed in parts a, b, c, d with "no trade" with part e one "with trade" in which the government "shuts down the home technology" and lives entirely from returns on foreign assets.

Can you say which path the representative consumer would prefer?

* Hint: I think it could go either way for reasons that might be easier to appreciate after you look at part g.

g. Now return to the economy in part e with $k_0 > \bar{k}$ from part d. Assume that the planner is free to borrow or lend capital at the fixed gross interest rate of $\beta^{-1} = 1 + g$, as before.

But now assume the planner chooses the optimal amount of k_0^{-1} to sell off and so not

not necessarily sell off the entire k_0^{-1} and possibly continues to operate the technology.

i) Find the solution of the planning problem.

Hint: you might want to review your solution to problem 2 of lecture problems 4.

ii) Explain why it is optimal not to shut

down the technology. Hint: Starting from

having shut the technology down, think of

putting a small amount ε of capital into the technology — this earns $\gamma \varepsilon^\alpha$ and

costs $g\varepsilon$ in terms of forgone wheat. because $0 < \alpha < 1$

$g\varepsilon < \gamma \varepsilon^\alpha$ for small ε . — The

technology is very productive for small ε —

so use it.

"

"

Practice problems 7

Monday, April 27, 2009
11:16 AM

1. Consider a consumer who wants to choose $\{c_t\}_{t=0}^T$ to

maximize $\sum_{t=0}^T \beta^t c_t$, $0 < \beta < 1$

subject to the intertemporal budget constraint

$$\sum_{t=0}^T R^{-t} [c_t - y_t] = 0, R > 1$$

where $c_t \geq 0$ and $y_t > 0$ for $t = 0, \dots, T$

Here $R > 1$ is the gross interest rate ($1 + r$), $r > 0$.

Assume R is constant. Here $\{y_t\}_{t=0}^T$ is an

exogenous sequence of "labor income".

a. Assume that $\beta R < 1$. Find the optimal path

$$\{c_t\}_{t=0}^T.$$

b. Assume that $\beta R > 1$. Find the optimal path

$$\{c_t\}_{t=0}^T.$$

c. Assume that $\beta R = 1$. Find an optimal path $\{c_t\}_{t=0}^T$.

2. Consider the Knesen-Wallace model of moral

Recall that when the bank takes maximal risk by investing all of its deposits in the state A claim, it exposes the insurance agency to a lot of risk.

When $Q_A = \frac{D}{P_A}$ and $Q_B = 0$, the expected value of the losses of the insurance agency are

$$\pi_B \underbrace{(r+i) D}_{\text{paid in state } B \text{ because } Q_B = 0}.$$

Can you think of a scheme for pricing deposit insurance that

- a. deters the bank from excessive risk-taking, and
- b. protects the deposit insurance agency from expected losses?

The idea is that the deposit insurance agency will charge the bank a fee for deposit insurance with the fee depending on (i) its total deposits, and (ii) its investment portfolio.

3. Term structure of interest rates.

This problem

assumes the

following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\gamma}, \quad \gamma > 0$$

where c_t is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma = 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

The technology is .

$$y_t = f(k_t) = z k_t^\alpha, \quad 0 < \alpha < 1$$

$$z > 0$$

where y_t is output per unit labor and

k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta)k_t + x_t, \quad 0 < \delta < 1$$

c_t = consumption per worker

The government finances its expenditures by levying some combination of a flat rate tax T_{ct} on the value of consumption goods purchased at t , a flat rate tax of T_{nt} on the value of labor earnings at t , a flat rate tax T_{kt} on earnings from capital at t , and a lump sum tax of T_{ht} in the t

consumption goods per worker at time t .

Let $\{q_t^0, r_t^0, w_t^0\}_{t=0}^\infty$ be a price system.

Here the superscript 0 means the time 0 price system.

a. Recall that we can represent

$$q_t^0 = q_{t,0}^0 m_{0,1} m_{1,2} \dots m_{t-1,t}$$

where $m_{t-1,t} = \frac{q_t^0}{q_{t-1}^0}$

$$\text{and } m_{t-1,t} \equiv \exp(-r_{t-1,t}) \approx \frac{1}{1+r_{t-1,t}}$$

Further recall that the t period long yield is

satisfies

$$q_t^0 = \exp(-t r_{0,t})$$

or

$$r_{0,t} = \frac{1}{t} [r_{0,1} + r_{1,2} + \dots + r_{t-1,t}]$$

Now suppose that at $t=0$, $k_0 = \bar{k}$, where

\bar{k} is the steady state aggregate for an economy with constant $g_t = \bar{g} > 0$ and all expenditures financed by lump sum taxes. Find

q_t^0 for this economy.

b. Plot $r_{t-1,t}$ for this economy for $t=1, 2, \dots, 10$.

c. Plot $r_{0,t}$ for this economy for $t=1, \dots, 10$.

(this is what Bloomberg plots)

d. Now assume that at time 0, starting from

$k_0 = \bar{k}$ for the steady state you computed in part a,

the government unexpectedly and permanently raises the tax rates

on income from capital $\gamma_{kt} = \gamma_b > 0$ to a

positive rate.

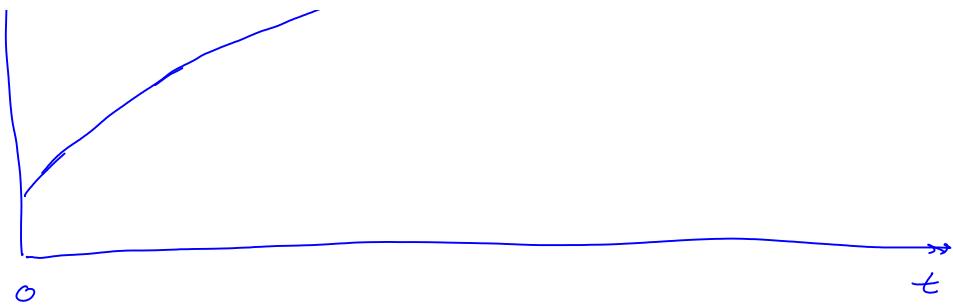
i) Plot $r_{t-1,t}$ for $t=1, \dots, 10$ for this economy. Explain how you got this outcome.

ii) Plot $r_{0,t}$ for $t=1, \dots, 10$ for this economy. Explain how you got this outcome.

4. The following problem assumes the same economic environment as the previous problem-i.e., the "growth model" with fiscal policy.

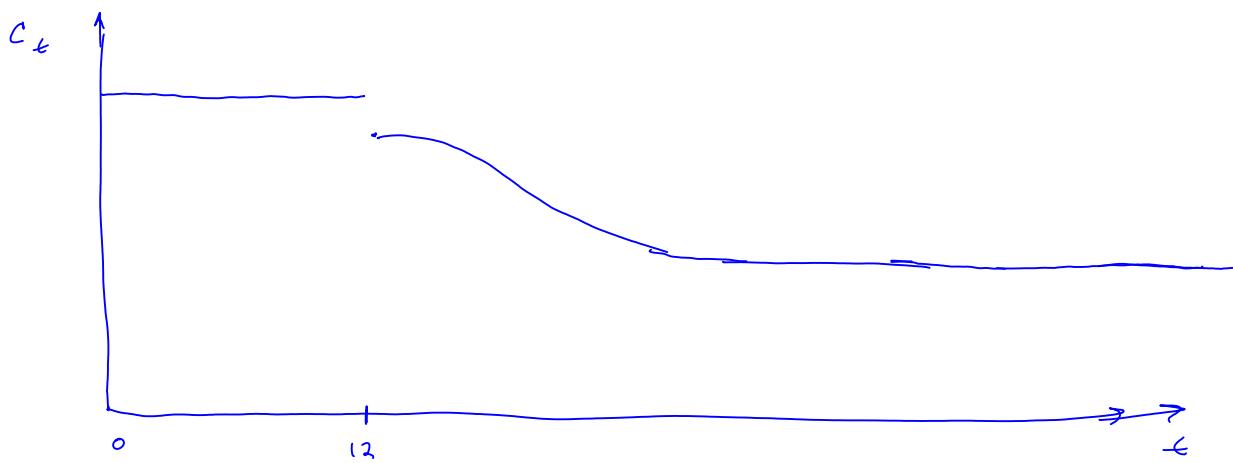
Suppose that you observe the following path for consumption per capita





Say what you can about likely behavior over time of k_t , R_t $[((1-\delta) + (1-\gamma_{k_t}) f'(k_t))]$, g_t , and γ_{k_t} . (You are free to make up any story you like that is consistent with the model.)

5. Assume the same economic environment as in the previous two problems. Assume that someone has observed the following time path for c_t :



- a. Describe a consistent set of assumptions about fiscal policy that explains this time path for c_t . In doing so, please distinguish carefully between changes in taxes and expenditures that are foreseen versus unforeseen.

versus unforeseen.

- b. Decide what is happening to k_t , R_t , and g_t over time. Make whatever assumptions you must to get a complete but consistent story - here "consistent" means "consistent with the economic environment we have assumed."

Practice problems 8

Monday, May 04, 2009
8:45 AM

1. Consider the optimal growth model with a representative consumer with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t, \quad 0 < \beta < 1,$$

with technology

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \quad \delta \in (0, 1)$$

$$c_t \geq 0, \quad k_0 > 0 \text{ given}$$

$$f' > 0, \quad f'' < 0, \quad \lim_{k \rightarrow 0} f'(k) = +\infty, \quad \lim_{k \rightarrow \infty} f'(k) = 0$$

Here c_t = consumption / person

k_t = capital / worker

Let \bar{k} solve $f'(\bar{k}) = g + \delta$.

- a. For k_0 given formulate and solve the optimal planning problem.

- b. For $k_0 > \bar{k}$, describe the optimal time path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

- c. For $k_0 < \bar{k}$, describe the optimal time path of (c_t, k_{t+1})

- d. Let the saving rate s_t be defined as the s_t that satisfies

$$k_{t+1} = s_t f(k_t) + (1-\delta) k_t$$

(in the Solow model, we assumed that $s_t = s$, an exogenous constant.) Here s_t in general varies along a $\{c_t, k_{t+1}\}$. Say what you can about how s_t varies as a function of k_t .

Hint: Form the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ c_t + \lambda_t [f(k_t) + (1-\delta)k_t - c_t - k_{t+1}] \right\}$$

FONC:

$$(1) \quad c_t: 1 - \lambda_t \leq 0, = 0 \text{ if } c_t > 0, t \geq 0$$

$$(2) \quad k_t: \lambda_t [f'(k_t) + (1-\delta)] - \beta^{-1} \lambda_{t-1} = 0, t \geq 1$$

$$(1) \Rightarrow \lambda_t \geq 1, \lambda_t = 1 \Rightarrow c_t > 0$$

$$\lambda_t > 1 \Rightarrow c_t = 0$$

$$(2) \Rightarrow \lambda_t = \frac{\beta^{-1}}{f'(k_t) + (1-\delta)} \lambda_{t-1}, \beta = \frac{1}{1+\rho}$$

$$\Rightarrow \lambda_t = \frac{(1+\rho)}{f'(k_t) + (1-\delta)} \lambda_{t-1}$$

this, $\frac{\lambda_t}{\lambda_{t-1}} < 1$ when $f'(k_t) + (1-\delta) > (1+\rho)$
or

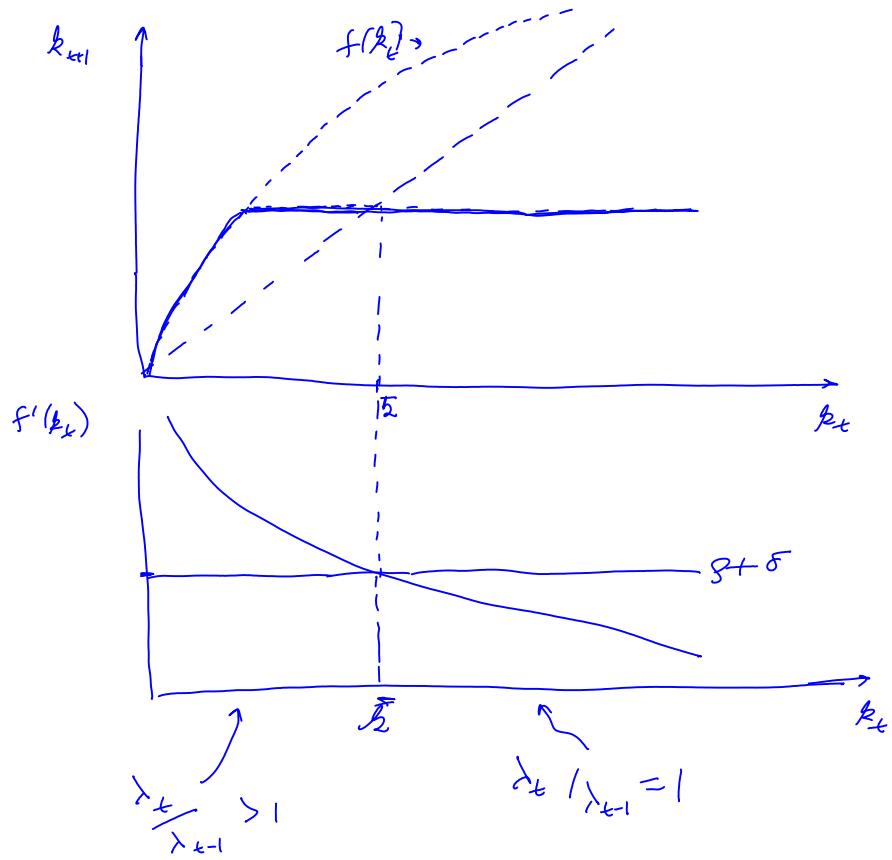
$$f'(k_t) > \rho + \delta$$

$$\frac{\lambda_t}{\lambda_{t-1}} = 1 \quad \text{when} \quad f'(k_t) = g + \delta$$

$$\frac{\lambda_t}{\lambda_{t-1}} > 1 \quad \text{when} \quad f'(k_t) < g + \delta$$

These outcomes indicate the following optimal path.

when $f'(k_t) > g + \delta$, $\lambda_t > \lambda_{t-1}$.



Optimal policy is :

if $k_0 > \bar{k}$,

set c_0 to solve

$$\bar{k} + c_0 = f(k_0) + (1-\delta)k_0$$

so that you jump to \bar{k} in one step.

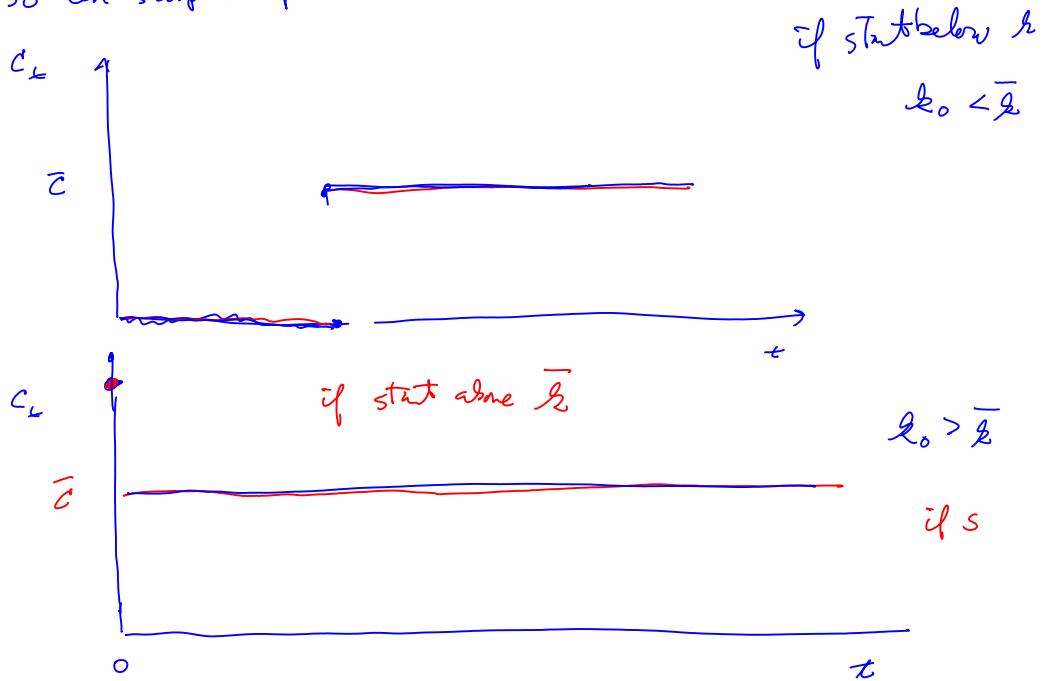
$U^* \quad U^* T \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

if $k_0 < \bar{k}$,

save everything until you get to \bar{k} \Rightarrow

get to \bar{k} as fast as you can.

So consumption path is:



Note: with preference $\sum_{t=0}^{\infty} \beta^t c_t$ - consumer

is willing to substitute consumption across time easily —

leads to a very unsmooth consumption path

under the optimal plan.

Compare above optimal growth model with the Solow model. The Solow model has the same technology

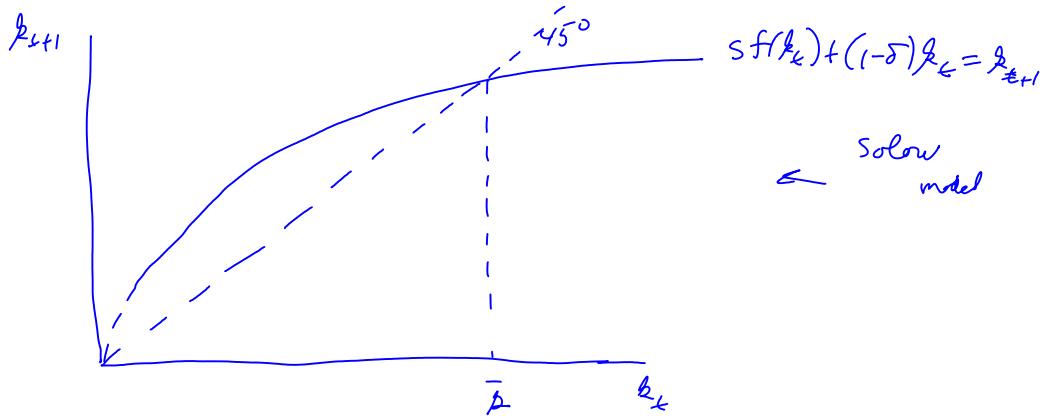
$$c_t + k_{t+1} = (1-\delta)k_t + f(k_t)$$

$$\stackrel{\sigma}{\approx} k_{t+1} = (1-\delta)k_t + (f(k_t) - c_t)$$

The Solow model assumes

$$\text{saving} \equiv f(k_t) - c_t = s f(k_t)$$

where s is a constant saving rate.



Steady state :

$$(*) \quad s f(\bar{k}) = \delta \bar{k} \quad \text{at steady state of Solow model}$$

(*) is eqn that determines steady state \bar{k} of Solow model
as function of assumed constant saving rate s .

In the above optimal growth model with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t \quad (\text{i.e. } u(c_t) = c_t)$$

the saving rate is not constant. Instead of (*), the
steady state capital stock is determined by
the usual equation

$$f'(\bar{k}) = g + \delta$$

The saving rate(s) adjust to make this happen.

Of course, at the steady state of the optimal growth model

$$k_{t+1} = (-\delta)k_t + (f(k_t) - c_t) \Rightarrow$$

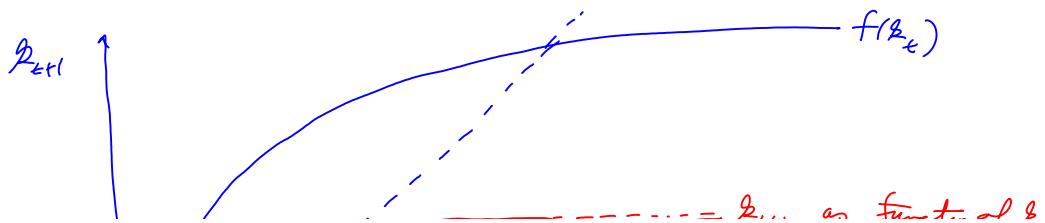
$$\delta \bar{k} = f(\bar{k}) - \bar{c} \equiv \bar{s} f(\bar{k}) \quad \text{[where } \bar{s} \text{ is saving rate at the steady state]}$$

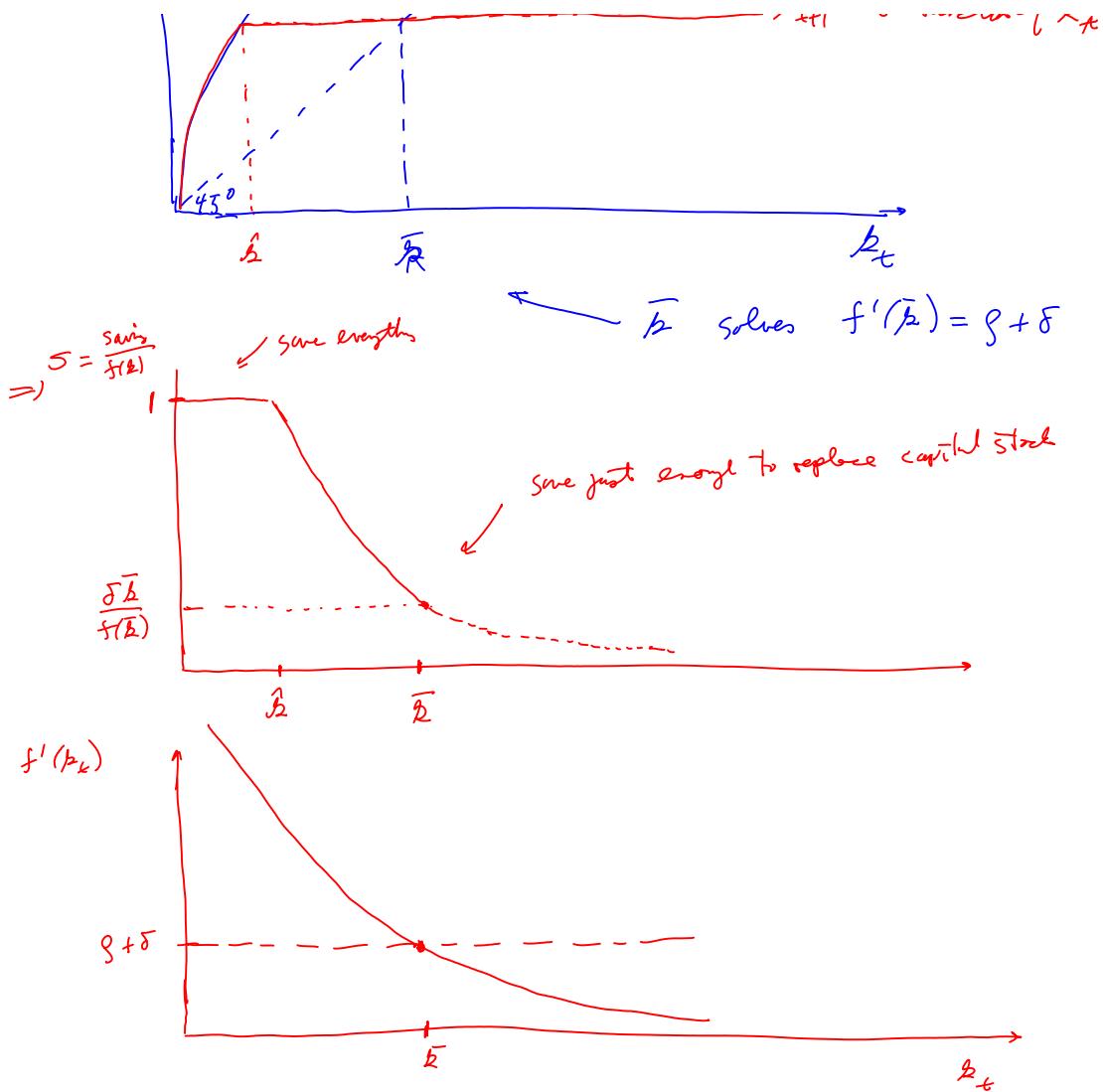
$$\bar{s} = \frac{\delta \bar{k}}{f(\bar{k})}$$

but here \bar{s} is an outcome - to be solved for as a function of \bar{k} - and not a determinant of \bar{k} as it is in the Solow model.

What about the saving rate $\frac{f(k) - c}{f(k)}$

away from the steady state? We can deduce the answer from our earlier picture:





Note how the rate of return on capital
co-varies with the saving rate here. Compare
this with the constant saving rate assumed
in the Solow model.

Practice problems 9

Tuesday, May 05, 2009
9:28 AM

1. Consider a competitive equilibrium version of the practice problems

8, problem 1 economy. There is a government that chooses sequences $\{g_t, \gamma_{kt}, \gamma_{ht}\}_{t=0}^{\infty}$

$\begin{matrix} g \\ \gamma_{kt} \\ \gamma_{ht} \end{matrix} \begin{matrix} \uparrow \\ \text{govt} \\ \text{expenditures} \end{matrix} \begin{matrix} \uparrow \\ \text{capital} \\ \text{tax} \end{matrix} \begin{matrix} \uparrow \\ \text{lump sum tax} \end{matrix}$

and a price system $\{q_t, w_t, r_t\}_{t=0}^{\infty}$.

Suppose we start from a prior steady state $k_0 = \bar{k}$

in which the government financed a constant $g_t = \bar{g}$

with lump sum taxes and $\gamma_k = 0$. At

$t=0$ the government announces a one-and-for all increase in γ_k to a constant level $\gamma_{k0} = \bar{\gamma}_k > 0$.

It adjusts lump sum taxes to leave the government budget balances. It leaves $g_t = \bar{g}$ as before.

Describe equilibrium paths of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $\{q_t\}_{t=0}^{\infty}$.

Hint: The household problem is

$$L = \sum_{t=0}^{\infty} \beta^t c_t + \mu \left[\sum_{t=0}^{\infty} \left(r_t (1 - \gamma_{k0}) k_t + w_t n_t - q_t \gamma_{ht} \right. \right. \\ \left. \left. - q_t (c_t + k_{t+1} - (1 - \delta) k_t) \right) \right]$$

Final:

FONC:

$$c_t: \beta^t - \gamma q_{ft} \leq 0, \Rightarrow \text{if } c_t > 0, t \geq 0$$

$$k_t: r_t(1-\gamma_{kt}) + q_{ft}(1-\delta) - q_{ft+1} = 0, t \geq 1$$

$$\Rightarrow (+) q_{ft} = r_{t+1}(1-\gamma_{k,t+1}) + q_{t+1}(1-\delta)$$

Firm problem: \Rightarrow

$$(+1) q_{ft} f'(k_t) = c_t \\ q_{ft} [f(k_t) - k_t f'(k_t)] = w_t$$

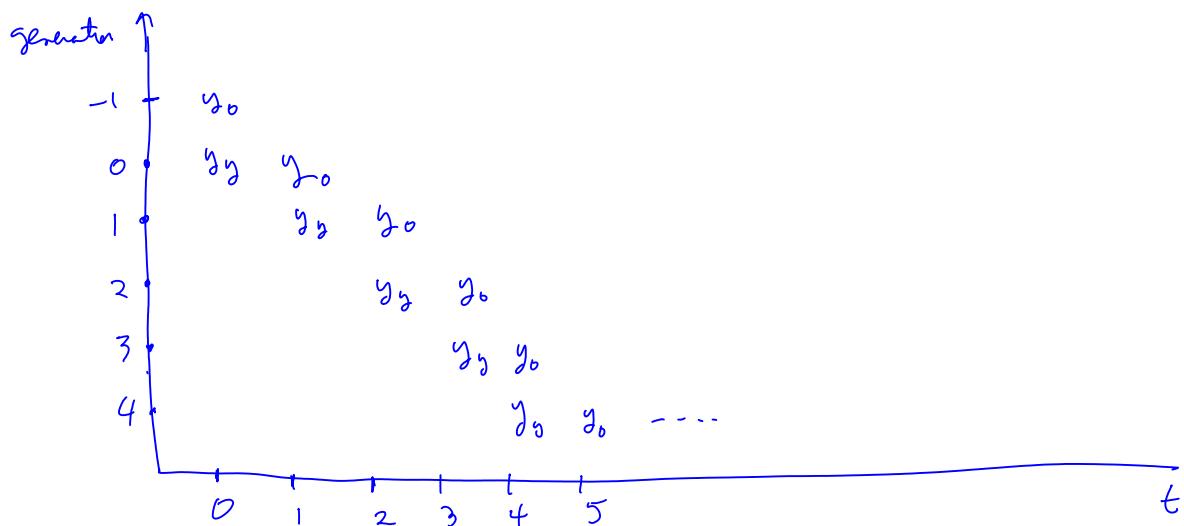
Substitute (+1) into (+) \Rightarrow

$$q_{ft} = q_{ft+1} f'(k_{t+1})(1-\gamma_{k,t+1}) + q_{t+1}(1-\delta) \\ = q_{t+1} [f'(k_{t+1})(1-\gamma_{k,t+1}) + (1-\delta)]$$

Practice problems 5

Tuesday, April 14, 2009
5:08 PM

- Consider the following simple overlapping generations model. At each date $t \geq 0$ there are born N two period lived agents. At $t=0$ there are also N old people - these are the "initial old". This is a "pure exchange" - no production" economy. The good arrives exogenously ("from heaven") each period. Each old person receives $y_o > 0$ of the consumption good and each young person receives $y_y > y_o > 0$ units of the consumption good. The good cannot be stored, but it can be exchanged. The endowment pattern is thus



—

The aggregate endowment at $t \geq 0$ is the

$$N y_0 + N y_o.$$

For $t \geq 0$, each young person has preferences described by the following utility function:

$$\ln(c_{yt}) + \ln(c_{ot+1})$$

where c_{yt} = consumption of a young person at t
 c_{ot+1} = consumption of an old person at t .

If a time t young person could save a_{t+1} and earn $a_{t+1}(1+r_{t+1})$ next period by doing so, this person would face the following intertemporal optimum problem:

$$\max_{c_{yt}, c_{ot+1}} \ln c_{yt} + \ln c_{ot+1}$$

subject to

$$c_{yt} + a_{t+1} = y_y \quad (1)$$

$$c_{ot+1} = (1+r_{t+1}) a_{t+1} + y_o \quad (2)$$

If $a_{t+1} > 0$, the person is a lender or saver

If $a_{t+1} < 0$, the person is a borrower.

Form the Lagrangian -

$$\begin{aligned} L = & \ln c_{yt} + \ln c_{ot+1} \\ & + \lambda \left[y_y + \frac{y_o}{1+r_{t+1}} - c_{yt} - \frac{c_{ot+1}}{1+r_{t+1}} \right] \end{aligned}$$

$$1 + r_{t+1} \quad 1 + c_{t+1} \rightarrow$$

Note:- we have solved (2) for a_{t+1} and substituted into (1) to get the single inter-temporal constraint

$$y_2 + \underbrace{\frac{y_0}{1+r_{t+1}}}_{\text{PV of endowment}} = c_{yt} + \underbrace{\frac{c_{0,t+1}}{1+r_{t+1}}}_{\text{PV of consumption}}$$

The first order conditions are:

$$\begin{aligned} \frac{1}{c_{yt}} &= \lambda \\ \frac{1}{c_{0,t+1}} &= \lambda \frac{1}{1+r_{t+1}} \end{aligned} \quad \Rightarrow \quad \frac{1}{c_{0,t+1}} = \frac{1}{c_{yt}} (1+r_{t+1})^{-1}$$

or

$$(*) \quad c_{0,t+1} = c_{yt} (1+r_{t+1})$$

and the intertemporal constraint.

Substitute (*) into the intertemporal constraint

$$c_{yt} + c_{yt} = y_2 + \frac{y_0}{1+r_{t+1}}$$

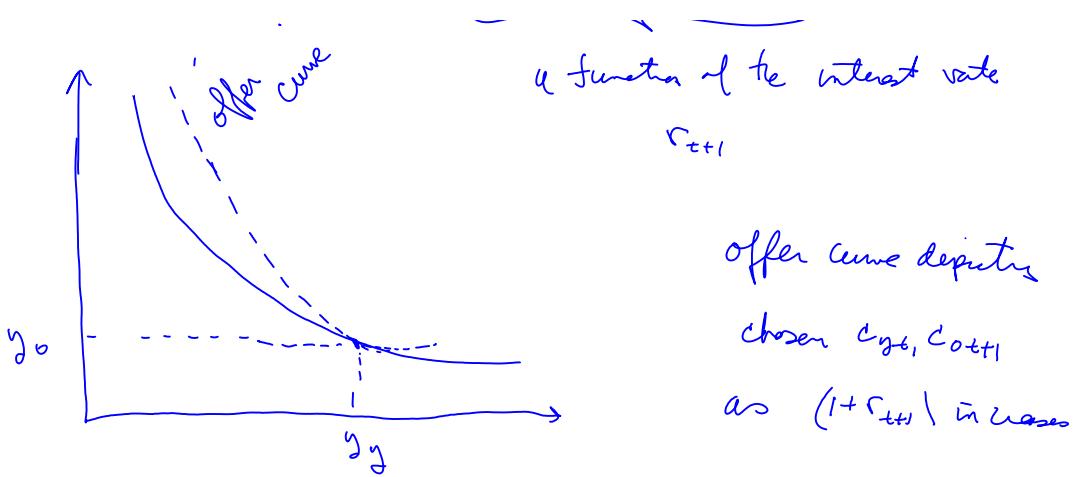
$$c_{yt} = \frac{1}{2} \left[y_2 + \frac{y_0}{1+r_{t+1}} \right]$$

\Rightarrow

$$\text{Savings} = a_{t+1} = y_2 - c_{yt} = \frac{1}{2} \left[y_2 - \frac{y_0}{(1+r_{t+1})} \right]$$

Aggregate savings of the young:

$$N(y_2 - c_{yt}) = \underbrace{\frac{N}{2} \left[y_2 - \frac{y_0}{(1+r_{t+1})} \right]}_{\text{Agg. Savings}}$$



Note: The interest rate that reconciles the

consumer to consume his endowment, i.e., to

$$\text{set } (c_{y_t}, c_{0,t+1}) = (y_y, y_0)$$

is found by setting

$$a_{t+1} = \frac{1}{2} \left(y_y - \frac{y_0}{1+r_{t+1}} \right) = 0 \quad (\text{savings} = 0)$$

$$\Rightarrow y_y = \frac{y_0}{1+r_{t+1}}$$

$$\Rightarrow 1+r_{t+1} = \frac{y_0}{y_y} < 1$$

$$\Rightarrow r_{t+1} < 1$$

The interest rate $1+r_{t+1} = \frac{y_0}{y_y}$ makes the

consumer content to consume his endowment

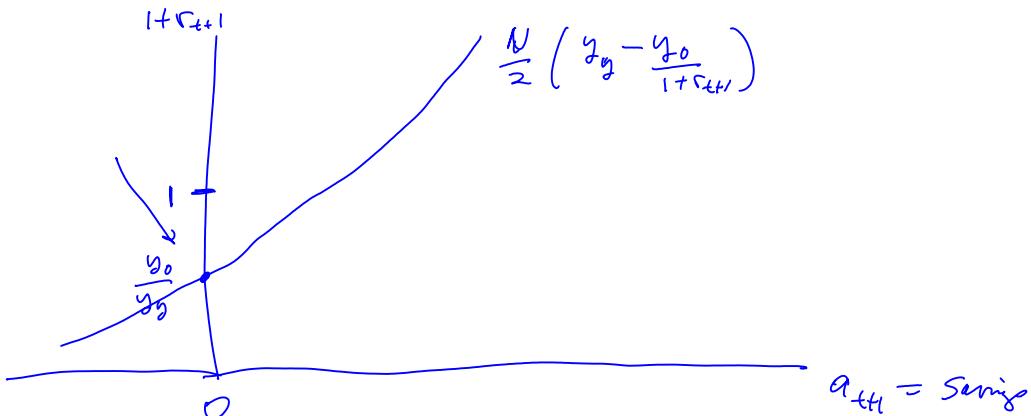
$1+r_{t+1} = \frac{y_0}{y_y} = \text{slope of indifference curve}$

through the endowment point.

(This is another version of the midterm question!)

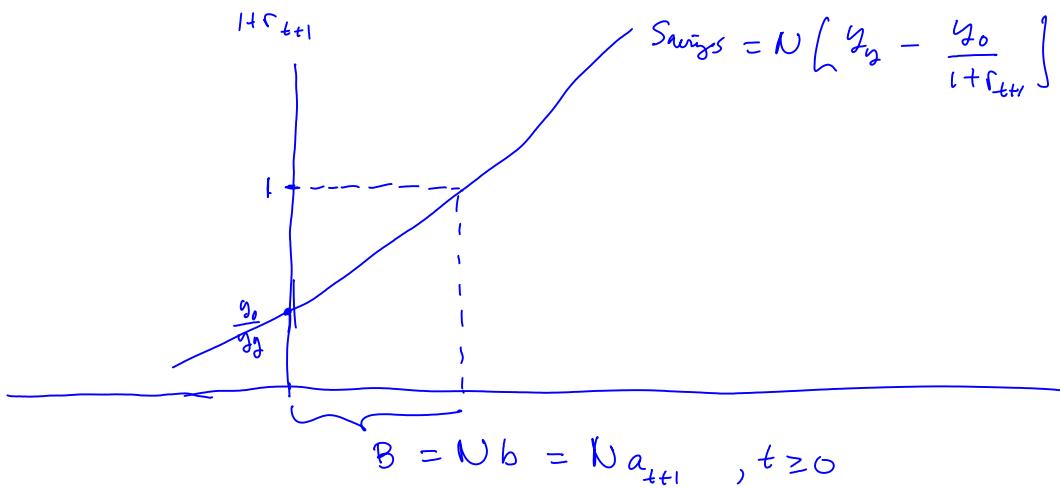
Remark: In the economy without a government - no social security or government debt, there is no one for a young person to borrow or lend with - why - because the only people who could repay are other young people - same cohort - and all of them have the same preferences & the same endowments. (If we introduced within-generation heterogeneity either of preferences or endowments, then we could generate non zero saving for some and non zero borrowing for others. Aggregate saving would still be zero).

So the situation looks like this:



$(1 + r_{t+1}) \hat{=} \frac{y_0}{y_y} < 1$ is a competitive equilibrium
 interest rate with zero

Interest rate with zero
saving.



Now consider the following scheme. At time $t=0$ the government gives each old person b units of a bond where $b = \frac{1}{2} [y_1 + y_0] - y_0$

$$= \frac{1}{2} [y_1 - y_0]$$

That is, An old person at $t=0$ sells the bond to a young person at time 0 in exchange for $\frac{1}{2} [y_1 - y_0]$, goods. So the old person ends up consuming $\frac{1}{2} [y_1 + y_0] > y_0$ at $t=0$.

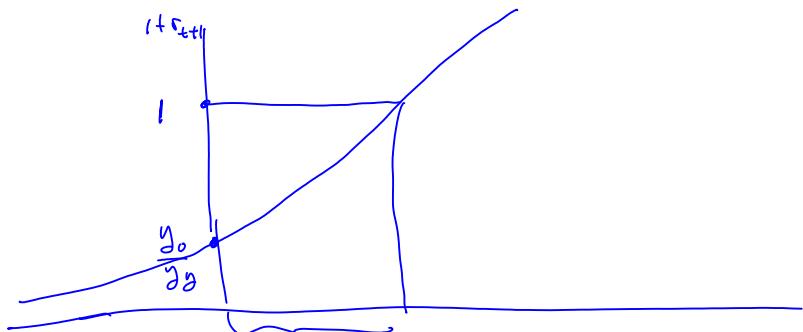
The young person at $t=0$ then sets $a_1 = \frac{1}{2} [y_1 - y_0] = b$, carries it into period 1, and sells it to another young person for $(1+r_{t+1}) b = b$ (so r_{t+1}) time 1 consumption goods. This enables the

time 1 consumption goods. This enables the time 0 young person to consume

$$(c_{y0}, c_{01}) = \left(\frac{1}{2} [y_0 + y_1], \frac{1}{2} [y_0 + y_1] \right)$$

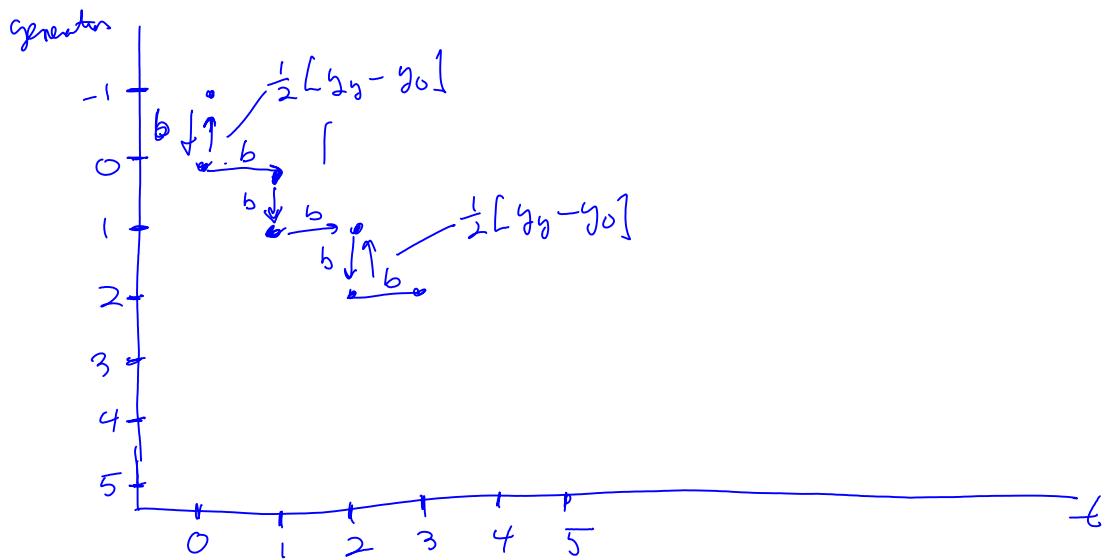
and thus completely smooth consumption over his

lifetime. Thus we have



$$B = Nb = \text{dissaving of old} = \text{saving of young}$$

Each new generation of young people purchases government debt from the old people. So the picture is



The government debt gets passed from generation to generation in exchange for goods.

This government debt scheme is equivalent to a social security scheme. Explain.

Dynamics -

generalization of model to allow time-varying (r_{t+1}, a_{t+1}) . Previous allocation will be a special case - a steady state with $r_{t+1} = 0$.

Saving function:

$$(1) \quad a_{t+1} = \frac{1}{2} \left[y_0 - \frac{1}{1+r_{t+1}} y_0 \right]$$

government budget

$$(2) \quad a_{t+1} = a_t (1+r_t), \quad t \geq 0$$

a_0 = transfer to initial old.

define $R_t = (1+r_t)$

$$(1) + (2) \Rightarrow$$

$$\begin{aligned} \frac{1}{2} \left[y_0 - \frac{y_0}{R_{t+1}} \right] &= \frac{1}{2} \left[y_0 - \frac{y_0}{R_t} \right] R_t \\ &= \frac{1}{2} [y_0 R_t - y_0] \end{aligned}$$

Notice: there are two steady states:

$$(1) \quad R_{t+1} = R_t = 1 \quad \text{and} \quad a_{t+1} = a_t = \bar{a} = \frac{1}{2} [y_0 - y_0]$$

$$(2) \quad R_{t+1} = R_t = \frac{y_0}{y_0} \quad \text{and} \quad a_{t+1} = a_t = \bar{a} = 0$$

above we "chose" the first one. Let's study the robustness of the choice and also we

whether there are also non-steady state equilibria.

To study: the dynamic system

$$\left[y_y - \frac{y_0}{R_{t+1}} \right] = [y_y R_t - y_0] \Rightarrow$$

after several lines of algebra

$$R_{t+1} = \frac{1}{\left(1 - \frac{y_y}{y_0} (R_t - 1) \right)}$$
$$= f(R_t)$$

note: this has two steady states

$$\bar{R} = f(\bar{R}) \Rightarrow \bar{R} = 1 \quad \text{or} \quad \frac{y_0}{y_y} < 1$$

Compute:

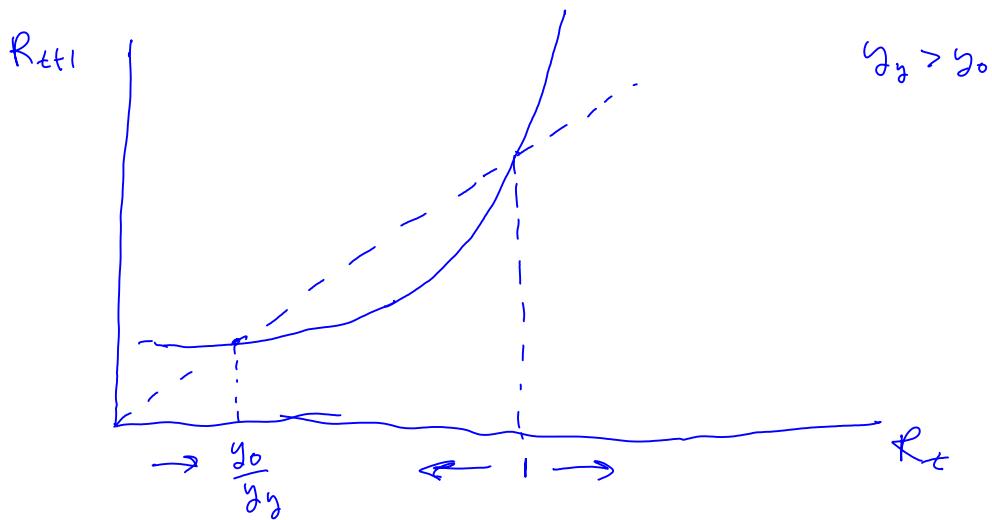
$$\frac{d f(R)}{d R} = \frac{y_y}{\left(y_0 ((y_y \cdot (R - 1)) / y_0 - 1) \right)^2}$$

we can calculate that:

$$\left. \frac{d f(R)}{d R} \right|_{R=1} = \frac{y_y}{y_0} > 1$$

$$\left. \frac{d f(R)}{d R} \right|_{R=\frac{y_0}{y_y}} = \frac{y_0}{y_y} < 1$$

∴ the picture is like this.



$\frac{y_0}{y_y}$ is stable

1 is unstable

If government sets $a_1 = \frac{1}{2} [y_y - y_0]$

and transfers $\frac{1}{2} [y_y - y_0]$ to the initial old,

the equilibrium has $a_{t+1} = a_1$ for $t \geq 0$

and $R_{t+1} = 1$ for all $t \geq 0$.

government "rolls over" a constant debt.

If $0 < \bar{a}_1 < \frac{1}{2} [y_y - y_0]$,

saving at 1 must equal

$$\bar{a}_1 = \frac{1}{2} \left[y_y - \frac{y_0}{R_1} \right] < \frac{1}{2} [y_y - y_0]$$

$$\Rightarrow -\frac{y_0}{R_1} < -y_0$$

$$\Rightarrow \frac{y_0}{R_1} > y_0$$

$$\Rightarrow \frac{1}{R_1} > 1 \Rightarrow \underline{\underline{R_1 < 1}}.$$

$$\therefore a_2 = R_1 \bar{a}_1 < \bar{a}_1$$

$$\text{and so on } \Rightarrow a_{t+1} \rightarrow 0$$

$$\text{and } R_{t+1} \rightarrow \frac{y_0}{y_y},$$

Thus, not issuing enough government debt at $t=0$ causes the value of debt to fall over time toward zero and "wedge" the "social security" like transfer scheme.

Why not try setting

$$a_1 > \frac{1}{2} [y_y - y_0] ?$$

It won't work. It would require an $R_1 > 1$ and $R_2 > R_1, \dots$, which would lead $a_{t+1} \rightarrow +\infty$.

That is not feasible: Eventually the value of government debt would exceed the total endowment in the economy. This is not possible because each period, the current old are selling the entire government debt to the young and consuming the entire proceeds. These proceeds are bounded above by $[y_y + y_0] = \text{total endowment of economy at time } t$.

Remark: an isomorphism ...

We can set

$$a_{t+1} = \frac{M}{P_t}, \quad R_{t+1} = \frac{P_t}{P_{t+1}}$$

P_t = price level at t - $\frac{\$}{\text{the } t \text{ goods}}$

M = unbaked fuel money - $\$$ the government gives to the old

$$\frac{M_t}{P_t} = \text{real balances} \sim \frac{\$}{\$/\text{one good}} \sim \text{the } t \text{ goods}$$

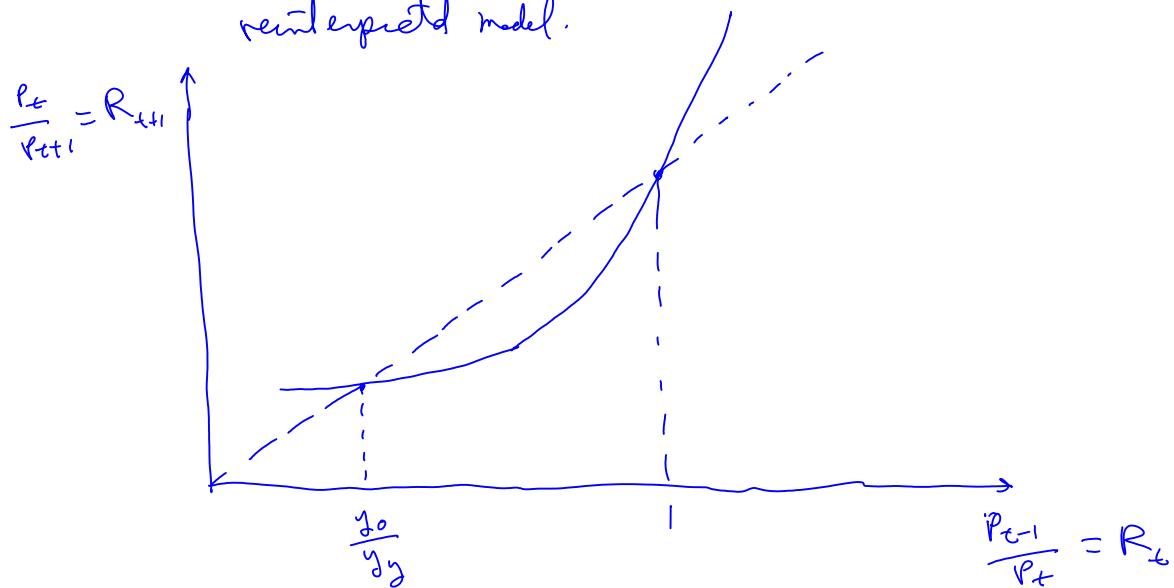
= Savings

$$R_{t+1} = \frac{P_t}{P_{t+1}} \sim \frac{\text{the } t+1 \text{ goods}}{\text{the } t \text{ goods}} = \text{gross real return on money}$$

$R_{t+1} > 1$ if there is deflation

< 1 if there is inflation.

The above analysis applies to this
reinterpreted model.



saving function can be written:

$$\frac{M}{P_t} = \frac{1}{2} \left[y_y - \frac{y_o}{R_{t+1}} \right]$$

$$= \frac{1}{2} \left[y_y - y_o \frac{P_{t+1}}{P_t} \right]$$

\Rightarrow

$$M = \frac{1}{2} [P_t y_y - y_o P_{t+1}]$$

$$2M = y_y P_t - y_o P_{t+1}$$

(*) $\Rightarrow P_t = \frac{2}{y_y} M + \frac{y_o}{y_y} P_{t+1}, t \geq 0$

Solution of this difference equation is

$$P_t = \underbrace{\frac{2}{y_y} \left[\frac{1}{1 - \frac{y_o}{y_y}} \right] M}_{\text{fundamental solution}} + \underbrace{C \left(\frac{y_y}{y_o} \right)^t}_{\text{bubble}}$$

for any $C > 0$

Note: if $C = 0 \Rightarrow$

$$\frac{M}{P_t} = \frac{y_y}{2} \left[1 - \frac{y_o}{y_y} \right]$$

$$= \frac{1}{2} [y_y - y_o] = \bar{a}_1 \quad \text{in our government debt model.}$$

and $P_t = \text{constant}$

$$P_t = M \neq \left(\frac{1}{2} (y_y - y_0) \right)$$

↑ quantity theory of money.

$\uparrow M \Rightarrow \uparrow P$ proportionately

But if $C > 0$

$$\frac{M}{P_t} \rightarrow 0 \quad \text{and}$$

$$\frac{P_{t+1}}{P_t} \rightarrow \frac{y_y}{y_0} > 1$$

\Rightarrow exploding price drive value of
money to zero.

Explain in terms of "fictitious" character of money.

- describe how equation

$$P_t = \frac{2}{y_y} M + \frac{y_0}{y_y} P_{t+1}, \quad t \geq 0$$

expresses how current price level depends on
expectation of future price level (value of money)

$$\frac{M}{P_t} = \frac{1}{2} \left[y_y - \frac{y_0}{(P_t/P_{t+1})} \right]$$

demand for real balances vary
inversely with expected inflation.

Scratch

Sunday, March 08, 2009
7:28 PM

Homework problem :- answer + 0 problem

$$c_0 + \beta c_1 + \frac{A}{1-\gamma} (1-h_0)^{1-\gamma} - \beta \frac{A}{1-\gamma} (1-h_1)^{1-\gamma}$$

$h \in (0, 1)$

$$\text{s.t. } c_0 + \beta c_1 \leq \omega_0 h_0 + \beta \omega_1 h_1$$

$$L = c_0 + \beta c_1 + \frac{A}{1-\gamma} (1-h_0)^{1-\gamma} + \beta \frac{A}{1-\gamma} (1-h_1)^{1-\gamma}$$

$$+ \lambda [\omega_0 h_0 + \beta \omega_1 h_1 - c_0 - \beta c_1] \quad , c_0, c_1 > 0$$

FONC:

$$c_0 : 1 - \lambda \leq 0 \quad = 0 \text{ if } c_0 > 0 \quad \Rightarrow \left. \begin{array}{l} 1 = \lambda \\ \text{some} \end{array} \right\} \quad \text{because some}$$

$$c_1 : 1 - \lambda \leq 0 \quad , = 0 \text{ if } c_1 > 0 \quad \underline{\text{some}} \quad \text{time}$$

$$h_0 : A (1-h_0)^{-\gamma} + \lambda \omega_0 = 0 \quad \text{interior}$$

$$h_1 : \beta A (1-h_1)^{-\gamma} + \lambda \beta \omega_1 = 0 \quad "$$

$$\Rightarrow A (1-h_0)^{-\gamma} = -\omega_0$$

$$A (1-h_1)^{-\gamma} = -\omega_1$$

$$(1-h_0)^{-\gamma} = -\frac{\omega_0}{A}$$

$$(1-h_0) = \left(-\frac{\omega_0}{A} \right)^{-\frac{1}{\gamma}}$$

e.g. $\gamma = 1$

$$(1-h_0) = \left(-\frac{\omega_0}{A} \right)^{-1} = -\frac{A}{\omega_0}$$

$$h_0 = 1 + \frac{A}{\omega_0}$$

$$A (1-h_0)^{-\gamma} = -\omega_0$$

$$(1-h_0)^{-\gamma} = -\frac{\omega_0}{A}$$

$$(1-h_0) = \left(-\frac{\omega_0}{A} \right)^{-\frac{1}{\gamma}}$$

e.g. $\gamma = 1$

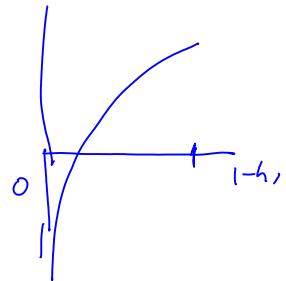
$$(1-h_0) = \left(-\frac{\omega_0}{A} \right)^{-1}$$

$$1-h_0 = -\frac{A}{\omega_0}$$

$$1 + \frac{A}{\omega_0} = h_0$$

$$\ln(1-h_0), h_0=1 \Rightarrow \ln(0)=-\infty.$$

$$L = c_1 + \beta c_2 + \ln(1-h_1) + \beta \ln(1-h_2) \\ + \lambda [w_1 h_1 + \beta w_2 h_2 - c_1 - \beta c_2]$$



FORC:

$$c_1: 1 - \lambda \leq 0 \Rightarrow c_1 > 0$$

$$c_2: 1 - \lambda \leq 0 \Rightarrow c_2 > 0$$

$$h_1: \frac{-1}{1-h_1} + w_1 = 0 \Rightarrow w_1 = \frac{1}{1-h_1}$$

$$1-h_1 = \frac{1}{w_1}$$

$$h_2: \frac{-1}{1-h_2} + w_2 = 0 \Rightarrow h_1 = 1 - \frac{1}{w_1}$$

$$h_2 = 1 - \frac{1}{w_2}$$

$$c_1 + \beta c_2 = w_1(1 - \frac{1}{w_1}) + \beta w_2(1 - \frac{1}{w_2}), h \geq 0$$

$$= (w_1 - 1) + \beta (w_2 - 1)$$

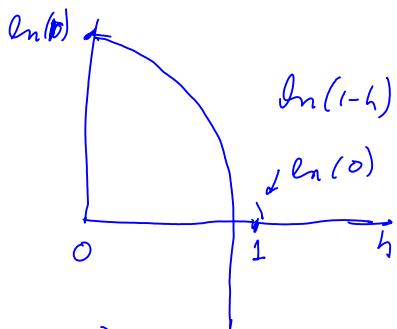
$$w_1 = 1, w_2 = 1 \Rightarrow$$

$$h_1 = 1 - \frac{1}{w_1} = 0,$$

$$w_1 \geq 1, w_2 \geq 1.$$

$$\ln(1-h_1) = g(h)$$

$$g'(h) = \frac{-1}{(1-h)} < 0, g''(h) = \frac{1(-1)}{(1-h)^2} < 0$$



Scratch

Tuesday, March 24, 2009
3:41 PM

Wilkinson problem 10.

$$Y_t = z K_t$$

$$\frac{Y_t}{N_t} = z \frac{K_t}{N_t}$$

$$\frac{K_{t+1}}{N_{t+1}} - \frac{N_{t+1}}{N_t} = (1-\delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}$$

$$k_{t+1}(1+n) = (1-\delta)k_t + \gamma_t$$

$$z f(k_t) = z k_t \Rightarrow$$

$$f(k_t) = \frac{Y_t}{N_t} \\ = z \frac{K_t}{N_t} = z k_t$$

$$k_{t+1} = \frac{s}{(1+n)} z k_t + \frac{(1-\delta)}{1+n} k_t$$

$$k_{t+1} = \left[\frac{s z}{1+n} + \frac{(1-\delta)}{1+n} \right] k_t$$

$$k_{t+1} = \left[\frac{s z + (1-\delta)}{1+n} \right] k_t$$

New Problem

Wednesday, March 10, 2010
5:57 PM

1. (15 points)

A consumer chooses $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad u' > 0, \quad u'' < 0$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} q_t^0 c_t \leq \sum_{t=0}^{\infty} q_t^0 y_t$$

where $\{y_t\}_{t=0}^{\infty}$ is an exogenous endowment

and q_t^0 = the price of one unit of consumption at time t , measured in units of time 0 consumption,

so

$$q_0^0 = 1 \quad (\text{a normalization})$$

Assume that

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma} \quad , \quad \gamma > 0, \quad \gamma \neq 1$$

$$= \ln c \quad \text{if } \gamma = 1$$

(Remember that $q_t^0 = \frac{1}{(1 + g_{0t})^t}$

where g_{0t} is the yield to maturity on a t -period zero-coupon bond)

Consider a "Lucas model" where there is a

representative consumer and prices $\{q_t^0\}_{t=0}^\infty$ must adjust to induce the consumer to set $c_t = y_t \quad \forall t \geq 0$.

a. Suppose $y_t = y_0 \delta^t, \quad 0 < \delta < \frac{1}{\beta}$.

Compute q_t^0 . Compute f_{0t} .

b. In the special case of $\gamma = 1$ (log utility),

compute q_t^0 and f_{0t} .

c. When $\delta = 1$, compute q_t^0 .

d. Is the one-period interest rate f_{01} .

"high" or "low" when $\delta > i$?

e. Is the one-period interest rate f_{01} "high" or "low"

when δ