Treasuries and Tax Farms*

Wei Jiang† Thomas J. Sargent‡ Neng Wang§ Jinqiang Yang¶

August 22, 2025

Abstract

We use continuous-time recursive contracts theory to extend Barro's (1979) tax-smoothing model to include explicit randomness, an impatient representative consumer, access to an outside market in a Shiller (1994) macro security, and the option to default on debts. A benevolent Department of Treasury's optimal financing problem is dual to a selfish tax farmer's problem. The two problems imply the same taxation and risk-management policies and deliver the same expected utilities to a representative consumer. The tax farmer promises the representative consumer an expected discounted present value of utility that is a one-to-one function of the value of the government's initial debt.

Keywords: Treasury, tax farm, duality, limited commitment, default, debt limit.

*We thank Michael Bordo and Yuliy Sannikov for useful discussion of aspects of our problem.

†Department of Industrial Engineering and Decision Analytics, Hong Kong University of Science and Technology. E-mail: weijiang@ust.hk

‡New York University. Email: thomas.sargent@nyu.edu

§Cheung Kong Graduate School of Business (CKGSB). E-mail: newang@gmail.com

¶Shanghai University of Finance and Economics (SUFE). Email: yang.jinqiang@mail.sufe.edu.cn

The art of taxation consists in so plucking the goose as to obtain the largest possible amount of feathers with the smallest possible amount of hissing.

Jean-Baptiste Colbert and/or Anne-Robert-Jacques Turgot

What a government spends the public pays for.

Keynes (1923, p. 68)

1 Introduction

In our model, Keynes' "public" is an infinitely lived representative consumer who, before taxes, owns a stochastic gross domestic product process $\{\text{GDP}_s\}_{s=0}^{\infty}$. The model starts at time 0 with government debt B_0 for government expenditures $\{\Gamma_s\}_{s=-\infty}^{0-}$ that past taxes had not payed for. The public must also pay for a flow of prospective government expenditures $\{\Gamma_s\}_{s=0}^{\infty}$ with taxes $\{\mathcal{T}_s\}_{s=0}^{\infty}$ whose collection imposes a flow of deadweight tax-collection costs $\{\Theta_s\}_{s=0}^{\infty}$. The representative consumer is left with a net present value of W_0 at time 0 after deducting taxes and deadweight losses from GDP.

We analyze two alternative arrangements for financing the government. One is a Department of Treasury that designs a tax and debt management policy that maximizes W_0. The Department of Treasury's problem generalizes one studied by Barro (1979). The other arrangement is a tax farmer who at time 0 pays the government a lump sum fee P_0 and promises to service B_0 and to pay for government expenditures $\{\Gamma_s\}_{s=0}^{\infty}$. In return, the tax farmer retains all remaining tax revenues.

We compare these arrangements in a setting that extends the small open economy of Barro (1979) in the following ways.

- In the tradition of Aguiar, Amador, and Gopinath (2009), our representative consumer is impatient relative to financiers who live outside our small economy.[1] We can call those outside financiers "the market".

- In a limited commitment tradition of Eaton and Gersovitz (1981), Aguiar and Gopinath (2006), and Arellano (2008), our Department of Treasury can default on financial obligations, at the cost of being permanently excluded from capital markets and of damaging GDP prospects. But unlike those papers, the Treasury Department in our model

[1] Managers are limpatient relative to investors in dynamic contracting and corporate finance literatures, e.g, DeMarzo and Sannikov (2006), Biais et al. (2007), DeMarzo and Fishman (2007), and DeMarzo et al. (2012),

chooses tax and debt management policies to maximize a representative consumer's utility functional, while our tax farmer maximizes the net present value of its profits subject to the constraint that tax payers prefer to pay it rather than a budget-balancing government-run tax authority.

- In the tradition of Kehoe and Levine (1993) and Alvarez and Jermann (2001), the fiscal authority – either a Department of Treasury or a tax farmer – has access to a complete set of history-contingent securities that are priced by the outside financiers.
- Although net government debt is risk free, it bears a risk premium when the revenue stream that ultimately backs it is stochastic and exposed to systematic shocks. The fiscal policy manager can use a Shiller (1994) macro security to insure itself against risk in GDP growth rates. Costs for buying that insurance appear in the debt transition equation and leads to an adjustment of an "$r - g$" term featured by Blanchard (2019) and Mehrotra and Sergeyev (2021). We adopt a "small open economy" assumption that there is an exogenous stochastic discount factor (SDF) process that is not affected by the government's tax and borrowing policy.[2]

Department of Treasury and tax farmer constrained optimization problems are duals with optimal plans that imply identical taxation and fiscal payment processes. Thus, a Department of Treasury and a tax farmer are equally good guardians of public welfare. Impatience of the representative consumer relative to outside investors tilts government debt issues forward and postpones tax payments and tax rate increases.[3] Tax administrators' inability to commit, combined with deadweight losses, pushes up the marginal costs of collecting taxes.

Section 2 presents a brief history of tax farming. Section 3 sets out our model's economic environment. Section 4 poses and solves the tax farmer's problem. Section 5 then poses and solves the Department of Treasury's problem, after which section 6 describes implications

[2]As our exogenous SDF process, we use a geometric Brownian motion process that resembles the *endogenous* SDF that emerges from the equilibrium asset-pricing model of Lucas (1978). It also resembles SDF processes that appear in the portfolio-choice model of Merton (1971) and the option pricing model of Black and Scholes (1973). Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019, 2020) use a complete-market economy with an exogenous SDF process that differs from ours in ways tailored to their quantitative applications.

[3]Jiang et al. (2026) analyze a Department of Treasury problem related to but distinct from ours here. Their representative consumer is not impatient relative to"the market." In their model, risk-free government debt bears a convenience yield not present in our model here. That gives government debt a special "privilege" nor present here. This difference in specifications of fundamentals leads to different outcomes in terms of optimal taxation and debt-management policies. For example, the impatience parameter in our model affects the representative consumer's utility both before and after stopping time $T^{\mathcal{D}} \geqslant 0$. In contrast, the convenience yield parameter appears only in the drift of debt-GDP ratio dynamics before an analogous stopping time in in Jiang et al. (2026). Jiang et al. do not pose or analyze a tax farmer problem.

that follow from the tax farmer and Department of Treasury problems being duals. Section 7 describes worthwhile possible extensions.

2 Brief History of Tax Farming

A government that hires a tax farmer transfers risks, administrative costs, and management difficulties to the tax farmer. There were tax farmers at least as far back as ancient Egypt. In the Roman Republic (590 BC to 27BC), wealthy citizens called *publicani* purchased rights to collect specific taxes in specific regions. Byzantine emperors and Islamic caliphs used tax farmers. Tax farms thrived in 16th and 17th century Europe. In France, *fermiers généraux* (general farmers) collected customs duties and indirect taxes on salt and tobacco. England used tax farmers to collect customs and excise duties. The Ottoman Empire used an *iltizam* system to collect taxes.

Adam Smith said that tax farming was not “the best and most frugal way of levying a tax” because a tax farmer charged the government not only for its administrative costs but also “a certain profit proportioned at least to the advance which he makes, to the risk which he runs, to the trouble which he is at, and to the knowledge and skill which it requires to manage so very complicated a concern” (Smith, 1776, Book V, Ch. 2). Smith recommended that a government department should collect taxes. He said that “the exorbitant profits of the farmers-general might be added to the revenue of the state” if France would replace its tax farms.[4]

In 1789, French revolutionaries, attentive students of Adam Smith, abolished tax farms. They did that before they set in place other ways to collect taxes. That postponed resolution of the fiscal crisis that helped ignited the Revolution. After the Coup of 18 Brumaire 1799, Napoleon organized government agencies to collect taxes, not only from French citizens, but also from citizens of other countries that French armies had conquered.[5] Britain phased out tax farming gradually during the late 18th and early 19th centuries.

Our model can be used as a benchmark against which to assess Adam Smith’s criticisms of tax farming. In our model, the contest between a privatized tax farm and a government administered Department of Treasury ends in a tie. Viewing Adam Smith’s criticism of tax farms from the perspective our model would lead us to search for features of our environment

[4]Adam Smith’s criticism of tax farming belies the myth that he always advocated privatization.

[5]Sargent and Velde (1995) describe how tax farmers operated in 17th and 18th century France before 1789, and how the 1789 reforms prolonged the French fiscal crisis. In 1794, the Committee on Public Safety executed Antoine Lavoisier because he was a tax farmer, not because he was a chemist.

that, if altered, would break that tie and make a tax farm socially inferior to a Department of Treasury. This would require that we understand how our model makes a tax farm and Department of Treasury equally reliable ways of managing the serious incentives to default that are present in our environment. In section 7 we speculate about complications that, if added to our environment, would break our model's duality tie.

3 Economic Environment

Stopping time and GDP. There is a stopping time $T^{\mathcal{D}} \geqslant 0$ that in section 4 is chosen by a profit maximizing tax farm and in section 5 is chosen by a benevolent Treasury department. Gross Domestic Product $\{GDP_t; t \geqslant 0\}$ obeys

$$\mathrm{GDP}_t = \begin{cases} Y_t, & t < T^{\mathcal{D}} \\ \widehat{Y}_t, & t \geqslant T^{\mathcal{D}}, \end{cases} \tag{1}$$

where $\widehat{Y}_t = \alpha Y_t$, $\alpha \in (0,1)$, and $\{Y_t; t \geqslant 0\}$ is a geometric Brownian motion process:

$$\frac{dY_t}{Y_t} = g dt + \sigma d\mathcal{Z}_t^Y , \tag{2}$$

where g sets a mean instantaneous rate of growth, $\sigma > 0$ is growth volatility, $\mathcal{Z}_t^Y$ is a standard Brownian motion, and $Y_0 > 0$.

Stochastic discount factor. A representative outside investor has a stochastic discount factor process (SDF) $\{\mathbb{M}_t\}$ with multiplicative increments

$$\frac{d\mathbb{M}_t}{\mathbb{M}_t} = -r dt - \eta d\mathcal{Z}_t^m , \quad \mathbb{M}_0 = 1 , \tag{3}$$

where $\mathcal{Z}_t^m$ is a standard Brownian motion that represents an aggregate/systematic shock affecting (world) capital markets and r is the risk-free rate. There is a time-invariant correlation coefficient ρ between the GDP shock $d\mathcal{Z}_t^Y$ and $d\mathcal{Z}_t^m$. Absence of arbitrage opportunities requires that the drift of $d\mathbb{M}_t/\mathbb{M}_t$ equals $-r$ (see Duffie, 2001). The diffusion coefficient of $d\mathbb{M}_t/\mathbb{M}_t$ equals $-\eta$, where η is the market price of risk $\mathcal{Z}_t^m$.[6]

[6] See Black and Scholes (1973) and Merton (1973) for explanations of the market price of risk. The $[\mathcal{Z}_t^m, \mathcal{Z}_t^Y]^\top$ process is a bivariate Brownian motion with a covariance matrix $\left[\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right] t$. If $\rho = 1$, the country's GDP shock is systematic. If $\rho = 0$, the country's GDP shock is idiosyncratic.

Government spending and taxes. At time 0, a government owes real government debt $B_0 \geqslant 0$ measured in units of time 0 goods and also finances a spending process

$$\Gamma_t = \gamma Y_t\,, \quad t \geqslant 0, \tag{4}$$

where $\gamma \in (0,1)$. Tax rates τ_t and $\hat{\tau}_t$ before and after stopping time $T^{\mathcal{D}}$ satisfy

$$\tau_t \leqslant \overline{\tau} \quad \text{and} \quad \hat{\tau}_t \leqslant \overline{\tau}\,, \tag{5}$$

where $\overline{\tau} \leqslant 1-\gamma$ is a maximal politically feasible tax rate on GDP.[7] Tax collections at time t thus satisfy

$$\text{Taxes} = \begin{cases} \mathcal{T}_t = \tau_t Y_t, & t < T^{\mathcal{D}}\,, \\ \hat{\mathcal{T}}_t = \hat{\tau}_t \hat{Y}_t, & t \geqslant T^{\mathcal{D}}\,. \end{cases} \tag{6}$$

Deadweight output losses from collecting taxes satisfy

$$\text{Taxes Distortion Costs} = \begin{cases} \Theta_t = \Theta(\mathcal{T}_t, Y_t) = \theta(\tau_t) Y_t\,, & t < T^{\mathcal{D}}, \\ \hat{\Theta}_t = \hat{\Theta}(\hat{\mathcal{T}}_t, \hat{Y}_t) = \hat{\theta}(\hat{\tau}_t)\hat{Y}_t\,, & t \geqslant T^{\mathcal{D}}, \end{cases} \tag{7}$$

where the scaled deadweight loss functions $\theta(\tau)$ and $\hat{\theta}(\hat{\tau}_t)$ are increasing, convex, and smooth.

Public finance after $T^{\mathcal{D}}$. After stopping time $T^{\mathcal{D}}$, the government permanently enters a fiscal "autarky regime" in which it has zero government debt and no access to outside investors. In this regime, the government sets its primary budget deficit to zero each period. Thus, in the post-$T^{\mathcal{D}}$ balanced primary budget regime:

$$\Gamma_t = \hat{\mathcal{T}}_t = \hat{\tau}_t \hat{Y}_t, \quad t \geqslant T^{\mathcal{D}}. \tag{8}$$

Public finance before $T^{\mathcal{D}}$. Before stopping time $T^{\mathcal{D}}$ the fiscal authority, either a tax farm or a Department of Treasury, has access to outside financiers who price all domestic GDP streams. The government's financier can finance expenditures Γ_t by issuing risk-free debt and dynamically trading a Shiller (1994) macro security whose payouts equal the country's

[7] Keynes (1923, pp.56–62) and Keynes (1931) described political considerations that shape an upper bound like $\overline{\tau}$ and used it to infer limits on a country's government debt-GDP ratio.

GDP.[8] The time t price S_t of the Shiller macro security is

$$S_t = \mathbb{E}_t \left[\int_t^{\infty} \frac{\mathbb{M}_s}{\mathbb{M}_t} Y_s ds \right] , \tag{9}$$

where $\{\mathbb{M}_t\}$ is the stochastic discount factor process given in (3). Applying Ito's lemma to (9), the cum-dividend return on the Shiller security is:

$$dR_t \equiv \frac{dS_t + Y_t dt}{S_t} = (r + \lambda)\, dt + \sigma d\mathcal{Z}_t^Y , \tag{10}$$

where $\lambda = \rho\eta\sigma$ is a risk premium.

Representative consumer. A representative domestic consumer discounts certain future payoffs at rate $(\zeta + r)$. The representative consumer evaluates risks in the same ways that a representative outside investor (a.k.a. "the market") does and hence applies the same market price η to aggregate risk $\mathcal{Z}_t^m$.[9] When $\zeta = 0$, the representative consumer and the market are equally patient, in which case, they use the same SDF $\mathbb{M}_t$ to value payouts. We adopt a common assumption in the sovereign debt literature (e.g., Aguiar and Gopinath (2006)) that $\zeta > 0$. This makes the representative consumer prefer to postpone paying taxes and to front load consuming. This means that the representative consumer uses $e^{-\zeta t}\mathbb{M}_t$ to evaluate risky payoffs.

Before $T^{\mathcal{D}}$, the representative consumer, who is the sole tax payer, receives GDP, pays taxes $\mathcal{T}_s ds$, possibly receives a lump-sum payment dJ_s, and suffers deadweight loss $\Theta_s ds$ over a small time interval ds, thus resulting in a net income flow equal to $(Y_s - \mathcal{T}_s - \Theta_s)ds + dJ_s$ over ds. The representative consumer's continuation value W_t at time t equals the present value of after-tax, after-transfer income:[10]

$$W_t = \mathbb{E}_t \Bigg[\int_t^{T^{\mathcal{D}}} e^{-\zeta(s-t)} \frac{\mathbb{M}_s}{\mathbb{M}_t} \left[dJ_s + \left(Y_s - (\mathcal{T}_s + \Theta_s)\right) ds \right] + e^{-\zeta(T^{\mathcal{D}}-t)} \frac{\mathbb{M}_{T^{\mathcal{D}}}}{\mathbb{M}_t} \widehat{W}_{T^{\mathcal{D}}} \Bigg], \; t < T^{\mathcal{D}} , \tag{11}$$

[8] Jiang et al. (2024) show that the presence of a complete set of Arrow history-contingent securities implies the existence of Shiller's macro security.

[9] The representative consumer and investors use the same Radon-Nikodym derivative that links physical measure $\mathbb{P}$ to risk-neutral measure $\widetilde{\mathbb{P}}$ (Duffie, 2001). With complete markets, this Radon-Nikodym derivative is unique.

[10] Continuation value $\{W_t; t \geqslant 0\}$ will be the state variable in a recursive version of the tax farmer's decision problem. See DeMarzo and Sannikov (2006) and Sannikov (2008) for pioneering work on continuous-time recursive contracting formulations. Ai and Li (2015) and Bolton, Wang, and Yang (2019) deploy recursive continuous-time formulations of contracting problems to study limited commitment problems in corporate finance.

where $\widehat{W}_{T^{\mathcal{D}}}$ denotes the continuation value at the stopping time $T^{\mathcal{D}}$. We can use the Martingale Representation Theorem to represent the dynamics of $\{W_t\}$ as

$$dW_t = \left[(\zeta + r)W_t - (Y_t - \mathcal{T}_t - \Theta_t)\right] dt - dJ_t - \Phi_t \left(dR_t - rdt\right), \tag{12}$$

where $\{\Phi_t; t \geqslant 0\}$ determines the volatility of the representative consumer's continuation value before stopping time $T^{\mathcal{D}}$.[11]

At stopping time $T^{\mathcal{D}}$, output immediately drops to $\widehat{Y}_{T^{\mathcal{D}}} = \alpha Y_{T^{\mathcal{D}}}$. Thereafter, at $t > T^{\mathcal{D}}$, the government pays for Γ_t with time t tax collections, so $\widehat{\mathcal{T}}_t = \Gamma_t$ and the government's primary budget deficit is zero. In this "balanced primary budget regime" at $t > T^{\mathcal{D}}$, the representative consumer's continuation value is

$$\widehat{W}_t = \widehat{W}(\widehat{Y}_t) = \mathbb{E}_t\left[\int_t^{\infty} e^{-\zeta(s-t)} \frac{\mathbb{M}_s}{\mathbb{M}_t} \left(\widehat{Y}_s - \widehat{\mathcal{T}}_s - \widehat{\Theta}(\widehat{\mathcal{T}}_s, \widehat{Y}_s)\right) dt\right]. \tag{13}$$

4 A Tax Farmer

At time $t = 0$, a tax farmer offers to pay the government a lump sum equal to B_0 and then to pay for the government's expenditure process $\{\Gamma_s\}_{s=0}^{T^{\mathcal{D}}}$. The tax farmer chooses stopping time $T^{\mathcal{D}} > 0$, tax revenue process $\{\mathcal{T}_s\}_{s=0}^{T^{\mathcal{D}}}$, and (undiscounted) cumulative transfer payment $\{J_s\}_{s=0}^{T^{\mathcal{D}}}$ to the representative consumer, together with state-contingent Shiller macro security holdings $\{\Phi_s\}_{s=0}^{T^{\mathcal{D}}}$. The tax farmer has access to outside financing sources that the government lacks.[12] The tax farmer maximizes the following risk-adjusted present value of its net income stream $(\mathcal{T}_s - \Gamma_s)\, ds - dJ_s$:

$$F_0 = \mathbb{E}_0 \left[\int_t^{T^{\mathcal{D}}} \frac{\mathbb{M}_s}{\mathbb{M}_t} \left[(\mathcal{T}_s - \Gamma_s)\, ds - dJ_s\right]\right]. \tag{14}$$

The tax farmer devises a tax plan and a stopping time that maximizes (14) subject to the participation constraints

$$W_t \geqslant \widehat{W}_t\,, \quad \forall\ t < T^{\mathcal{D}} \tag{15}$$

[11] Jiang et al. (2024) show that a continuation value like W_t in equation (11) is the indirect utility function for a consumer who chooses a consumption-portfolio plan that maximizes expected discounted utility of consumption, where the instantaneous utility function has constant relative risk aversion.

[12] This is a counterpart to assumptions that made by Green (1987), Phelan and Townsend (1991), and Atkeson (1991) make about outside financiers.

that describe the representative consumer's desire to sustain its relationship with the tax farmer rather than dealing with post-$T^{\mathcal{D}}$ budget balancing government tax collector. The gap $W_t - \widehat{W}_t$ measures how much utility the representative consumer gains from the intertemporal tax smoothing that the tax collector provides relative to a post-stopping-time-$T^{\mathcal{D}}$ budget-balancing tax adminstrators.

Let $\underline{W}_t = \underline{W}(Y_t)$ denote a minimal present value at which the representative consumer is willing to participate. Effective participation constraints are

$$W_t \geqslant \underline{W}(Y_t), \quad t \geqslant 0\,. \tag{16}$$

Constraint (16) requires that the lower bound $\underline{W}(Y_t)$ on W_t in an interior region is greater than or equal to the representative consumer's value $\widehat{W}(\widehat{Y}_t)$ in the balanced-budget regime:

$$W_t \geqslant \underline{W}(Y_t) \geqslant \widehat{W}(\widehat{Y}_t)\,. \tag{17}$$

Inequality $\underline{W}(Y_t) \geqslant \widehat{W}(\widehat{Y}_t)$ holds with equality when tax rate constraint (5) does not bind. Otherwise, constraint (5) on the tax rate pins down lower boundary $\underline{W}(Y_t)$.

4.1 Optimal Tax Policies

We first study the balanced-budget regime that prevails when $t \geqslant T^{\mathcal{D}}$. When promised value W_t is lower than threshold $\underline{W}(Y_t)$, the representative consumer terminates its relationship with the tax farmer and consents to pay taxes directly to a Department of Treasury that administers a balanced-budget regime. The representative consumer's present value at $t \geqslant T^{\mathcal{D}}$ satisfies the following differential equation (see Appendix A)

$$(\zeta + r)\widehat{W}(\widehat{Y}) = \widehat{Y} - \widehat{\mathcal{T}} - \widehat{\Theta}(\widehat{\mathcal{T}}, \widehat{Y}) + \widetilde{g}\,\widehat{Y}\widehat{W}'(\widehat{Y}) + \frac{\sigma^2\widehat{Y}^2}{2}\widehat{W}''(\widehat{Y})\,, \tag{18}$$

where $\widetilde{g} = (g - \lambda)$ is the risk-adjusted GDP (expected) growth rate. Let $\widehat{w}_t = \widehat{W}(\widehat{Y}_t)/\widehat{Y}_t$. The scaled promised value $\widehat{w}$ satisfies

$$(\zeta + r)\widehat{w} = 1 - \gamma/\alpha - \widehat{\theta}(\gamma/\alpha) + \widetilde{g}\widehat{w}\,, \tag{19}$$

so that

$$\widehat{w} = \frac{1 - \gamma/\alpha - \widehat{\theta}(\gamma/\alpha)}{\zeta + r + \lambda - g}\,. \tag{20}$$

We maintain:

Assumption 4.1. $r + \zeta + \lambda > g$, $\kappa \geqslant 1$, $\alpha \leqslant 1$, *and* $1 - \gamma/\alpha - \widehat{\theta}(\gamma/\alpha) \geqslant 0$.

4.2 Lump-sum Payout Region $W_t > \overline{W}(Y_t)$

We turn now to the tax farmer's choice of a lump-sum payout to the representative consumer and an associated upper boundary $\overline{W}(Y)$ for W. It is costly for the tax farmer to defer payments to the representative consumer because the representative consumer is less patient than the tax farmer (i.e., $\zeta > 0$). At each date, the tax farmer pays the representative consumer something now and promises to pay him more later. Postponing payment by one unit instantaneously increases W_t by one unit, thereby relaxing participation constraint (16). The tax farmer's choice between paying an impatient representative consumer immediately and relaxing its financing constraint implies existence of an endogenous threshold $\overline{W}_t = \overline{W}(Y_t)$ above which it is optimal for the tax farmer immediately to pay a lump-sum and below which it is better to postpone payments. Therefore, we set

$$dJ_t = \max\{W_t - \overline{W}(Y_t), 0\}\,. \tag{21}$$

Let $F(W_t, Y_t)$ denote the tax farmer's optimal value function. Whenever $W_t > \overline{W}(Y_t)$, the tax farmer immediately pays the representative consumer a lump sum and

$$F(W_t, Y_t) = F(\overline{W}(Y_t), Y_t) - \left(W_t - \overline{W}(Y_t)\right)\,. \tag{22}$$

Threshold $\overline{W}$ solves

$$\max_{\overline{W}} \quad F(\overline{W}, Y) + \overline{W}\,. \tag{23}$$

Later we shall show that the tax farmer would pay a lump sum only at time $t = 0$.

4.3 Interior Region $W_t \in [\underline{W}(Y_t), \overline{W}(Y_t)]$

In an interior region $W \in [\underline{W}, \overline{W}]$, the tax farmer optimally sets $dJ_t = 0$ and value function $F(W, Y)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rF = \max_{\mathcal{T} \leqslant \overline{\tau} Y, \Phi} & (\mathcal{T} - \Gamma) + ((\zeta + r)W - (Y - \mathcal{T} - \Theta(\mathcal{T}, Y)))\, F_W + \widetilde{g} Y F_Y \\ & + \frac{\sigma^2 Y^2 F_{YY}}{2} + \frac{\sigma^2 \Phi^2 F_{WW}}{2} - \sigma^2 \Phi Y F_{WY}. \end{aligned} \tag{24}$$

Optimal tax and hedging policies. The tax farmer's first-order necessary condition for tax revenue $\mathcal{T}$ equates the marginal cost $1+\Theta_{\mathcal{T}}(\mathcal{T},Y)$ of taxing the representative consumer with marginal benefit $-1/F_W(W,Y)>0$ to the tax farmer:

$$1+\Theta_{\mathcal{T}}(\mathcal{T},Y) = -1/F_W(W,Y)\,. \tag{25}$$

Following a prescription of Merton (1971), the tax farmer's purchases Φ of the Shiller security satisfy

$$\Phi = \frac{YF_{WY}(W,Y)}{F_{WW}(W,Y)}\,. \tag{26}$$

By using first-order necessary conditions (25)-(26) to simplify (24), we obtain the following first-order partial differential equation:

$$rF = \max_{\mathcal{T}\leqslant\overline{\tau}Y}(\mathcal{T}-\Gamma) + ((\zeta+r)W-(Y-\mathcal{T}-\Theta(\mathcal{T},Y)))\,F_W + \widetilde{g}YF_Y\,. \tag{27}$$

By optimally managing its risk, the tax farmer makes the representative consumer's present value become risk free. This explains why no diffusion terms associated with F_{WW}, F_{YY} or F_{WY} appear in (27).

4.4 Scaled Variables

Tax farmer's scaled value function. Let $w_t = W_t/Y_t$ and define

$$F(W_t,Y_t) = f(w_t)\cdot Y_t\,. \tag{28}$$

Let $\overline{w}_t = \overline{W}(Y_t)/Y_t$ denote the upper boundary of scaled value w. It turns out that $\overline{w}_t$ is constant, so we can drop its time subscript. The optimal scaled lump-sum transfer $dj_t = dJ_t/Y_t$ to a representative consumer who has been promised w_t at t is

$$dj_t = \max\{w_t-\overline{w}_t, 0\}. \tag{29}$$

Interior region: $w_t \in [\underline{w},\overline{w}]$. Here $dj_t = 0$. Let $\tau_t = \tau(w_t) = \mathcal{T}_t/Y_t$ denote the optimal tax rate. Substituting (28) into (24) and simplifying yields the following implicit equation for $\theta(w)$:

$$1+\theta'(\tau(w)) = -1/f'(w). \tag{30}$$

Similarly, let $\phi_t = \phi(w_t) = \Phi_t/Y_t$. We then obtain

$$\phi(w) = -w\,. \tag{31}$$

Applying Ito's lemma to $w_t = W_t/Y_t$, where W_t is given in (12) and Y_t is given in (2), and using optimal tax policy (30) and optimal hedging strategy (31), we discover the following deterministic dynamics for the scaled promised value w_t:

$$dw_t = \dot{w}_t dt = \mu^w(w_t)dt = \left[(\zeta + r + \lambda - g)w_t - (1 - \tau_t - \theta(\tau_t))\right] dt\,. \tag{32}$$

Substituting $F(W_t, Y_t) = f(w_t) \cdot Y_t$ from (28) and conditions (30) and (31) for optimal policy functions for $\theta(w)$ and $\phi(w)$ into HJB equation (24), we obtain a first-order nonlinear differential equation for the tax farmer's scaled value $f(w)$:

$$(r - \widetilde{g})f(w) = \tau(w) - \gamma + \left[(\zeta + r - \widetilde{g})w - (1 - \tau(w) - \theta(\tau(w)))\right] f'(w)\,. \tag{33}$$

Lump-sum payout region: $w > \overline{w}$**.** Here the tax farmer's (scaled) value function is $f(w) = f(\overline{w}) + \overline{w} - w$. The upper boundary $\overline{w}$ is constant and solves

$$\max_{\overline{w}} \quad f(\overline{w}) + \overline{w}\,. \tag{34}$$

Balanced-budget region and participation constraint. Given the scaled default value defined in (20), the scaled promised outside value $\underline{w}$ is

$$\underline{w} = \alpha\widehat{w}\,, \text{ when the tax constraint (5) does not bind.} \tag{35}$$

Otherwise, constraint (5) on the maximum tax rate binds at the boundary and $\underline{w}$ satisfies:

$$\tau(\underline{w}) = \overline{\tau}\,, \text{ when constraint (5) on tax rates binds.} \tag{36}$$

The following zero-drift condition for w at $\underline{w}$ ensures that $w \geqslant \underline{w}$:

$$\mu^w(\underline{w}) = (\zeta + r + \lambda - g)\underline{w} - (1 - \tau(\underline{w}) - \theta(\tau(\underline{w}))) = 0. \tag{37}$$

The following theorem describes the contract that the tax farmer offers to the representative consumer.

Theorem 4.2. *Under the conditions $r + \lambda + \zeta > g$, $\kappa \geqslant 1$, $\alpha \leqslant 1$, and $1 - \gamma/\alpha - \widehat{\theta}(\gamma/\alpha) \geqslant 0$ maintained in Assumption 4.1, the scaled value function $f(w)$ in the no-default regime satisfies the nonlinear first-order differential equation* (33) *subject to the zero-drift condition (37) and either restriction* (35) *or restriction* (36) *on scaled promised value $\underline{w}$. The scaled value $\widehat{w}$ in the balanced-budget regime is given by (20). The lump-sum payout boundary $\overline{w}$ is given by (34), and the optimal lump-sum payout policy, dj_t, is given by (29). The optimal tax rate policy $\tau(w)$ is given by (30) and scaled promised value $\{w_t\}$ evolves deterministically at rate $\dot{w}_t$ described by (32).*

5 The Department of Treasury

We now describe how a government administered Department of Treasury would choose tax and financing policies if it has access to the same outside financing sources that are available to our section 4 tax farmer. The Department of Treasury chooses stopping time $T^{\mathcal{D}}$ as part of its financing plan. The Department of Treasury owes debt B_0 at $t = 0$ and faces the intertemporal budget constraint:

$$B_0 + \mathbb{E}_0 \int_0^{T^{\mathcal{D}}} \mathbb{M}_t dU_t \leqslant \mathbb{E}_0 \int_0^{T^{\mathcal{D}}} \mathbb{M}_t(\mathcal{T}_t - \Gamma_t)dt \,, \tag{38}$$

where dU_t is a non-negative incremental lump-sum payment to the representative consumer over dt. At $t < T^{\mathcal{D}}$, the Treasury's risk-free debt B_t evolves according to

$$dB_t = (\Gamma_t - \mathcal{T}_t)dt + rB_t dt + dU_t - \Psi_t \left(dR_t - rdt\right). \tag{39}$$

The last term describes how the government's purchase Ψ_t of the Shiller security affects dB_t.

The representative consumer receives flow payments $\{Y_t - \mathcal{T}_t - \Theta_t\}$ in the no-default regime before stopping time $T^{\mathcal{D}}$, $\{\widehat{Y}_t - \widehat{\mathcal{T}}_t - \widehat{\Theta}_t\}$ in the balanced-budget regime after $T^{\mathcal{D}}$, and a cumulative government transfer payment process $\{U_t; t \geqslant 0\}$. The Department of Treasury chooses its holdings $\{\Psi_t\}$ of the Shiller macro security and a tax revenue process $\{\mathcal{T}_t\}$ to maximize

$$P_t = \mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} e^{-\zeta(s-t)} \frac{\mathbb{M}_s}{\mathbb{M}_t} \left(dU_s + (Y_s - \mathcal{T}_s - \Theta_s)\right) ds + e^{-\zeta(T^{\mathcal{D}}-t)} \frac{\mathbb{M}_{T^{\mathcal{D}}}}{\mathbb{M}_t} \widehat{P}_{T^{\mathcal{D}}} \right], \tag{40}$$

where $\widehat{P}_{T^{\mathcal{D}}} = \widehat{W}_{T^{\mathcal{D}}}$ from equation (13). Optimal policies $\{\mathcal{T}_t\}, \{\widehat{\mathcal{T}}_t\}$, and $\{U_t\}$ depend on the history of GDP shocks $\{\mathcal{Z}_t^Y\}$.

In the no-default regime that prevails before stopping time $T^{\mathcal{D}}$, the Department of Treasury's optimal value function $P(B, Y)$ solves the HJB equation:

$$(\zeta + r)P = \max_{\mathcal{T} \leqslant \bar{\tau} Y, \Psi} \ (Y - \mathcal{T} - \Theta(\mathcal{T}, Y)) + [rB + \Gamma - \mathcal{T}\ ] P_B + \widetilde{g} Y P_Y + \frac{\sigma^2 Y^2}{2} P_{YY} + \frac{\sigma^2 \Psi^2}{2} P_{BB} - \sigma^2 \Psi Y P_{BY} \,. \tag{41}$$

The first term on the right side of (24), $(Y - \mathcal{T} - \Theta(\mathcal{T}, Y))$, is the net payment flow to the representative consumer. The second and third terms are drift and diffusion volatility effects of increasing debt B on $P(B, Y)$. The fourth and fifth terms express effects of drift and volatility of GDP on $P(B, Y)$. The sixth term captures effects of the Shiller macro security demands on $P(B, Y)$. The Department of Treasury chooses to make a lump sum payment dU to the representative consumer only at $t = 0$.

Let $b_t = B_t/Y_t$ denote the debt-to-GDP ratio, $p(b_t) = P_t/Y_t$ denote the scaled 's value, and $\widehat{p} = \widehat{P}_{T^{\mathcal{D}}}/Y_{T^{\mathcal{D}}}$ denote the scaled default value. Let $\varphi_t = \mathcal{T}_t/Y_t$ denote the Department of Treasury's tax rate, $\psi_t = \Psi_t/Y_t$ denote the scaled hedging demand, and $du_t = dU_t/Y_t$ denote the scaled lump-sum transfer.

The following theorem describes an optimal Department of Treasury plan.

Theorem 5.1. *Under conditions that $r + \lambda + \zeta > g$, $\kappa \geqslant 1$, $\alpha \leqslant 1$, and $1 - \gamma/\alpha - \widehat{\theta}(\gamma/\alpha) \geqslant 0$ given in Assumption (4.1), the scaled value function $p(b)$ in the no-default regime satisfies the first-order nonlinear differential equation:*

$$[\zeta + (r - \widetilde{g})]\, p(b) = 1 - \varphi(b) - \theta(\varphi(b)) + [(r - \widetilde{g})b + \gamma - \varphi(b)]\, p'(b) \,, \tag{42}$$

subject to the debt-sustainability condition

$$\bar{b} = \frac{\varphi(\bar{b}) - \gamma}{r - \widetilde{g}} \tag{43}$$

and one of the following restrictions on scaled government debt capacity $\bar{b}$:

$$p(\bar{b}) = \alpha \widehat{p}\,, \textit{ when constraint (5) on tax rates doesn't bind}; \tag{44}$$

$$\varphi(\bar{b}) = \bar{\tau}\,, \textit{ when the constraint (5) on tax rates binds}. \tag{45}$$

The scaled value $\widehat{p}$ in the balanced-budget regime is

$$\widehat{p} = \frac{1 - \gamma/\alpha - \widehat{\theta}(\gamma/\alpha)}{\zeta + (r + \lambda - g)} \,. \tag{46}$$

The lump-sum debt issue boundary $\underline{b}$ is described by

$$\max_{\underline{b}} \; p(\underline{b}) + \underline{b} \,, \tag{47}$$

and the optimal lump-sum transfer du_t satisfies

$$du_t = \max\{\underline{b} - b_t, 0\} \,. \tag{48}$$

The optimal tax rate policy $\varphi(b)$ satisfies

$$1 + \theta'(\varphi(b)) = -p'(b) \tag{49}$$

and the optimal hedging policy $\psi(b)$ is

$$\psi(b) = -b \,. \tag{50}$$

The debt-output ratio $\{b_t\}$ evolves deterministically at rate

$$\dot{b}_t = \mu^b(b_t) = \gamma - \varphi(b_t) + (r + \lambda - g)b_t \,. \tag{51}$$

6 Duality

The section 4 tax farmer and the section 5 Department of Treasury choose identical plans for tax rates and payouts to the representative consumer. Equality of those choices confirm that the section 4 tax farming problem is the mathematical dual to the section 5 Treasury Department problem.

6.1 Identical Tax, Payout, and Hedging Policies

The same tax and risk-management policies solve the tax farmer's value maximization problem (14) and the Department of Treasury's debt management problem (40): (i) the tax farmer's value $F(\underline{W}, Y)$ when its participation constraint binds equals maximum debt capacity $\overline{B}(Y)$ in the Department of Treasury's problem: $F(\underline{W}, Y) = \overline{B}(Y)$; (ii) the tax farmer's value $F(\overline{W}, Y)$ when it makes a lump sum payment to the representative consumer equals the lumpy debt-issuance boundary $\underline{B}(Y)$ in the Department of Treasury's problem: $F(\overline{W}, Y) = \underline{B}(Y)$; (iii) the tax farmer's value function $F(W, Y)$ relates to the Department of Treasury's value function $P(B, Y)$ through the equations $P(B_t, Y_t) = W_t$ and $B_t = F(W_t, Y_t)$.

The scaled state variable w in tax farmer's problem equals the scaled value $p(b)$ in the Department of Treasury's problem. The scaled state variable b in the Department of Treasury's problem equals in the scaled value $f(w)$ in the tax farmer's problem. Thus,

$$w = p(b) \quad \text{and} \quad b = f(w). \tag{52}$$

Together these equations imply $f \circ p(b) = b$. The composition of the Department of Treasury's value function $p(\cdot)$ with the tax farmer's value function $f(\cdot)$ is an identity function. Table 1 summarizes one-to-one mappings between state variables, value functions, and policies in the tax farmer and Department of Treasury optimization problems. Equation (52) asserts that the government's debt-to-GDP ratio b_t is a one-to-one function of the scaled promised value w_t that the tax farmer promises to deliver to the representative consumer.

Boundary conditions are

$$f(\underline{w}) = \overline{b}\,, \tag{53}$$

and

$$f(\overline{w}) = \underline{b}\,. \tag{54}$$

Note that that by substituting $b = f(w)$ into the ODE for $p(b)$, we obtain the ODE for $f(w)$, and vice versa. Substituting (52) and (53) into the constraint (43) for $\overline{b}$ and ODE (42) for $p(b)$, we obtain the constraint (37) for w, and ODE (19) for the default value $\widehat{w}$. Substituting (52) and (54) into the constraint (47) for $\underline{b}$ yields constraint (34) for $\overline{w}$. Substituting (52) into the optimal tax policy (49) in the Department of Treasury's problem yields the optimal tax policy (30) in the tax farmer's problem. Since $w = p(b)$ is the inverse function of $b = f(w)$, substituting (52) into the tax farmer's optimal hedging policy (31) and the Department of Treasury's optimal hedging policy (50) yields

$$\phi(w) = -p \circ f(w) = -w, \quad \text{and} \quad \psi(b) = -f \circ p(b) = -b\,. \tag{55}$$

Table 1: Tax Farmer and Department of Treasury Problems

	Tax Farmer	Treasury
A. State variables	w	b
Drift	$\dot{w}_t$ from (32)	$\dot{b}_t$ from (51)
Admissible region	$w \in [\underline{w}, \overline{w}]$	$b \in [\underline{b}, \bar{b}]$
B. Value functions	$f(w)$	$p(b)$
Interior region	ODE from (33)	ODE from (42)
C. Policy rules		
Payout boundaries	$\overline{w}$ from (34)	$\underline{b}$ from (47)
Lump-sum transfers	dj from (29)	du from (48)
Hedging policies	$\phi(w)$ from (31)	$\psi(b)$ from (50)
Tax rates	$\tau(w)$ from (30)	$\varphi(b)$ from (49)
D. Limited commitment		
Boundary conditions	$\mu^w(\underline{w}) = 0$	$\mu^b(\bar{b}) = 0$
Values at $T^{\mathcal{D}}$	$\widehat{w}$ from (20)	$\widehat{p}$ from (46)
Non-binding upper bound on τ	$\underline{w} = \alpha\widehat{w}$	$p(\bar{b}) = \alpha\widehat{p}$
Binding upper bound on τ	$\tau(\underline{w}) = \overline{\tau}$	$\varphi(\bar{b}) = \overline{\tau}$

With free entry into an auction to be the tax farmer just prior to time $t = 0$, the government would receive a winning bid that promises the representative consumer a (scaled) value $w_0 = p(b_0) = f^{-1}(b_0)$.[13]

6.2 The Two Problems in Graphs

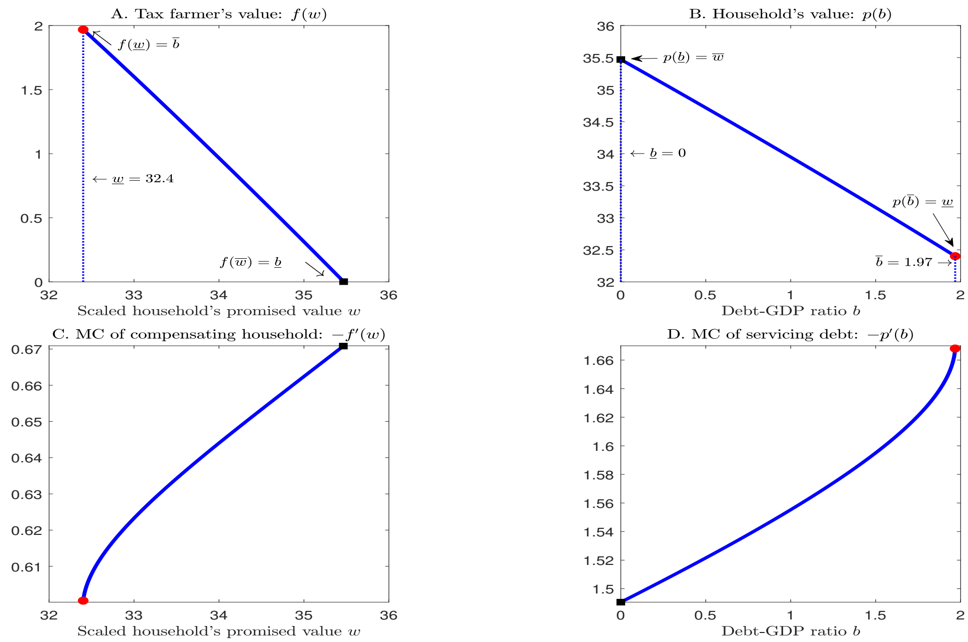
Figure 1 portrays relationships between the tax farmer's and the Department of Treasury's problems for some possible parameter values.[14] Panels A and C of Figure 1 plot the tax farmer's scaled value $f(w)$ and the marginal cost (MC) $-f'(w) = -F_W(W, Y)$ of inducing the representative consumer to stay with the tax farmer. The tax farmer's value function $f(w)$ is decreasing and concave in the (scaled) representative consumer's present value w. As the tax farmer's participation constraint limits it more and more, the MC $-f'(w)$ of inducing the representative consumer to stay put increases.

Panels B and D of Figure 1 illustrates the Department of Treasury's plan by plotting the representative consumer's value $p(b)$ and the marginal cost (MC) of servicing debt $-p'(b) =$

[13]The auction extracts from the tax farmer all of the "exorbitant profits" that concerned Adam Smith.

[14]To create these graphs, we assume $\theta(\tau) = \varpi\tau^2/2, \widehat{\theta}(\cdot) = \kappa\theta(\cdot)$ and set $r = 1\%, \zeta = 0.1\%, g = 2\%, \gamma = 20\%, \lambda = 3\%, \kappa = 1, \alpha = 0.94, \bar{\tau} = 0.3, \varpi = 2.8$.

Figure 1: **Tax farmer's value $f(w)$, representative consumer's value $p(b)$, marginal cost of compensating representative consumer $-f'(w)$, and marginal cost of (servicing) debt $-p'(b)$.** Debt capacity is $\overline{b} = 1.97$ and there is no jumpy debt issuance: $\underline{b} = 0$.

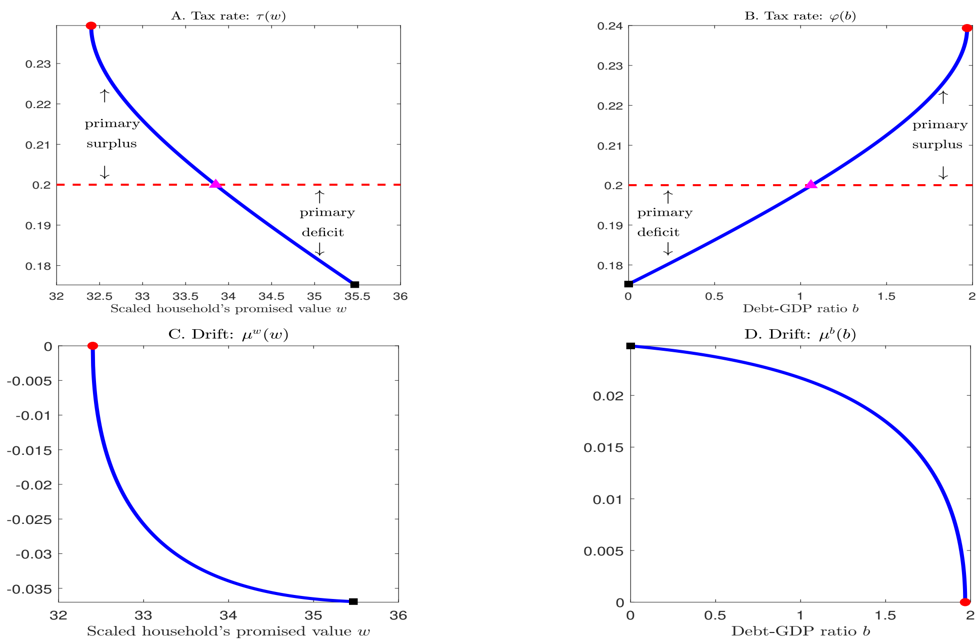


$-P_B(B, Y)$, respectively. The representative consumer's value $p(b)$ is decreasing and concave in b because as b increases, the present value of taxes required to service b constrain the representative consumer more. As b increases from its lower bound $b = 0$ to the Department of Treasury's debt capacity $b = \overline{b} = 1.97$, $p(b)$ decreases from $p(0) = 35.5$ to $p(\overline{b}) = 32.4$ and the MC of servicing debt $-p'(b)$ increases from $-p'(0) = 1.49$ to $-p'(\overline{b}) = -p'(1.97) = 1.67$ (panel D). That the MC of servicing debt exceeds one reflects costs of tax distortions as well as the Department of Treasury's option not to service its debts.

Panel B illustrates the Department of Treasury's plan by rotating the tax farmer's solution given in panel A (away from its plane) and swapping x and y axes. The red dot in panel A corresponds to the red dot in panel B: $p(\overline{b}) = \underline{w}$ and $f(\underline{w}) = \overline{b}$. Similarly, the black square in panel A corresponds to the black square in panel B. Indeed, for all $b \in [0, \overline{b}]$, we have $f \circ p(b) = b$.

It follows that $f \circ p(b) = b$ and $p'(w) \times f'(b) = 1$. Since tax distortions make the Department of Treasury's MC of servicing debt exceed one ($-f'(b) > 1$), the tax farmer's marginal cost (MC) of retaining the representative consumer by cutting taxes is less than one $-p'(w) < 1$. As w increases from $\underline{w} = 32.4$ to $\overline{w} = 35.5$, MC $-f'(w)$ increases from

Figure 2: **Optimal tax rate $\tau(w)$ for tax farmer, optimal tax rate $\varphi(b)$ for the Department of Treasury, drift of scaled promised value $\mu^w(w)$, and drift of debt-GDP ratio $\mu^b(b)$.** Bounds on promised values $\overline{w} = 32.4$ are $\underline{w} = 35.5$; there are no lump sum payments.



$-f'(w) = 0.60$ at $w = \underline{w} = 32.4$ to $-f'(w) = 0.67$ at $w = \overline{w} = 35.5$. The MC of cutting taxes is less than one for the tax farmer because cutting taxes also reduces distortions and relaxes its participation constraint. The higher is the representative consumer's value w, the less financially constrained the consumer is and the smaller are the benefits coming from reduced distortions brought by lower tax rates.

Panels A and B of Figure 2 plot optimal tax rate functions $\tau(w)$ and $\varphi(b)$ associated with the two plans. The optimal tax rate $\tau(w)$ chosen by the tax farmer decreases with w and reaches its maximal value $\tau(\underline{w}) = 0.24$ at the lower bound $\underline{w} = 32.4$ (panel A). For sufficiently high w, the tax farmer keeps taxes low and runs a primary deficit. When promised value $w < 33.8$, the tax farmer runs a primary surplus and increases the tax rate at an increasing rate in order to lift the drift of w (panel C). The tax farmer's plan makes the representative consumer's continuation value converge from above to $\underline{w} = 32.4$.

Panel B shows that the Department of Treasury's tax rate, $\varphi(b)$, increases with b. This happens because the Department of Treasury's incentive to acquire resources from the representative consumer increases as the debt-GDP ratio b increases. Red dots in panels A and

B describe the same outcomes as do black squares.

Panels C and D plot drifts of w and b, respectively. Note how the rate $\dot{w}_t$ at which the scaled promised value w increases decreases with the level of w_t. As w decreases, the representative consumer's value w from staying with the tax farmer rather than defecting to the $t > T^{\mathcal{D}}$ budget-balancing government decreases at a slower and slower rate until it reaches zero at $\underline{w}$: $\mu^w(\underline{w}) = 0$ (panel C). This occurs because the representative consumer will not participate when promised a present value that is too small. Correspondingly, the drift of debt-GDP ratio $\mu^b(b_t)$ decreases as b_t increases. As b increases, the marginal cost of servicing debt $-p'(b)$ and the tax rate $\varphi(b)$ both increase, so the debt-GDP ratio increases more slowly (i.e., $\dot{b}_t$ decreases) until it eventually reaches zero at debt capacity: $\mu^b(\bar{b}) = 0$ (panel D).

We turn now to our model's transition dynamics. The tax farmer makes the scaled promised value w_t evolve deterministically at a rate $\dot{w}_t = \mu^w(w_t)$ described by (32). The minimal scaled promised value $\underline{w}$ is reached in finite time. Starting from w_0, the time $T(w_0 \to \underline{w})$ that it takes to reach $\underline{w}$ is

$$T(w_0 \to \underline{w}) := \int_{w_0}^{\underline{w}} \frac{dw_t}{\dot{w}_t} = \int_{w_0}^{\underline{w}} \frac{1}{(\zeta + r + \lambda - g)w_t - (1 - \tau_t - \theta(\tau_t))} dw_t. \tag{56}$$

Similarly, starting from a given b_0, the time $T(b_0 \to \bar{b})$ that it takes to reach debt-limit $\bar{b}$ is

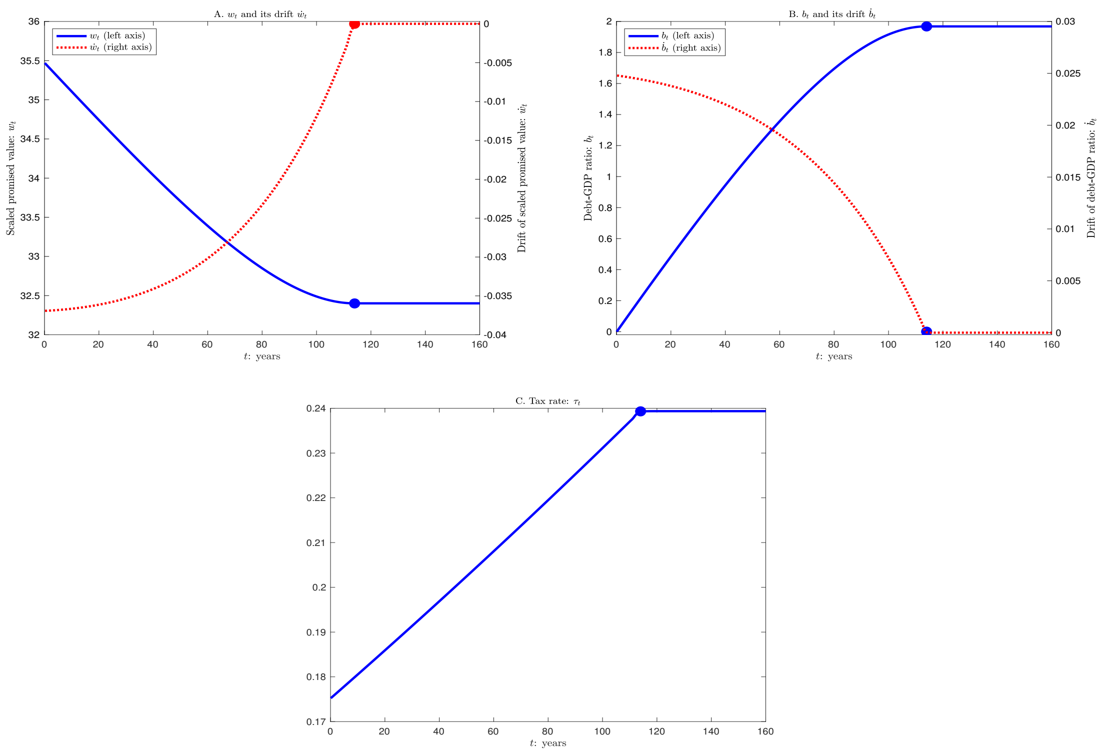
$$T(b_0 \to \bar{b}) := \int_{b_0}^{\bar{b}} \frac{db_t}{\dot{b}_t} = \int_{b_0}^{\bar{b}} \frac{1}{\gamma - \varphi(b_t) + (r + \lambda - g)b_t} db_t. \tag{57}$$

By setting $b_0 = f(w_0)$, we discover that the maximal debt-GDP ratio $\bar{b}$ is reached in the same amount of time that it takes to reach $\underline{w}$.

Setting value $w_0 = 35.5$ promised by the tax farmer corresponds to setting $b_0 = f(w_0) = 0$ owed by the Treasury. For these settings for w_0 and b_0, panels A and B of Figure 3 plot the scaled promised value w_t and drift $\dot{w}_t$ for the tax farmer's plan together with the corresponding debt-GDP ratio b_t and its drift $\dot{b}_t$ for the Department of Treasury's plan. The tax farmer gradually lowers the value it promises the representative consumer until it reaches $\underline{w} = 32.4$ at $t = 118$. The rate at which the tax farmer decreases the representative consumer's promised value w_t starts at $-\dot{w}_0 = 0.036$, then decreases until it equals 0 at $t = 118$. Promised value $w_t \geqslant 0$ evolves as it does because of how the representative consumer's impatience induces the tax authority to postpone higher tax rates and the associated deadweight tax collection losses. Panel B plots the corresponding b_t and $\dot{b}_t$ for the Department of

Treasury's debt management problem. Finally, starting from $\tau_0 = 17.5\%$ at $t = 0$, panel C illustrates how both plans make the tax rate τ_t increase gradually to $\tau(\underline{w}) = 24\%$ at $t = 120$.

Figure 3: **Scaled representative consumer's promised value w_t and debt-GDP ratio b_t dynamics, and transition of optimal tax rates τ_t.** The initial level of w is set at $w_0 = 35.5$, which corresponds to $b_0 = 0$.



7 Concluding Remarks

We have presented an economic environment in which a Department of Treasury and a tax farmer, neither of which can commit to keeping its promises, but either of which would have access to outside financiers, would administer identical taxation and debt management policies. Impatience of the domestic representative consumer relative to outside creditors drives both the timing of tax collections and debt issuance patterns. The two arrangements achieve the same welfare outcomes. Default options and tax distortions interact to determine sustainable debt limits and dynamics of tax rates, government risk-free debt, and the representative consumer's continuation value.

That our tax farmer's and our Department of Treasury's problems are duals shapes how we might want use it to interpret historical evidence like that sketched in section 2. In our model, the contest between a tax farm and a Department of Treasury is a toss-up. Section 2 indicated that tax farming was widespread before 1800, but that after 1800 more and more governments chose to set up Departments of Treasury. To explain that pattern, we would have to add complications to the economic environment that capture aspects of administering and collecting taxes that make it "so very complicated a concern," in Adam Smith's words. We would have somehow to give tax farmers a relative advantage in managing those concerns in Europe before 1800, but to give the state a relative advantage after 1800. Features of our model environment that could be altered include (i) the absence from our model of "agency" or "moral hazard" problems of the tax authority vis a vis people that actually collect taxes; (ii) the identical deadweight loss functions $\Theta(\cdot,\cdot), \widehat{\Theta}(\cdot,\cdot)$ that we ascribe to tax farmers and Departments of Treasury; (iii) the equal consequences under both arrangements after stopping time $T^{\mathcal{D}}$; or (iv) incomplete markets in the sense, that neither the tax farmer nor the Department of Treasury has access to the Shiller security but only instead, say, a single non-state-contingent security in the spirit of Aiyagari et al. (2002). It would be worthwhile to perturb our model in one or more of these directions and then to watch how that breaks duality between the tax farmer and Department of Treasury problems, making one or the other socially better. Structuring research along these lines could take advantage of evidence assembled by Fukuyama (2014) about origins of efficient state bureaucracies.

Cross-continent historical evidence could shed light on considerations missing from our model that tilt outcomes for or against tax farming. Thus, the fact that tax farming was rarely used in ancient China reflects a long-standing administrative philosophy that fostered a professional bureaucracy that collected revenues for the state. A notable exception occurred during the Mongol Yuan Dynasty (1271-1368). They were foreigners who imported Moslem tax farming arrangements into China. But those arrangements did not last. The Ming Dynasty (1368-1644) restored traditional Chinese practices that relied on state-run firms to raise revenues, for example, national monopolies of salt and iron production. Fukuyama (2014) emphasizes that China had effective state bureaucracies long before European countries.

A Technical Details

HJB equation for $F(W,Y)$. Consider the interior region where $dJ_t = 0$. Using Ito's formula, we obtain the following SDF-adjusted dynamics for the tax farmer's value function $F(W_t, Y_t)$:

$$d(\mathbb{M}_t F(W_t, Y_t)) = \mathbb{M}_t dF(W_t, Y_t) + F(W_t, Y_t) d\mathbb{M}_t + < d\mathbb{M}_t, dF(W_t, Y_t) >, \tag{58}$$

where the SDF $\{\mathbb{M}_t; t \geqslant 0\}$ is given in (3) and

$$\begin{aligned} dF(W_t, Y_t) &= F_W dW_t + \frac{F_{WW}}{2} < dW_t, dW_t > + F_Y dY_t + \frac{F_{YY}}{2} < dY_t, dY_t > + F_{WY} < dW_t, dY_t > \\ &= \left[\left((\zeta + r) W_t + (Y - \mathcal{T}_t - \Theta(\mathcal{T}_t, Y_t)) - \lambda \Phi_t \right) F_W + g Y_t F_Y \right] dt \\ &\quad + \left[\frac{\sigma^2 Y_t^2 F_{YY}}{2} + \frac{\sigma^2 \Phi_t^2 F_{WW}}{2} - \sigma^2 \Phi_t Y_t F_{WY} \right] dt - \sigma \Phi_t F_W d\mathcal{Z}_t^Y + \sigma Y_t F_Y d\mathcal{Z}_t^Y . \end{aligned} \tag{59}$$

The process defined by

$$\int_0^t \mathbb{M}_s \left(\mathcal{T}_s - \Gamma_s \right) ds - \mathbb{M}_s dJ_s + \mathbb{M}_t F(W_t, Y_t)$$

is a martingale under physical measure $\mathbb{P}$ so its drift under $\mathbb{P}$ is zero:

$$\mathbb{E}_t \left[d \left(\mathbb{M}_t F(W_t, Y_t) \right) \right] + \mathbb{M}_t \left(\mathcal{T}_t - \Gamma_t \right) dt = 0. \tag{60}$$

Simplifying (60) gives HJB equation (24) for value function $F(W_t, Y_t)$. First-order conditions (FOCs) for tax and risk management policies, respectively, are given in (25) and (26).

HJB equation for $\widehat{W}(\widehat{Y})$. To obtain (18), we construct the following Martingale using definition (13):

$$\int_0^t e^{-\zeta s} \mathbb{M}_s \left(\widehat{Y}_s - \widehat{\mathcal{T}}_s - \widehat{\Theta}(\widehat{\mathcal{T}}_s, \widehat{Y}_s) \right) ds + e^{-\zeta t} \widehat{W}(\widehat{Y}_t) \mathbb{M}_t .$$

Applying the Martingale representation theorem, we obtain

$$\mathbb{E}_t \left[e^{-\zeta t} \mathbb{M}_t \left(\widehat{Y}_t - \widehat{\mathcal{T}}_t - \widehat{\Theta}(\widehat{\mathcal{T}}_t, \widehat{Y}_t) \right) dt + d \left(e^{-\zeta t} \widehat{W}(\widehat{Y}_t) \mathbb{M}_t \right) \right] = 0 . \tag{61}$$

(18) can be obtained directly from applying Ito's lemma to (61).

References

Aguiar, Mark and Gita Gopinath. 2006. Defaultable debt, interest rates and the current account. *Journal of International Economics* 69 (1):64–83.

Aguiar, Mark, Manuel Amador, and Gita Gopinath. 2009. Investment cycles and sovereign debt overhang. *The Review of Economic Studies* 76 (1):1–31.

Ai, Hengjie and Rui Li. 2015. Investment and CEO compensation under limited commitment. *Journal of Financial Economics* 116 (3):452–472.

Aiyagari, S Rao, Albert Marcet, Thomas J Sargent, and Juha Seppälä. 2002. Optimal taxation without state-contingent debt. *Journal of Political Economy* 110 (6):1220–1254.

Alvarez, Fernando and Urban J Jermann. 2001. Quantitative asset pricing implications of endogenous solvency constraints. *The Review of Financial Studies* 14 (4):1117–1151.

Arellano, Cristina. 2008. Default risk and income fluctuations in emerging economies. *American Economic Review* 98 (3):690–712.

Atkeson, Andrew. 1991. International Lending with Moral Hazard and Risk of Repudiation. *Econometrica* 59 (4):1069–1089.

Barro, Robert J. 1979. On the determination of the public debt. *Journal of Political Economy* 87 (5, Part 1):940–971.

Biais, Bruno, Thomas Mariotti, Guillaume Plantin, and Jean-Charles Rochet. 2007. Dynamic security design: Convergence to continuous time and asset pricing implications. *The Review of Economic Studies* 74 (2):345–390.

Black, Fischer and Myron Scholes. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 637–654.

Blanchard, Olivier. 2019. Public Debt and Low Interest Rates. *American Economic Review* 109 (4):1197–1229.

Bolton, Patrick, Neng Wang, and Jinqiang Yang. 2019. Optimal contracting, corporate finance, and valuation with inalienable human capital. *The Journal of Finance* 74 (3):1363–1429.

DeMarzo, Peter M and Michael J Fishman. 2007. Optimal long-term financial contracting. *The Review of Financial Studies* 20 (6):2079–2128.

DeMarzo, Peter M and Yuliy Sannikov. 2006. Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance* 61 (6):2681–2724.

DeMarzo, Peter M, Michael J Fishman, Zhiguo He, and Neng Wang. 2012. Dynamic agency and the q theory of investment. *The Journal of Finance* 67 (6):2295–2340.

Duffie, Darrell. 2001. *Dynamic asset pricing theory*. Princeton University Press.

Eaton, Jonathan and Mark Gersovitz. 1981. Debt with potential repudiation: Theoretical and empirical analysis. *The Review of Economic Studies* 48 (2):289–309.

Fukuyama, Francis. 2014. *Political order and political decay: From the industrial revolution to the globalization of democracy*. Macmillan.

Green, Edward J. 1987. Lending and the Smoothing of Uninsurable Income. In *Contractual Arrangements for Intertemporal Trade, Minnesota Studies in Macroeconomic Series*, edited by Edward C. Prescott and Neil Wallace, 3–25. Minneapolis, Minnesota: University of Minnesota.

Jiang, Wei, J. Thomas Sargent, Neng Wang, and Jinqiang Yang. 2024. Managing government debt. *Proceedings of the National Academy of Sciences* 121 (11).

Jiang, Wei, Thomas J. Sargent, Neng Wang, and Jinqiang Yang. 2026. A p Theory of Taxes and Debt Management. *The Journal of Finance* Forthcoming.

Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z Xiaolan. 2019. The US public debt valuation puzzle. Tech. rep., National Bureau of Economic Research.

———. 2020. Manufacturing risk-free government debt. Tech. rep., National Bureau of Economic Research.

Kehoe, Timothy J and David K Levine. 1993. Debt-constrained asset markets. *The Review of Economic Studies* 60 (4):865–888.

Keynes, John Maynard. 1923. *A Tract on Monetary Reform*. London: MacMillan and Company.

———. 1931. *Essays in Persuasion*, chap. An open letter to the French Minister of Finance (1926). R. & R. Clark,Limited, Edinburgh.

Lucas, Robert E, Jr. 1978. Asset prices in an exchange economy. *Econometrica: Journal of the Econometric Society* 46 (6):1429–1445.

Mehrotra, Neil R and Dmitriy Sergeyev. 2021. Debt sustainability in a low interest rate world. *Journal of Monetary Economics* 124:S1–S18.

Merton, Robert C. 1971. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* 3:373–413.

———. 1973. Theory of rational option pricing. *The Bell Journal of economics and management science* 141–183.

Phelan, Christopher and Robert M. Townsend. 1991. Computing Multi-Period, Information-Constrained Optima. *Review of Economic Studies* 58 (5):853–881.

Sannikov, Yuliy. 2008. A continuous-time version of the principal-agent problem. *The Review of Economic Studies* 75 (3):957–984.

Sargent, Thomas J and François R Velde. 1995. Macroeconomic features of the French Revolution. *Journal of Political Economy* 103 (3):474–518.

Shiller, Robert J. 1994. *Macro markets: creating institutions for managing society's largest economic risks*. OUP Oxford.

Smith, Adam. 1776. *An Inquiry into the Nature and Causes of the Wealth of Nations*. London: W. Strahan and T. Cadell.