Robust control of forward-looking models

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ABSTRACT

This paper shows how to formulate and compute robust Ramsey (aka Stackelberg) plans for linear models with forward looking private agents. The leader and the followers share a common approximating model and both have preferences for robust decision rules because both doubt the model. Since their preferences differ, the leader's and followers' decision rules are fragile to different misspecifications of the approximating model. We define a Stackelberg equilibrium with robust decision makers in which the leader and follower have different worst-case models despite sharing a common approximating model. To compute a Stackelberg equilibrium we formulate a Bellman equation that is associated with an artificial single-agent robust control problem. The artificial Bellman equation contains a description of implementability constraints that include Euler equations that describe the worst-case analysis of the followers. As an example, the paper analyzes a model of a monopoly facing a competitive fringe.

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1. Introduction

This paper is a prolegomenon to subsequent work that analyzes the quantitative effects of preferences for robustness on Ramsey policies in forward-looking monetary policy models like those of Clarida, Gali, and Gertler (1999, 2000), King and Wolman (1999), Goodfriend and King (1997), Rotemberg and Woodford (1997), and Woodford (1998). Such work requires an equilibrium concept that somehow renders consistent the multiplicity of models that are in play when decision makers fear misspecification. This paper formulates such an equilibrium concept for Ramsey or Stackelberg problems in which both the leader and follower doubt the model specification and therefore prefer robust decision rules. As a consistency condition that extends rational expectations to our setting, we impose that the leader and follower share a common approximating model. The same appeals to economy and discipline that recommend rational expectations in environments without model uncertainty can be used to justify our imposing a common approximating model here.

Robust control theory instructs decision makers to investigate the fragility of decision rules by conducting worst-case analyses. When both types of agent prefer robustness, the approximating model for each agent must include a description of the robust decision rules of the other type of agent, and of how they respond to his own actions. Though they share a common approximating model, because their preferences may differ, the different types of agent may *not* share the same worst-case model. In order completely to describe the common approximating model of the two types of agents, the Stackelberg leader requires an adequate description of the dynamics of the worst case shocks of the followers.

Without preferences for robustness, it is known that these Stackelberg or Ramsey problems can be solved by forming a Lagrangian in which a sequence of multipliers adheres to a sequence of the followers' Euler equations. The followers' Euler equations become 'implementability constraints' that require the leader's decision at time t to confirm forecasts that had informed followers' earlier decisions. The Lagrange multipliers on the implementability constraints make the leader's actions depend on the history of the economy and allow a recursive representation of the leader's history-dependent Stackelberg plan.

To compute a Stackelberg plan when the followers have a preference for robustness, similar procedures apply, except that we have to augment the implementability constraints with Euler equations that characterize the follower's choice of a worst case model. We attach implementability multipliers to these constraints too, then proceed much as we would in the analysis without preferences for robustness.[1]

[1] As noted below, this strategy exploits the remarkable Bellman-Isaacs condition that characterizes the zero-sum two-person game that the follower uses to devise a robust decision rule.

1.1. Motivation

Milton Friedman expressed an enduring concern when he recommended that designers of macroeconomic policy rules acknowledge model uncertainty. His style of analysis revealed that he meant a kind of model uncertainty that could not be formalized in terms of objective or subjective probability distributions over models. For despite his willingness to use Savage's (1954) axioms in other work (see Friedman and Savage (1948)), sometimes he declined to use them in his writings on monetary policy. Instead Friedman sometimes indicated that he knew too little about the dynamic structure of the economy to allow him to specify the personal probabilities needed to rationalize policy choices using Savage's method. To support his preference for policy rules that don't depend on detailed knowledge about the workings of the economy, Friedman mentioned unknown long and variable lags in the effects of policies.

Recently Bennett McCallum and others have taken up Friedman's theme by embracing a research program designed to evaluate the robustness of monetary policy rules across sets of alternative models. In his comment on Rotemberg and Woodford (1997), McCallum repeated his preference for using multiple models when analyzing a proposed rule:[2]

> I have favored a research strategy centering around a rule's *robustness*, in the following sense: Because there is a great deal of professional disagreement as to the proper specification of a structural macroeconomic model, it seems likely to be more fruitful to strive to design a policy rule that works reasonably well in a variety of plausible quantitative models, rather than to derive a rule that is optimal in any one particular model. (McCallum, 1997, p. 355).

McCallum also criticized analyses of monetary policy rules that unrealistically assume that latent variables like 'potential GNP' or 'trend productivity growth' are observed without error by the policy maker.[3]

This paper describes how robust control can be used to address some of McCallum's concerns about robustness.[4] We extend earlier work on robust control theory to solve Ramsey or Stackelberg problems – i.e., problems of optimal government policy where the government can commit itself, but where neither the government nor the private agents doubt the common (rational expectations) model that they share. Our use of robust control theory in the context of model uncertainty and our desire to put private agents and the government on the same footing, with both having doubts about model misspecification, induces us to model robustness differently than do McCallum and his co-workers. They take a small set of particular models and evaluate the performance of a given rule across those models. Typically, at least some of those models are rational expectations models in which the agents forecast using the model itself. In contrast, we explicitly specify only a single model and require all of the agents inside the model to share it. This commonality of models lends our approach much of the same discipline that rational expectations models acquire by having only a single model in play. But here the decision makers all treat their common model as an approximation. Each agent surrounds the approximating model with

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[2] Blinder (1998, pp. 12-13) advocates a procedure for assessing the robustness of proposed decisions.

[3] Several of the contributions in Taylor (1999) share McCallum's concern for robustness.

[4] In related work (Hansen and Sargent, (2004)), we take up issues about robust filtering that pertains to signal extraction in the presence of doubts about model specification. We hope soon to apply this work to McCallum's concern about the implications of latent variables for designing policy.

a continuum of unspecified alternative models that fit the data nearly as well. Both the government and the agents inside the model explicitly confront their concern about model misspecification in making forecasts and designing policies. We allow possibly different but also possibly equal concerns about model misspecification within the private sector and the government, modelled as different sets of models surrounding the approximating model. As under rational expectations, we propose an equilibrium concept in which the private sector and the government share the same approximating model of the stochastic variables shaking the economy. But both types of agent have doubts about that model in the form of possibly different penumbras of alternative models that surround the approximating models.[5] These alternative models can be difficult to distinguish statistically from the approximating model based on finite data sets, so that the government's and public's concerns about model misspecification are well founded.

1.2. A word about practicality

While the multiplicity of models that any analysis of misspecification must acknowledge can make robust control theory seem intimidating at first encounter, it is actually easy to use. One purpose of this paper is to convince the reader of this. Robust decision rules are justified and can be computed as the Markov perfect equilibrium of a zero-sum two person game. The zero-sum feature means that there is only one value function that can be computed by solving a Bellman equation that is as simple as the ones we now routinely solve in macroeconomics. Because we work in a linear-quadratic context, a matrix Riccati equation is associated with the Bellman equation. For us, the key step in computing an equilibrium is to solve that Riccati equation, which is easy.

1.3. Related literature

Brunner and Meltzer (1969) and von zur Muehlen (1982) were early advocates of zero-sum two person games for representing model uncertainty and designing macroeconomic rules. Stock (1999), Sargent (1999), and Onatski and Stock (2002) have used versions of robust control theory to study robustness of purely backward looking macroeconomic models. They focused on whether a concern for robustness would make policy rules more or less aggressive in response to shocks. Blanchard and Khan (1980), Whiteman (1983), and Anderson and Moore (1985) are early sources on solving control problems with forward-looking private sectors.[6] Without a concern for robustness, Kydland and Prescott (1980), Hansen, Epple, and Roberds (1985), Miller and Salmon (1985a, 1985b), Oudiz and Sachs (1985), Sargent (1987), Currie and Levine (1987), Pearlman, Currie, and Levine (1986), Pearlman (1992), Woodford (1998), King and Wolman (1999), and Marcet and Marimon (1999) have solved Stackelberg or Ramsey problems using Lagrangian formulations. Pearlman, Currie, and Levine (1986), Pearlman (1992) and Svensson and Woodford (2000) study the control of forward looking models where part of the state is unknown and must be filtered. De Jong, Ingram, and Whiteman (1996), Otrok (In press), and others study

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[5] These differences are parameterized by the robustness parameters Θ and θ that we assign to the leader and the follower below.

[6] Anderson, Hansen, McGrattan, and Sargent (1996) describe efficient computational algorithms for such models.

the Bayesian estimation of forward looking models. They summarize the econometrician's doubts about parameter values with a prior distribution, meanwhile attributing no doubts about parameter values to the private agents in their models. Mark Giannoni (2002) studies robustness in a forward looking macro model. He models the policy maker as knowing all parameters except two, for which he knows only bounds. The policy maker then computes the $\min - \max$ policy rules. Kasa (2002) also studies robust policy in a forward looking model. Onatski (2000) designs simple (not history dependent) robust policy rules for a forward looking monetary model. Christiano and Gust (1999) study robustness from the viewpoint of the determinacy and stability of rules under nearby parameters. They adopt a perspective of robust control theorists like Başar and Bernhard (1995) and Zhou, Doyle, and Glover (1996), who are interested in finding rules that stabilize a system under the largest set of departures from a reference model.[7]

1.4. Organization

The remainder of this paper is organized as follows. Section 2 states a Stackelberg problem in which decision makers fear model misspecification and therefore want robustness. Section 3 describes how to solve the robust Stackelberg problem by properly rearranging and reinterpreting some state variables and some Lagrange multipliers after having solved a robust linear regulator. As an example, section 5 describes a dynamic model of a monopolist facing a competitive fringe. Section 6 tells our plans to apply our equilibrium concept to 'new synthesis' macro models.

2. The robust Stackelberg problem

This section defines a robust Stackelberg problem where the Stackelberg leader is concerned about model misspecification. In macroeconomic problems, the Stackelberg leader is often a government and the Stackelberg follower is a representative agent within a private sector. In section 5, we present an application with an interpretation of the two players as a monopolist and a competitive fringe.

Let z_t be an $n_z \times 1$ vector of natural state variables, x_t an $n_x \times 1$ vector of endogenous variables free to jump at t, and U_t a vector of the leader's controls. The z_t vector is inherited from the past. The model determines the 'jump variables' x_t at time t. Included in x_t are prices and quantities that adjust to clear markets at time t. Let $y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}$.

[7] Some of our previous work about robustness is contained in Hansen, Sargent, and Tallarini (1999) and Hansen, Sargent, and Wang (2002), both of which apply single agent robust decision theory to representative agent pricing models. Anderson (1998)'s work that computes Pareto problems when agents have differing degrees of risk-sensitivity can be interpreted as one of the few applications of robustness to environments with heterogenous agents. For important and useful references on robustness, see Jacobson (1973), Whittle (1990, 1996), Başar and Bernhard (1995), Glover and Doyle (1988), Mustafa and Glover (1990), and Zhou, Glover, and Doyle (1996).

Define the Stackelberg leader's one-period loss function[8]

$$r(y, U) = y'Qy + U'RU. \tag{2.1}$$

The leader wants to maximize

$$-\sum_{t=0}^{\infty} \beta^t r(y_t, U_t). \tag{2.2}$$

The leader makes policy in light of a set of models indexed by a vector of specification errors W_{t+1} around its approximating model:

$$\begin{bmatrix} I & 0 \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \hat{B}U_t + \hat{C}W_{t+1}. \tag{2.3}$$

We assume that the matrix on the left is invertible, so that[9]

$$\begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + BU_t + CW_{t+1} \tag{2.4}$$

or

$$y_{t+1} = Ay_t + BU_t + CW_{t+1}. \tag{2.5}$$

2.1. Characterizing the followers

The followers' behavior is summarized by the second block of equations of (2.3) or (2.4). These typically include the first-order conditions of private agents' optimization problem (i.e., their Euler equations). These equations summarize the forward looking aspect of the followers' behavior. The particular structure of these equations and the variables composing x_t depend on the followers' optimization problems, and in particular, on whether we impute a concern about robustness to them. As we shall see later, if we impute a motive for robustness to the followers, then it is necessary to include what we shall denote as w_{t+1}, the specification errors of the followers, among the variables in x_t. In section 5, we'll display a concrete example.

[8] The problem assumes that there are no cross products between states and controls in the return function. A simple transformation converts a problem whose return function has cross products into an equivalent problem that has no cross products. See Anderson, Hansen, McGrattan, and Sargent (1996).

[9] We have assumed that the matrix on the left of (2.3) is invertible for ease of presentation. However, by appropriately using the invariant subspace methods described in Anderson, Hansen, McGrattan, and Sargent (1996), it is straightforward to adapt the computational method when this assumption is violated.

2.2. Modelling specification errors

Returning to (2.3) or (2.4), the vector W_{t+1} of unknown specification errors can feed back, possibly nonlinearly, on the history y^t, which lets the W_{t+1} sequence represent misspecified dynamics. The leader regards its approximating model (which has $W_{t+1} = 0$) as a good approximation to the unknown true model in the sense that the unknown W_{t+1} sequence satisfies

$$\sum_{t=0}^{\infty} \beta^{t+1} W'_{t+1} W_{t+1} \leq \eta_0 \tag{2.6}$$

where $\eta_0 > 0$.

A certainty equivalence principle discussed by Hansen and Sargent (2004) allows us to work with non stochastic approximating and distorted models. We would attain the same decision rule if we were to replace x_{t+1} with the forecast $E_t x_{t+1}$ and to add a shock process $\hat{C}\epsilon_{t+1}$ to the right side of (2.3) or $C\epsilon_{t+1}$ to the right side of (2.4), where ϵ_{t+1} is an i.i.d. random vector with mean of zero and identity covariance matrix.[10]

Let X^t denote the history of any variable X from 0 to t. Kydland and Prescott (1980), Miller and Salmon (1985a, 1985b), Oudiz and Sachs (1985), Hansen, Epple, and Roberds (1985), Pearlman, Currie and Levine (1986), Sargent (1987), Pearlman (1992) and others have studied non-robust (i.e., $\eta_0 = 0$) versions of the following problem:

Definition 2.1. For $\eta_0 > 0$, the *constraint version of the Stackelberg problem* is to extremize (2.2) subject to (2.5) by finding a sequence of decision rules expressing U_t and W_{t+1} as sequences of functions mapping the time t history of the state z^t into the time t decision. The leader chooses these decision rules at time 0 and commits to them forever.

Definition 2.2. When $\eta_0 > 0$, the decision rule for U_t that solves the Stackelberg problem is called a robust *Stackelberg plan* or robust *Ramsey plan.*

Note that the decision rules are designed to depend on the history of the true state z_t and not on the history of the jump variable x_t. For a non-robust version of the problem, the forementioned authors show that the optimal rule is history-dependent, meaning that U_t, W_{t+1} depend not only on z_t but also on lags of it. The history dependence comes from two sources: (a) the leader's ability to commit to a sequence of rules at time 0,[11] and (b) the forward-looking behavior of the followers that is embedded in the second block of equations in (2.3) or (2.4).

Fortunately, there is a recursive way of expressing this history dependence by having decisions U_t, W_{t+1} depend linearly only on the current value z_t and on a new component of the state vector, μ_{xt}. The component μ_{xt} is a vector of Lagrange multipliers on the last n_x equations of (2.3) or (2.4). Part of the solution of the problem in Definition 2.2 is then

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[10] If $C\epsilon_{t+1}$ were added to the right side of (2.5), we would take the expectation of (2.6). As discussed in Hansen and Sargent (2004), we could also reformulate the problem as one in which distorted shock distributions are selected instead of just shifts in the conditional means. When ϵ is normal, the resulting worst case shock distribution will be a shifted normal with an enhanced covariance matrix. Since a version of certainty equivalence applies, it is only the conditional mean distortion that is pertinent in computing robust decision rules. The analysis that follows thus focuses exclusively on the conditional mean distortion.

[11] The leader would make different choices were it to choose sequentially, that is, were it to set U_t at time t rather than at time 0.

a law of motion expressing μ_{xt+1} as a linear function of (z_t, μ_{xt}). The history dependence of the leader's plan is expressed in the dynamics of μ_{xt}. These multipliers track past leader promises about current and future settings of U. At time 0, if there are no past promises to honor, it is appropriate for the leader to initialize the multipliers to zero (this maximizes its criterion function). The multipliers take non zero values thereafter, reflecting the subsequent costs to the leader of adhering to its commitments.

2.3. Multiplier version of the robust Stackelberg problem

Hansen and Sargent (2004) show that it is usually more convenient to solve a *multiplier game* rather than a *constraint game*. Essentially, a multiplier game is obtained by attaching a non-negative Lagrange multiplier Θ to constraint (2.6) and formulating the constraint problem in terms of a Lagrangian. Accordingly, we use:

Definition 2.3. The *multiplier version of the robust Stackelberg problem* is the zero-sum two-player game:

$$\max_{\{U_t\}_{t=0}^{\infty}} \min_{\{W_{t+1}\}_{t=0}^{\infty}} -\sum_{t=0}^{\infty} \beta^t \left\{ r\left(y_t, U_t\right) - \beta \Theta W_{t+1}' W_{t+1} \right\} \tag{2.7}$$

where the extremization is subject to (2.5) and $\underline{\Theta} < \Theta < \infty$.

Recall again that the followers' behavior is embedded in Euler equations that are included in (2.5).

3. Solving the robust Stackelberg problem

This section describes a three step algorithm for solving a multiplier version of the robust Stackelberg problem.

3.1. Step 1: solve a robust linear regulator

Step 1 temporarily disregards the forward looking aspect of the problem (step 3 will take account of that) and notes that superficially the multiplier version of the robust Stackelberg problem (2.7), (2.5) has the form of a robust linear regulator problem. Mechanically, we can solve this artificial robust linear regulator by noting that associated with problem (2.7) is the Bellman equation[12]

$$v\left(y\right) = \max_{U} \min_{W} \left\{ -r\left(y, U\right) + \beta \Theta W' W + \beta v\left(y^*\right) \right\}, \tag{3.1}$$

where y^* denotes next period's value of the state and the extremization is subject to the transition law $y^* = Ay + BU + CW$. The solution has the form $v(y) = -y'Py$, where P is a fixed point of the operator $T \circ \mathcal{D}$ defined by

$$T\left(P\right) = Q + \beta A'PA - \beta^2 A'PB \left(R + \beta B'PB\right)^{-1} B'PA \tag{3.2}$$

$$\mathcal{D}\left(P\right) = P + \Theta^{-1} PC \left(I - \Theta^{-1} C'PC\right)^{-1} C'P. \tag{3.3}$$

[12] This Bellman equation is closely related to the one on page 87 of Kydland and Prescott (1980), because for the problems that we have in mind, Kydland and Prescott's λ can be shown to be an invertible function of the jump variables.

Thus, the Bellman equation (3.1) leads to the Riccati equation

$$P = T \circ \mathcal{D}(P). \tag{3.4}$$

Here the T operator emerges from the maximization over U on the right side of (3.1), while the $\mathcal{D}$ operator emerges from the minimization over W. The extremizing decision rules are given by $U_t = -F_1 y_t$ where

$$F_1 = \beta \left(R + \beta B'\mathcal{D}(P) B\right)^{-1} B'\mathcal{D}(P) A \tag{3.5}$$

and $W_{t+1} = -F_2 y_t$ where

$$F_2 = -\Theta^{-1}\left(I - \Theta^{-1} C'PC\right)^{-1} C'P(A - BF_1). \tag{3.6}$$

(See Hansen and Sargent (2004) or Başar and Bernhard (1995).)

The next steps recognize how the solution of the Riccati equation $P = T \circ \mathcal{D}$ encodes objects that solve the robust Stackelberg problem. That will tell us how to manipulate the decision rules for U_t and W_{t+1} (linear functions identified by the vectors (3.5) and (3.6)) to get the solution of the robust Stackelberg problem.

3.2. Step 2: use the stabilizing properties of shadow price Py_t

At this point we use P to describe how shadow prices on the transition law relate to the artificial state vector $y_t = [\, z_t' \quad x_t' \,]'$ (we say 'artificial' because x_t is a vector of jump variables.) Linear quadratic dynamic programming problems can also be solved with Lagrangian methods (see Anderson, Hansen, McGrattan, and Sargent (1996) and Hansen and Sargent (2004).) Thus, another way to solve the multiplier version of the robust Stackelberg problem (2.7), (2.5) is to form the Lagrangian:

$$\mathcal{L} = -\sum_{t=0}^{\infty} \beta^t \left[y_t' Q y_t + U_t' R U_t + 2\beta \mu_{t+1}' (A y_t + B U_t + C W_{t+1} - y_{t+1}) - \beta \Theta W_{t+1}' W_{t+1} \right]. \tag{3.7}$$

We want to maximize (3.7) with respect to sequences for U_t and y_{t+1} and minimize it with respect to a sequence for W_{t+1}. The first-order conditions with respect to U_t, y_t, W_{t+1}, respectively, are:

$$0 = RU_t + \beta B' \mu_{t+1} \tag{3.8a}$$

$$\mu_t = Q y_t + \beta A' \mu_{t+1} \tag{3.8b}$$

$$0 = \beta \Theta W_{t+1} - \beta C' \mu_{t+1}. \tag{3.8c}$$

Solving (3.8a) and (3.8c) for U_t and W_{t+1} and substituting into (2.5) gives

$$y_{t+1} = A y_t - \beta \left(B R^{-1} B' - \beta^{-1} \Theta^{-1} C C' \right) \mu_{t+1}. \tag{3.9}$$

Write (3.9) as

$$y_{t+1} = A y_t - \beta \tilde{B} \tilde{R}^{-1} \tilde{B}' \mu_{t+1} \tag{3.10}$$

where $\tilde{B} = [\,B \quad C\,]$ and $\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & -\beta\theta I \end{bmatrix}$. We can represent the system formed by (3.10) and (3.8b) as

$$\begin{bmatrix} I & \beta \tilde{B}\tilde{R}^{-1}\tilde{B}' \\ 0 & \beta A' \end{bmatrix} \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} y_t \\ \mu_t \end{bmatrix} \tag{3.11}$$

or

$$L^* \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = N \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}. \tag{3.12}$$

We want to find a 'stabilizing' solution of (3.12), i.e., one that satisfies

$$\sum_{t=0}^{\infty} \beta^t y_t' y_t < +\infty.$$

The stabilizing solution is attained by setting $\mu_0 = P y_0$, where P solves the matrix Riccati equation $P = T \circ \mathcal{D}(P)$. The solution for μ_0 replicates itself over time in the sense that

$$\mu_t = P y_t. \tag{3.13}$$

3.3. Key insight

In a typical robust linear regulator problem, y_0 is a state vector inherited from the past; the multiplier μ_0 jumps at $t = 0$ to satisfy $\mu_0 = P y_0$. See Anderson, Hansen, McGrattan, and Sargent (1996). But in the Stackelberg problem, pertinent components of *both* y_0 *and* μ_0 must adjust to satisfy $\mu_0 = P y_0$, as shown in step 3.

3.4. Step 3: convert implementation multipliers into state variables

Partition μ_t conformably with the partition of y_t' into $[\,z_t' \quad x_t'\,]$:[13]

$$\mu_t = \begin{bmatrix} \mu_{zt} \\ \mu_{xt} \end{bmatrix}.$$

For the robust Stackelberg problem, only the first n_z elements of y_t are predetermined while the remaining components are free to jump at t. And while the first n_z elements of μ_t are free to jump at t, the remaining components are not. The third step completes the solution of the robust Stackelberg problem by taking note of these facts. We convert the last n_x Lagrange multipliers μ_{xt} into state variables by using the following procedure *after* we have performed the key step of computing the P that solves the Riccati equation $P = T \circ \mathcal{D}(P)$.

Write the last n_x equations of (3.13) as

$$\mu_{xt} = P_{21} z_t + P_{22} x_t. \tag{3.14}$$

[13] This argument just adapts one in Pearlman (1992). The Lagrangian associated with the robust Stackelberg problem remains (3.7). Then the logic of section 3.2 implies that the stabilizing solution must satisfy (3.13). It is only in how we impose (3.13) that the solution diverges from that for the linear regulator.

The vector μ_{xt} becomes part of the state at t, while x_t is free to jump at t. Therefore, solve (3.14) for x_t in terms of (z_t, μ_{xt}):

$$x_t = -P_{22}^{-1} P_{21} z_t + P_{22}^{-1} \mu_{xt}. \tag{3.15}$$

Then we can write

$$y_t = \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{3.16}$$

and from (3.14)

$$\mu_{xt} = [\, P_{21} \quad P_{22} \,]\, y_t. \tag{3.17}$$

With these modifications, the key formulas (3.5), (3.6), and (3.4) from the optimal linear regulator for F and P, respectively, continue to apply. Using (3.16), the solutions for the control and worst case shock are

$$\begin{bmatrix} U_t \\ W_{t+1} \end{bmatrix} = \begin{bmatrix} -F_1 \\ -F_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{3.18}$$

Using the law of motion for y_{t+1} together with (3.16) and (3.17) allows us to represent our solution recursively as

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_{21} & P_{22} \end{bmatrix} (A - BF_1 - CF_2) \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{3.19a}$$

$$x_t = [\, -P_{22}^{-1} P_{21} \quad P_{22}^{-1} \,] \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{3.19b}$$

When the random shock ϵ_{t+1} is present, we must add

$$\begin{bmatrix} I & 0 \\ P_{21} & P_{22} \end{bmatrix} C \epsilon_{t+1} \tag{3.20}$$

to the right side of (3.19). Equation (3.19a) is the worst-case law of motion for z_t. To get the law of motion under the approximating model and the robust Stackelberg or Ramsey plan, we replace $(A - BF_1 - CF_2)$ with $A - BF_1$ in (3.19a). By doing so, we set the worst-case shock W_{t+1} to zero. Then we have the following description of the approximating model under the robust Stackelberg plan:

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_{21} & P_{22} \end{bmatrix} (A - BF_1) \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{3.21a}$$

$$x_t = [\, -P_{22}^{-1} P_{21} \quad P_{22}^{-1} \,] \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{3.21b}$$

Again, in the random case we must add (3.20) to the right side of (3.21). The difference equation (3.21a) is to be initialized from the given initial value for z_0 and the value $\mu_{x,0} = 0$. The latter setting reflects that at time 0 there are no past promises to keep.

In summary, we solve the robust Stackelberg problem by formulating a particular optimal linear regulator, solving the associated matrix Riccati equation (3.4) for P, computing F_1, F_2, and then partitioning P to obtain representation (3.21).[14]

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[14] For some purposes, it is useful to eliminate the implementation multipliers μ_{xt} and to express the decision rules for U_t and W_{t+1} as functions of z_t, z_{t-1} and U_{t-1}. Hansen and Sargent (2004) show how this can be done. By making the leader's control or 'instrument' feed back on itself, this form of the decision rule for U_t allows 'instrument-smoothing' to emerge as an optimal rule under commitment. This insight partly motivated Woodford (1998) to use his model as a tool to interpret empirical evidence about interest rate smoothing in the U.S.

By following the approaches of Kydland and Prescott (1980) and Marcet and Marimon (2000), Hansen and Sargent (2004, chapter 15) describe a closely related Bellman equation that can be used to compute a robust Ramsey plan.

4. Incorporating robustness for the followers

So far we have concentrated on getting a robust rule for the leader, taking as given the Euler equations that characterize the followers' behavior. In this section, we point out that by including the appropriate Euler equations for the followers among the implementability constraints, we can impute a concern for robustness to the followers as well as to the leader. For a representative follower example, we shall index the concern for robustness among the followers by a multiplier θ that can but need not equal the robustness parameter Θ of the leader.

4.1. An approach enabled by the Bellman-Isaacs condition

To apply the preceding results to a problem in which the Stackelberg leader and the Stackelberg followers both want robust decision rules, we have to include Euler equations for the follower that incorporate a concern about robustness. To formulate these implementability constraints concisely, we can rely on findings about the zero-sum two-player dynamic game that underlies the single-agent robust control problem. Başar and Bernhard (1995) and Hansen and Sargent (2004) show that the equilibrium outcomes are identical for several games with different timing protocols for the maximizing and minimizing players. Among these different timing protocols is one in which both players simultaneously choose entire sequences of state-contingent decisions at time 0. By using that timing protocol for the follower's two-person zero-sum game, we can represent the followers' decisions by the stabilizing solution of the follower's Euler equations for extremizing with respect to *both* his 'natural control' u_t *and* his pseudo-control w_{t+1}, the worst case shocks.[15] Then to formulate the robust Stackelberg problem, we can regard the first-order conditions of the competitive firm, *including* those for choosing the follower's worst-case shock process, as among the implementability conditions for the monopolist. This leads to an equilibrium of the game between the leader and the follower in which each understands the decision

[15] A Bellman-Isaacs condition on the value function described by Hansen and Sargent (2004) allows us to characterize the solution of the robust control problem in this way. A substantial and very important result is being used here. For general two-person games, the Markov-perfect equilibrium *cannot* be computed by stacking and solving the Euler equations for the two players. Doing that would produce a candidate equilibrium that would not be subgame perfect. But under the Bellman-Isaacs condition, which pertains to two-player zero-sum games, a Markov perfect equilibrium *can* be computed by stacking and solving the Euler equations. For proofs, see Başar and Bernhard (1995) and Hansen and Sargent (2004). Technically, the irrelevance of timing protocols for zero-sum two-player dynamic games is related to Chari, Kehoe, and Prescott's (1989, pp. 269–272) characterization of time-inconsistency in macroeconomics as pertaining only to situations in which there is conflict between a society's objective and those of the agents within it. Chari, Kehoe, and Prescott show that without such conflict, the existence of a single value function makes irrelevant the order of maximization. Comparing their result to the similar one based on the Bellman-Isaacs condition for two-player zero-sum dynamic games, it can be seen that to avoid time inconsistency requires only that objective functions of different decision makers be completely *aligned*, a condition that allows complete conflict.

rules of the other, and in which the leader takes into account how the follower's decisions respond to its own. To know how the follower responds, the leader has to keep track of how the worst case shocks of the follower respond to the leader's decisions. This impels us to include the worst case shock process of the followers in the state vector for the leader.

By following this recipe, we can construct an equilibrium in which leaders and followers share a common approximating model. However, differences in their preferences can lead them to slant their worst case models in different directions away from their common approximating model, as the two types of agents use their own worst-case analyses to investigate the fragility of alternative rules to possible misspecifications of that common approximating model. In the next section, we illustrate our equilibrium concept with an example.

5. A monopolist with a competitive fringe

As an example, this section studies an industry with a large firm that acts as a Stackelberg leader with respect to a competitive fringe. The industry produces a single nonstorable homogeneous good. One large firm called the monopolist produces Q_t and a representative firm in a competitive fringe produces q_t. We use q_t to denote the quantity chosen by the individual competitive firm and $\overline{q}_t$ to denote the equilibrium quantity. In equilibrium, $q_t = \overline{q}_t$, but it is necessary to distinguish between q_t and $\overline{q}_t$ in posing the optimum problem of the representative competitive firm. The representative firm in the competitive fringe takes Q_t and $\overline{q}_t$ as exogenous and chooses sequentially. In light of the responses of the representative firm in the competitive fringe, the monopolist commits to a policy at time 0, taking into account its ability to manipulate the price sequence *and* the worst case beliefs of the representative competitive firm through its quantity choices. Subject to the competitive fringe's best response, the monopolist views itself as choosing $\overline{q}_{t+1}$ *and* Q_{t+1} for $t \geq 0$, as well as the representative competitive firm's worst-case shock process w_{t+1} for $t \geq 0$.

Costs of production are $\mathcal{C}_t = eQ_t + .5gQ_t^2 + .5c(Q_{t+1} - Q_t)^2$ for the monopolist and $\sigma_t = dq_t + .5hq_t^2 + .5c(q_{t+1} - q_t)^2$ for the representative competitive firm, where $d > 0, e > 0, c > 0, g > 0, h > 0$ are cost parameters. There is a linear inverse demand curve

$$p_t = A_0 - A_1 \left(Q_t + \overline{q}_t\right) + v_t, \tag{5.1}$$

where A_0, A_1 are both positive and v_t is a disturbance to demand governed by

$$v_{t+1} = \rho v_t + C_v \check{\epsilon}_{t+1} \tag{5.2}$$

and where $|\rho| < 1$ and $\check{\epsilon}_{t+1}$ is an i.i.d. sequence of random variables with mean zero and variance 1. The monopolist and the representative competitive firm share equation (5.2) as their approximating model for the demand shock. The monopolist and the representative competitive firm both want decision rules that are robust to alternative specifications of the process for the demand shock. Because the monopolist and the representative firm in the competitive fringe potentially have different worst case models of the demand shock, we distinguish between them by letting v_t denote the process perceived by the representative firm, and V_t the process perceived by the monopolist. For the representative competitive firm, the alternative models of the demand shock have the form

$$v_{t+1} = \rho v_t + C_v \left(\epsilon_{t+1} + w_{t+1}\right). \tag{5.3}$$

For the monopolist, they have the form

$$V_{t+1} = \rho V_t + C_v \left(\tilde{\epsilon}_{t+1} + W_{t+1} \right). \tag{5.4}$$

It is appropriate to set initial conditions so that $V_0 = v_0$. Here w_{t+1}, W_{t+1} are specification errors for the representative competitive firm and the monopolist, respectively, and $\epsilon_{t+1}, \tilde{\epsilon}_{t+1}$ are other i.i.d. random processes with mean zero and variance 1. The distortions (w_{t+1}, W_{t+1}) can feed back on the history of the state of the market, namely, $(\overline{q}, Q, v, V)$. The distortions w_{t+1} and W_{t+1} will typically differ because the monopolist and the representative competitive firm have different objectives.

5.1. The competitive fringe

The representative competitive firm regards $\{Q_t, \overline{q}_t\}_{t=0}^{\infty}$ as given stochastic processes and chooses an output plan $\{q_{t+1}\}_{t=0}^{\infty}$ and shock distortion process $\{w_{t+1}\}_{t=0}^{\infty}$ to extremize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ p_t q_t - \sigma_t + \beta \theta w_{t+1}^2 \right\}, \quad \beta \in (0,1) \tag{5.5}$$

subject to q_0 given, where E_t is the mathematical expectation based on time t information evaluated with respect to a distorted model that includes (5.3). Here θ is the robustness parameter of the representative firm in the competitive fringe, which could differ from Θ, the robustness parameter of the monopolist. Let $u_t = q_{t+1} - q_t$. We take (u_t, w_{t+1}) as the representative competitive firm's composite control vector at t. Subject to (5.1) and (5.3), first order-conditions for extremizing (5.5) with respect to u_t, w_{t+1} are

$$\begin{aligned} u_t &= E_t \beta u_{t+1} - c^{-1} \beta h q_{t+1} + c^{-1} \beta E_t \left(p_{t+1} - d \right) \\ w_{t+1} &= -\frac{1}{2\theta} C_v q_{t+1} + \beta \rho E_t w_{t+2} \end{aligned} \tag{5.6}$$

for $t \geq 0$.

In more detail, we derive the first-order conditions (5.6) by forming the following Lagrangian for the representative firm in the competitive fringe:

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & \left[A_0 - A_1 \left(Q_t + \overline{q}_t \right) + v_t \right] q_t - \left[d q_t + .5 h q_t^2 + .5 c u_t^2 \right] \\ & \beta \theta w_{t+1}^2 + \ell_{1t} \left[q_t + u_t - q_{t+1} \right] + \ell_{2t} \left[\rho v_t + C_v w_{t+1} - v_{t+1} \right] \Big\}. \end{aligned} \tag{5.7}$$

Here $\{\ell_{1t}, \ell_{2t}\}$ are sequences of Lagrange multipliers. Taking $\{Q_t, \overline{q}_t\}_{t=0}^{\infty}$ as given, the representative firm maximizes L with respect $\{u_t, q_{t+1}\}_{t=0}^{\infty}$ and minimizes it with respect to $\{w_{t+1}, v_{t+1}\}_{t=0}^{\infty}$. Rearranging the first order conditions for (u_t, q_{t+1}) gives the first equation of (5.6), while rearranging the first-order conditions for (w_{t+1}, v_{t+1}) gives the second equation of (5.6), which from now on we call the Euler equation for w_{t+1}.

We can appeal to a certainty equivalence principle stated by Hansen and Sargent (2004) to justify working with a non-stochastic version of (5.6) that we form by dropping the expectation operator and the random terms $\check{\epsilon}_{t+1}$ and ϵ_{t+1} from (5.2) and (5.3).[16] Shift (5.1) forward one period, set $q_t = \overline{q}_t$ for all $t \geq 0$, and substitute for p_{t+1} in (5.6) to get

$$\begin{aligned} u_t &= \beta u_{t+1} - c^{-1}\beta h \overline{q}_{t+1} + c^{-1}\beta (A_0 - d) - c^{-1}\beta A_1 \overline{q}_{t+1} \\ &\quad - c^{-1}\beta A_1 Q_{t+1} + c^{-1}\beta v_{t+1} \\ w_{t+1} &= -\frac{1}{2\theta} C_v \overline{q}_{t+1} + \beta\rho w_{t+2}. \end{aligned} \tag{5.8}$$

Equation (5.8) combines the Euler equations of the representative firm in the competitive fringe with market clearing.[17] Note that v, and not V, appears in the first equation of (5.8). This reflects how the representative competitive firm's forecasts influence its decisions, a fact that the monopolist will acknowledge when he designs his policy.

5.2. The monopolist's problem

The monopolist views the sequence of Euler equations-cum-market-clearing conditions (5.8) as implementability constraints. We can represent the constraints impinging on the monopolist, including (5.8), in terms of the transition law:

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ A_0 - d & 1 & 0 & -A_1 & -A_1 - h & c & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\theta}C_v & 0 & \beta\rho \end{bmatrix} \begin{bmatrix} 1 \\ v_{t+1} \\ V_{t+1} \\ Q_{t+1} \\ \overline{q}_{t+1} \\ u_{t+1} \\ w_{t+2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & C_v \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c}{\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ v_t \\ V_t \\ Q_t \\ \overline{q}_t \\ u_t \\ w_{t+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U_t + \begin{bmatrix} 0 \\ 0 \\ C_v \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} W_{t+1}, \end{aligned} \tag{5.9}$$

where $U_t = Q_{t+1} - Q_t$ is the control of the monopolist. The last row portrays (5.8). Represent (5.9) as

$$y_{t+1} = Ay_t + BU_t + CW_{t+1}. \tag{5.10}$$

Although we have included (u_t, w_{t+1}) as components of the 'state' y_t in the monopolist's transition law (5.10), (u_t, w_{t+1}) are actually 'jump' variables that correspond to x_t in

16 We use a method that Sargent (1987) used to compute a rational expectations equilibrium. The key step is to eliminate price and output by setting $q_t = \overline{q}_t$ and substituting from the inverse demand curve and the production function into the firm's first-order conditions to get a difference equation in capital.

17 As shown in Sargent (1987) in the case without robustness, (5.8) is also the Euler equation for a fictitious planner who takes Q_t as exogenous and who chooses a sequence for $\{q_{t+1}\}_{t=0}^{\infty}$ to maximize the discounted sum of consumer and producer surplus. Given stable sequences $\{Q_t, v_t\}$, we could solve (5.8) and $u_t = \overline{q}_{t+1} - \overline{q}_t$ to express the competitive fringe's output sequence as a function of the monopolist's output sequence.

section 3. The analysis in section 3 implies that the solution of the monopolist's problem is encoded in the Riccati equation associated with a robust linear regulator that takes (5.10) as the transition law.

To match the setup of section 3, we partition y_t as $y_t' = [\,z_t' \quad x_t'\,]$ where $z_t' = [\,1 \quad v_t \quad V_t \quad Q_t \quad \overline{q}_t\,]$, $x_t' = [\,u_t' \quad w_{t+1}'\,]$, and let $\mu_{xt} = \begin{bmatrix} \mu_{ut} \\ \mu_{wt} \end{bmatrix}$ be the vector of multipliers associated with the Euler equations for (u_t, w_{t+1}). The monopolist's artificial optimal linear regulator problem can be expressed

$$\max_{\{U_t\}} \min_{\{W_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ p_t Q_t - \mathcal{C}_t + \beta\Theta W_{t+1}' W_{t+1} \right\}$$

or

$$\max_{\{U_t\}} \min_{\{W_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ \left(A_0 - A_1 \left(\overline{q}_t + Q_t\right) + V_t\right) Q_t - eQ_t - .5gQ_t^2 - .5cU_t^2 + \beta\Theta W_{t+1}^2 \right\} \tag{5.11}$$

subject to (5.10). Notice that the monopolist's perceived demand shock appears in (5.11). The monopolist's problem can be written

$$\max_{\{U_t\}} \min_{\{W_{t+1}\}} -\sum_{t=0}^{\infty} \beta^t \left\{ y_t' Q y_t + U_t' R U_t - \beta\Theta W_{t+1}^2 \right\} \tag{5.12}$$

subject to (5.10) where

$$Q = -\begin{bmatrix} 0 & 0 & 0 & \frac{A_0-e}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{A_0-e}{2} & 0 & \frac{1}{2} & -A_1 - .5g & -\frac{A_1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{A_1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $R = \frac{c}{2}$. The results of section 3 apply.

5.3. Representation of the monopolist's decision rule

We want to study the approximating model under the robust decision rules for the monopolist and the representative competitive firm. Subject to one adjustment, the approximating model under the robust rules is given by an equation of the form (3.21). The required adjustment comes from the need to take account of the fact that we have included the follower's Euler equation for w_{t+1} among the pseudo-state equations in (5.9). In particular, notice the (2,7) entry in the matrix multiplying the pseudo-state on the right of (5.9). It builds in the law of motion for the representative competitive firm's worst-case model for v. But now we want to build in the law of motion for v under the approximating model. Therefore, we must make sure at this point that (3.21) incorporates the approximating model for v_t, not the worst-case model. We can do this by appropriately adjusting A on the right side of (3.21a), namely, by 'zeroing out' the C_v term that appears in the (2,7) position of the matrix multiplying the pseudo-state on the right side of equation (5.9). It is important that we make this adjustment only *after* we have solved Bellman equation (3.1) for the robust Stackelberg plan.

Recall that $z_t = [\,1 \quad v_t \quad V_t \quad Q_t \quad \overline{q}_t\,]'$ and $x_t = [\,u_t \quad w_{t+1}\,]'$. The monopolist's decision rule has the representation

$$\begin{bmatrix} U_t \\ W_{t+1} \end{bmatrix} = o_1 \begin{bmatrix} 1 \\ v_t \\ V_t \\ Q_t \\ \overline{q}_t \end{bmatrix} + o_2 \begin{bmatrix} \mu_{ut} \\ \mu_{wt} \end{bmatrix}. \tag{5.13}$$

Equation (3.15), which describes the decisions of the representative competitive firm, has the form

$$\begin{bmatrix} u_t \\ w_{t+1} \end{bmatrix} = n_1 \begin{bmatrix} 1 \\ v_t \\ V_t \\ Q_t \\ \overline{q}_t \end{bmatrix} + n_2 \begin{bmatrix} \mu_{ut} \\ \mu_{wt} \end{bmatrix}. \tag{5.14}$$

Here n_1, n_2, o_1, o_2 are matrices to be defined by matching the formulas from section 3. In addition, (3.21a) gives the law of motion of $[\,z_t \quad \mu_{xt}\,]'$ with the Stackelberg plan under the approximating model.

5.4. Interpretation

The approximating model incorporates the robust decision rules for both types of firm, but, after the adjustment mentioned in ther precding subsection, adds neither $C_v W_{t+1}$ nor $C_v w_{t+1}$ to the right side of (3.21a). The absence of these terms from the right side of (3.21a) reflects that the W_{t+1} and w_{t+1} terms that emerge from the monopolist's and representative competitive firm's problems are not their 'predicted misspecifications', but are instead artifacts of their procedures for devising robust decision rules. Under the approximating model the selected W_{t+1} and w_{t+1} processes are just some of the firms' decision making tools. They do not affect the motion of v_t and V_t under the approximating model. Indeed, under the approximating model, $V_t \equiv v_t$.[18]

[18] To simulate the random version of the model, we would add $[\,0 \quad C_v \quad C_v \quad 0 \quad 0 \quad 0 \quad 0\,]'\,\epsilon_{t+1}$ to the right side of (5.9) and the appropriate counterpart to the right side of (3.21a). Note that the same innovation ϵ_{t+1} impinges on both V_t and v_t under the approximating model.

Table 5.1: Steady state values

(Θ, θ)	(∞, ∞)	$(\infty, 10)$	$(10, \infty)$	$(10, 10)$
p	50	49.69	50.2	49.9
q	30	29.45	30.2	29.65
Q	20	20.86	19.59	20.46
w	0	-1.23	0	-1.24
W	0	0	-.82	-.84

5.5. Numerical example

This section briefly describes a numerical example of the monopoly-competitive fringe model in which we start without preferences for robustness, then study the effects of successively turning on preferences for robustness for one type of agent, but not the other, and then turning them on for both.

For parameter settings $(A_0, A_1, \rho, C_v, c, d, e, g, h, \beta)=$ $(100, 1, .8, .2, 1, 20, 20, 1, 1, .95)$, Table 5.1 displays steady state values associated with four pairs of settings for Θ, θ. To represent little or no preference for robustness, we set θ or Θ equal to 100000. To activate preferences for robustness, we set θ or Θ equal to 10. Fig. 5.1, Fig. 5.2, and Fig. 5.3, display some impulse responses for the model with $(\Theta, \theta) = (10, 10)$ under the approximating model and the robust rule.

The first column of Table 5.1 serves as a benchmark, preferences for robustness having been turned off for both the monopolist and the competitive firms by setting $\Theta = \theta \approx +\infty$. The next two columns turn on a preference for robustness for one but not the other of the two types of agents, while the fourth column turns on a preference for robustness for both types. The entries in the table show that a main effect of turning on a preference for robustness is to make the steady state values of the worst case shocks w and W negative. In effect, firms' pessimistic forecasts about demand push their outputs down. In the middle two columns in the table in which preferences for robustness are turned on for one but not the other type of agent, the type with the preference for robustness produces less and the other type produces more than under the benchmark steady state without preferences for robustness. Hovever, when we activate a concern for robustness for *both* types of firms in the fourth column, the monopolist produces enough more in the steady state to drive the price below its value in the benchmark no-robustness case in the first column.

For $(\theta, \Theta) = (10, 10)$, Fig. 5.1, Fig. 5.2, and Fig. 5.3 show impulse responses to the demand innovation ϵ_t. (Impulse response functions for price and output associated with other pairs of (θ, Θ) are very similar; the main effects of activating robustness are to affect the constants, or very low frequency components, of prices and quantities. The worst case shocks embrace state-dependent pessimism about the state of demand, which is evidently mostly a very low frequency phenomenon, virtually a difference in unconditional means.) The impulse responses show that a demand innovation pushes the implementation multiplier μ_u down and μ_w up, and leads the monopolist to expand output while the representative competitive firm at first contracts and then expands output in subsequent periods. The response of price to a demand shock innovation is to rise on impact but then to decrease in subsequent periods in response to the increase in total supply $q + Q$ engineered by the monopolist. Note from Fig. 5.3 that both of the worst case shocks W and w fall in response to an innovation in demand. This and the negative unconditional means of w and W in Table 5.1 tell us that both types of firms' decision rules are most fragile in the direction of overestimating demand.

The steady state values of the multipliers μ_u, μ_w are negative. This reflects the cost to the monopolist of adhering to its plan. Time inconsistency is surfaces in the incentive the monopolist would have to reset the multipliers to zero after period 0 and thereby reinitialize its plan (see Hansen, Epple, and Roberts (1985)).

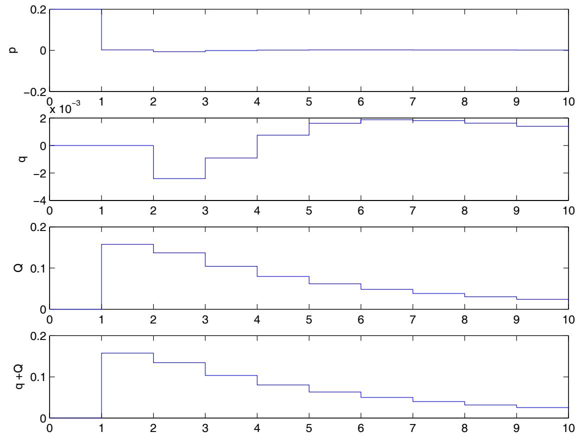

Figure 5.1: Impulse response of $p, q, Q, q + Q$ to innovation to demand shock ϵ

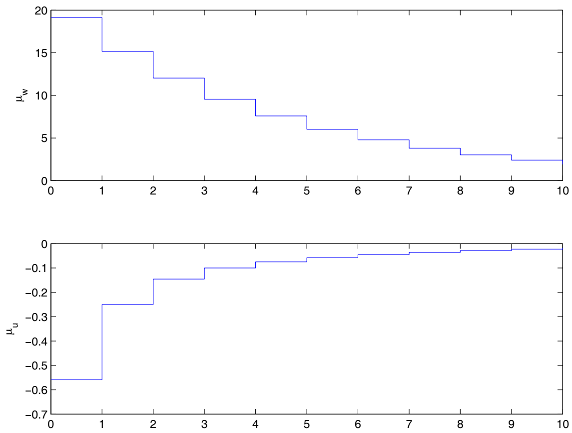

Figure 5.2: Impulse response of μ_w and μ_u to innovation in demand shock ϵ.

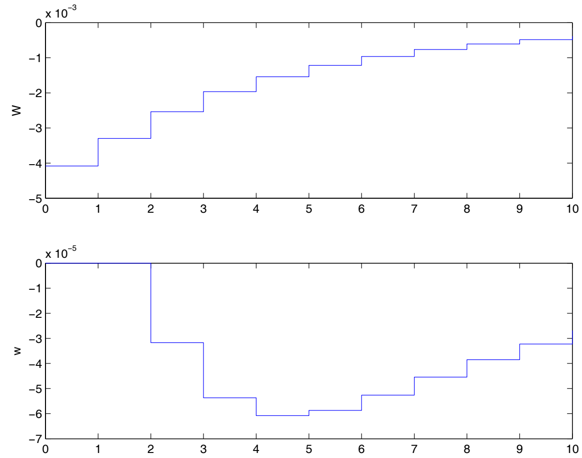

Figure 5.3: Impulse response of W and w to demand shock innovation ϵ.

6. Concluding remarks

We wrote this paper to define an appropriate equilibrium concept for studying the effects of concerns about model misspecification in dynamic macroeconomic and monetary policy models and because existing equilibrium concepts did not seem to allow us to impute doubts about the model specification to both the government and to private agents. The equilibrium concept in this paper will allow us to compute robust Ramsey plans for macroeconomics models with forward-looking agents, like the 'new synthesis' models of Clarida, Gali, and Gertler (1999), King and Wolman (1999), Rotemberg and Woodford (1997) and others.

The existing new synthesis literature without concerns for robustness computes Ramsey plans by formulating linear quadratic approximants that include implementability conditions in the form of consumption Euler equations for consumers and price-setting Euler equations for a typical monopolistically competitive price-setting firm. If in the spirit of this paper we wish to impute concerns about misspecification and the consequent desires for robustness to consumers and firms in such new synthesis models, we have to add some additional state variables and some additional implementability conditions for both the consumers and the firms. These additional state variables are specification error processes (worst case shocks) for the representative household and the representative firm, and the additional implementability conditions are Euler equations for those worst case shock processes.[19] Our Ramsey equilibrium concept pins down all of the endogenous (decision) variables in a common approximating model that all agents in the model share, but that all agents doubt.

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[19] To derive those Euler equations, it is necessary to have specified the objective functions of each type of agent.

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