

Optimal Fiscal Policy in a Linear Stochastic Economy*

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Introduction

Computation of optimal fiscal policies for Lucas and Stokey's (1983) economy requires repeated evaluations of the present value of the government's surplus, an object formally equivalent to an asset price. The functional equation for an asset price is typically difficult to solve. In this paper, we specify a linear-quadratic version of Lucas and Stokey's economy, making both asset pricing computations and optimal fiscal policy calculations easy. The key steps are described in Appendix for two basic kinds of stochastic process: a stochastic first-order linear difference equation and a Markov chain. We use the Lucas-Stokey economy to exhibit features of Lucas and Stokey's model, and how they compare to Barro's (1979) tax-smoothing model.¹

Review of Barro's model

Robert Barro (1979) formalized the idea that taxes should be smooth by saying that they should be a martingale, regardless of the stochastic process for government expenditures. Hansen, Sargent, and Roberds (1991) use the following linear quadratic model to formalize Barro's conclusions. The government chooses a rule for taxes to maximize the criterion

$$-E \sum_{t=0}^{\infty} \beta^t T_t^2 \tag{1}$$

subject to an initial condition B_0 and

$$\begin{aligned} \gamma(L) g_t &= \rho(L) w_t \\ B_{t+1} &= R(B_t + g_t - T_t), \end{aligned} \tag{2}$$

where T_t, g_t, b_t denote tax collections, government expenditures, and the stock of risk-free government debt, respectively, and where R is a gross risk-free rate of return on government debt and $\beta \in (0, 1)$ is a discount factor. In (2), $\gamma(L)$ and $\rho(L)$ are stable one-sided polynomials in nonnegative powers of the lag operator L , w_t is a scalar martingale difference sequence adapted to its own history. Under the assumption that $R\beta = 1$, the

¹ The asset pricing calculations emanate from Hansen (1987) and Hansen and Sargent (1999).

solution of this problem that satisfies the side condition $E_0 \sum_{t=0}^{\infty} \beta^t T_t^2 < +\infty$ is a rule for taxes of the form

$$T_t - T_{t-1} = [(1 - \beta) \rho(\beta) / \gamma(\beta)] w_t. \quad (3)$$

Using (3) with (2) shows that B_{t+1} is cointegrated with T_t .²

Equation (3) asserts the striking property that the serial correlation properties of taxes are independent of the serial correlation properties of government expenditures. That a random walk with small innovation variance appears smooth is the sense of ‘tax smoothing’ that emerges from Barro’s analysis. This outcome depends on the debt being risk-free.

The second equation of (2) can be written

$$\pi_{t+1} = B_{t+1} - R[B_t - (T_t - g_t)] \equiv 0, \quad (4)$$

where π_{t+1} is interpretable as the payoff on government debt in excess of the risk-free rate. Barro’s model has T_t adjust permanently by a small amount in response to a surprise in g_t , w_t , and has B_{t+1} make the rest of the adjustment to enforce (4) period by period. These adjustments make the cumulative excess payoff to government creditors be

$$\Pi_t = \sum_{s=1}^t \pi_s \equiv 0. \quad (5)$$

The adjustments are very different in Lucas and Stokey’s model.

Lucas and Stokey (1983) reexamined the optimal taxation problem in an equilibrium economy with complete markets, where the government issues state-contingent debt, not only the risk-free debt in (2). In their analysis, tax-smoothing in the form emphasized by Barro does not emerge. Taxes are not a martingale but rather have serial correlation properties that mirror those of government expenditures. A martingale lurks in their analysis, but as a counterpart to (5) for the cumulated excess payoff to the government’s creditors, not taxes, and only after appropriate adjustments for risk and risk aversion.

² See Hansen, Sargent, and Roberds (1991).

Lucas and Stokey's Model

We present a linear quadratic version of Lucas and Stokey's (1983) model of optimal taxation in an economy without capital and compute a variety of examples.

Exogenous processes and information

Let x_t be an exogenous information vector. We shall use x_t to drive exogenous stochastic processes $g_t, d_t, b_t, {}_0s_t$, representing, respectively, government expenditures, an endowment, a preference shock, and a stream of promised coupon payments owed by the government at the beginning of time 0:

$$g_t = S_g x_t \tag{6a}$$

$$d_t = S_d x_t \tag{6b}$$

$$b_t = S_b x_t \tag{6c}$$

$${}_0s_t = {}_0S_s x_t. \tag{6d}$$

We make one of two alternative assumptions about the underlying stochastic process x_t .

ASSUMPTION 1: The process x_t is an $n \times 1$ vector with given initial condition x_0 and is governed by

$$x_{t+1} = Ax_t + Cw_{t+1}. \tag{7}$$

Here $\{w_{t+1}\}$ is a martingale difference sequence adapted to its own past and to x_0 , and A is a stable matrix.

ASSUMPTION 2: The process x_t is an n state Markov chain with transition probabilities arranged in the $n \times n$ matrix P with $P_{ij} = \text{Prob}(x_{t+1} = \bar{x}_j | x_t = \bar{x}_i)$.

Technology

There is a technology for converting one unit of labor ℓ_t into one unit of a single nonstorable consumption good. Feasible allocations satisfy:

$$c_t + g_t = d_t + \ell_t. \quad (8)$$

Households

Markets are complete. At time 0, a representative consumer faces a scaled Arrow-Debreu price system³ $\{p_t^0\}$ and a flat rate tax on labor $\{\tau_t\}$. and chooses consumption and labor supply to maximize:

$$-.5E_0 \sum_{t=0}^{\infty} \beta^t \left[(c_t - b_t)^2 + \ell_t^2 \right] \quad (9)$$

subject to the time 0 budget constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 [d_t + (1 - \tau_t) \ell_t + {}_0s_t - c_t] = 0. \quad (10)$$

This states that the present value of consumption equals the present value of the endowment plus coupon payments on the initial government debt plus after tax labor earnings. The scaled Arrow-Debreu prices are ordinary state prices divided by discount factors and conditional probabilities. The scaled Arrow-Debreu price system is a stochastic process.

Government

The government's time 0 budget constraint is

$$E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 [(g_t + {}_0s_t) - \tau_t \ell_t] = 0. \quad (11)$$

Given the government expenditure process and the present value $E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 {}_0s_t$, a feasible tax process must satisfy (11).

³ The scaled Arrow-Debreu prices are ordinary state prices divided by probabilities and the time t power of the discount factor, transformations that permit representing values as conditional expectations of scaled prices times quantities. See Hansen (1987) and Hansen and Sargent (1999).

Equilibrium

Definition: L_0^2 is the space of random variables y_t measurable with respect to x_t and such that $E_0 \sum_{t=0}^{\infty} \beta^t y_t^2 < +\infty$.

Definitions: A *feasible allocation* is a stochastic process $\{c_t, \ell_t\}$ that satisfies (8). A *tax system* is a scalar stochastic process $\{\tau_t\}$. A *price system* is a stochastic process $\{p_t^0\}$. The time t elements of each of these processes are assumed to be measurable with respect to x_t , and to belong to L_0^2 .

Definition: An *equilibrium* is a feasible allocation, a price system, and a tax system that have the following properties:

- i. Given the tax and price systems, the allocation solves the household's problem.
- ii. Given the price system, the allocation and the tax system satisfy the government's budget constraint.

Properties

The first-order conditions for the household's problem imply that the equilibrium price system satisfies $p_t^0 = \mu(b_t - c_t)$, where μ is a numeraire that we set at $b_0 - c_0$. The preference specification permits the scaled Arrow-Debreu price p_t^0 to be expressed in terms of ratios of linear functions of the state:

$$p_t^0 = M_p x_t / M_p x_0,$$

where M_p is a matrix defined so that $M_p x_t = b_t - c_t$. The preference specification will make it possible to express government time t revenues as the ratio of a quadratic function of the state at t to a linear function of the state at 0. The forms of these prices and taxes, together with the other objects in (9), reduce the technical problem to evaluating geometric sums of a quadratic form in the state. For assumptions 1 and 2, Appendix A shows how to compute such sums.

Ramsey problem

There are many equilibria, indexed by tax systems. The Ramsey problem is to choose the tax system that delivers the equilibrium preferred by the representative household. The Ramsey problem assumes that at time 0 the government commits itself to the tax system, once and for all.

Definition: The *Ramsey problem* is to choose an equilibrium that maximizes the household's welfare (9). The allocation that solves this problem is called the *Ramsey allocation*, and the associated tax system is called the *Ramsey plan*.

Solution strategy

In solving the Ramsey problem, the government chooses all of the objects in an equilibrium, subject to the constraint on the equilibrium imposed by its budget constraint. Following a long line of researchers starting with Frank Ramsey (1929), we shall solve this problem using a 'first-order' approach that involves the following steps. The steps incorporate the properties required by the definition of equilibrium.

1. Obtain the first-order conditions for the household's problem and use them to express the tax system and the price system in terms of the allocation alone.
2. Substitute the expressions for the tax system and the price system obtained in step 1 into the government's budget constraint to obtain a single iso-perimetric restriction on allocations.
3. Use Lagrangian methods to find the feasible allocation that maximizes the utility of the representative household subject to the restriction derived in step 2. The maximizer is the Ramsey allocation.
4. Use the expressions from step 1 to find the associated Ramsey equilibrium price and tax systems by evaluating them at the Ramsey allocation.

Computation

We now execute these four steps. The problem is set so that the mathematics of linear systems can support a solution.

Step 1. The household's first order conditions imply

$$p_t^0 = (b_t - c_t) / (b_0 - c_0) \quad (12)$$

$$\tau_t = 1 - \frac{\ell_t}{b_t - c_t} \quad (13)$$

Step 2. Using (12) and (13) express (11) as

$$E_0 \sum_{t=0}^{\infty} \beta^t [(b_t - c_t)(g_t + {}_0s_t) - (b_t - c_t)\ell_t + \ell_t^2] = 0 \quad (14)$$

Equation (14) is often called the implementability constraint on the allocation.

Step 3. Consider the maximization problem associated with the Lagrangian:

$$\begin{aligned} J = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -0.5 [(c_t - b_t)^2 + \ell_t^2] \right. \\ \left. + \lambda_0 [(b_t - c_t)\ell_t - \ell_t^2 - (b_t - c_t)(g_t + {}_0s_t)] \right. \\ \left. + \mu_{0t} [d_t + \ell_t - c_t - g_t] \right\} \end{aligned}$$

where λ_0 is the multiplier associated with the government's budget constraint, and μ_{0t} is the multiplier associated with the time t feasibility condition. Obtain the first-order conditions:

$$c_t : -(c_t - b_t) + \lambda_0 [-\ell_t + (g_t + {}_0s_t)] = \mu_{0t} \quad (15a)$$

$$\ell_t : \ell_t - \lambda_0 [(b_t - c_t) - 2\ell_t] = \mu_{0t} \quad (15b)$$

$$d_t + \ell_t = c_t + g_t \quad (15c)$$

We want to solve equations (15a), (15b), (15c) and the government's budget constraint (11) for an *allocation*. Our strategy is to begin by taking λ_0 as given and to solve (15) for an allocation contingent on λ_0 . Then we shall use (11) to solve for λ_0 .

Using the feasibility constraint $c_t = d_t + \ell_t - g_t$, we can express (15a), (15b) as

$$\ell_t - \lambda_0 [(b_t - d_t - \ell_t + g_t) - 2\ell_t] = -(d_t + \ell_t - g_t - b_t) + \lambda_0 [-\ell_t + (g_t + {}_0s_t)]$$

or

$$\ell_t = \frac{1}{2} (b_t - d_t + g_t) - \frac{\lambda_0}{2 + 4\lambda_0} (b_t - d_t - {}_0s_t).$$

We also derive

$$c_t = \frac{1}{2} (b_t + d_t - g_t) - \frac{\lambda_0}{2 + 4\lambda_0} (b_t - d_t - {}_0s_t).$$

Define $\tilde{c}_t = (b_t + d_t - g_t)/2$, $\tilde{\ell}_t = (b_t - d_t + g_t)/2$ and $m_t = (b_t - d_t - {}_0s_t)/2$. We have:

$$\ell_t = \tilde{\ell}_t - \mu m_t \quad (16a)$$

$$c_t = \tilde{c}_t - \mu m_t \quad (16b)$$

where, for convenience, we define

$$\mu = \frac{\lambda_0}{1 + 2\lambda_0}. \quad (17)$$

Using (16), the general term of (14) can be written as:

$$\begin{aligned} & (b_t - \tilde{c}_t) (g_t + {}_0s_t) - (b_t - \tilde{c}_t) \tilde{\ell}_t + \tilde{\ell}_t^2 \\ & \quad - \mu m_t \left[-(g_t + {}_0s_t) + \tilde{\ell}_t - (b_t - \tilde{c}_t) + 2\tilde{\ell}_t \right] + \mu^2 a_t^2 \\ & = (b_t - \tilde{c}_t) (g_t + {}_0s_t) - 2m_t^2 \mu + 2m_t^2 \mu^2, \end{aligned}$$

where we used the fact that $2\tilde{\ell}_t = b_t - d_t + g_t$ and the fact that $\tilde{\ell}_t = b_t - \tilde{c}_t$ to reduce the bracketed factor in the second line.

This allows us to write (14) as:

$$a_0(x_0) (\mu^2 - \mu) + b_0(x_0) = 0 \quad (18)$$

where

$$\begin{aligned}
a_0(x_0) &= E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} (b_t - d_t - {}_0s_t)^2 \\
&= E_0 \sum_{t=0}^{\infty} \beta^t x_t' \frac{1}{2} [S_b - S_d - {}_0S_s]' [S_b - S_d - {}_0S_s] x_t
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
b_0(x_0) &= E_0 \sum_{t=0}^{\infty} \beta^t \left[(b_t - \tilde{c}_t) (g_t + {}_0s_t) - (b_t - \tilde{c}_t) \tilde{\ell}_t + \tilde{\ell}_t^2 \right] \\
&= E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} (b_t - d_t + g_t) (g_t + {}_0s_t) \\
&= E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} x_t' [S_b - S_d + S_g]' [S_g + {}_0S_s] x_t
\end{aligned} \tag{20}$$

$$\tag{21}$$

where the fact that $b_t - \tilde{c}_t = \tilde{\ell}_t$ was used. The 0 subscripts on the forms a_0 and b_0 denote their dependence on ${}_0S_s$. The coefficients in the polynomial expression of (18) only functions of x_0 alone because, given the law of motion for the exogenous state x_t , the infinite sums can be computed using the algorithms described in Appendix A.

Notice that $b_0(x_0)$, when expressed by (20), is simply the infinite sum on the left-hand side of (14) evaluated for the specific allocation $\{\tilde{c}_t, \tilde{\ell}_t\}$. That allocation solves the problem:

$$\max_{c, \ell} -0.5 \left[(c - b_t)^2 + \ell^2 \right]$$

subject to $c + g_t = \ell + d_t$. In other words, $\{\tilde{c}_t, \tilde{\ell}_t\}$ is the allocation that would be chosen by a social planner, or the Ramsey allocation when the government can resort to lump-sum taxation. The term $b_0(x_0)$ is the present-value of the government stream spending commitments $\{g_t + {}_0s_t\}$, evaluated at the prices corresponding to the $\{\tilde{c}_t, \tilde{\ell}_t\}$ allocation. If that present value is 0, distortionary taxation is not necessary, and $\mu = 0$ (that is, $\lambda_0 = 0$) solves (18): the government's budget constraint is not binding. One configuration for which $b_0(x_0) = 0$ is when $g_t = -{}_0s_t$ for all t , but there are many others. Because markets are complete, the timing of the government's claims on the household does not matter. If the government were able to acquire such claims on the private sector in a non-distortionary way, it would be able to implement a first-best allocation.

When the net present value of the government's commitments is positive, we must solve (18) for a μ in $(0, 1/2)$, corresponding to $\lambda_0 > 0$. The polynomial $a_0(x_0)\mu(1 - \mu)$ is bounded above by $a_0(x_0)/4$, which means that government commitments that are "too large" cannot be supported by a Ramsey plan. If $b_0(x_0) < a_0(x_0)/4$ there exists a unique solution μ in $(0, 1/2)$ and a unique $\lambda_0 > 0$. The Ramsey allocation can then be computed as:

$$\begin{aligned} c_t &= \tilde{c}_t - \mu m_t \\ &= \frac{1}{2} ([S_b + S_d - S_g] - \mu [S_b - S_d - {}_0S_s]) x_t \end{aligned} \quad (22a)$$

$$\begin{aligned} \ell_t &= \tilde{\ell}_t - \mu m_t \\ &= \frac{1}{2} ([S_b - S_d + S_g] - \mu [S_b - S_d - {}_0S_s]) x_t \end{aligned} \quad (22b)$$

and the Ramsey plan as:

$$\begin{aligned} \tau_t &= 1 - \frac{\ell_t}{b_t - c_t} \\ &= 1 - \frac{\tilde{\ell}_t - \mu m_t}{b_t - \tilde{c}_t + \mu m_t} \\ &= \frac{2\mu m_t}{\tilde{\ell}_t + \mu m_t} \\ &= \frac{2\mu [S_b - S_d - {}_0S_s] x_t}{([S_b - S_d + S_g] + \mu [S_b - S_d - {}_0S_s]) x_t}. \end{aligned} \quad (23)$$

Expression (23) shows how the stochastic properties of the tax rate mirror those for government expenditures when the endowment and the preference shocks are constant.

Martingale returns on government debt

Recursive formulation of the government budget

The government's budget constraint can be written

$$B_0 = E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 (\tau_t \ell_t - g_t) \quad (24)$$

where $B_0 \equiv E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 s_t$ is the time 0 present value of initial government debt obligations. Define

$$B_t \equiv E_t \sum_{j=0}^{\infty} \beta^j p_{t+j}^t (\tau_{t+j} \ell_{t+j} - g_{t+j}). \quad (25)$$

Along the Ramsey allocation, B_t can be computed as

$$B_t = \frac{E_t \sum_{j=0}^{\infty} \beta^j [(b_{t+j} - c_{t+j}) \ell_{t+j} - \ell_{t+j}^2 - (b_{t+j} - c_{t+j}) g_{t+j}]}{(b_t - c_t)}, \quad (26)$$

which can evidently be expressed as a function of the time t state x_t in particular, a quadratic form in x_t plus a constant divided by a linear form in x_t . The quantity B_t can be regarded as the time t value of government state contingent debt issued at $t - 1$ and priced at t .

The government budget constraint can be implemented recursively by issuing one-period state contingent debt represented as a stochastic process B_t that is measurable with respect to time t information. In particular, we can replace the single budget constraint (11) with the sequence of budget constraints for $t \geq 0$:

$$B_t = [\tau_t \ell_t - g_t] + \beta E_t [p_{t+1}^t B_{t+1}], \quad (27)$$

where $p_{t+1}^t = \frac{b_{t+1} - c_{t+1}}{b_t - c_t}$ is the scaled Arrow-Debreu state price for one-period ahead claims at time t . We can think of the optimal plan as being implemented as follows. The government comes into period t with state contingent debt worth B_t , all of which it buys back or 'redeems'. It pays for these redemptions and its time t net of interest deficit $g_t - \tau_t \ell_t$ by selling state contingent debt worth $E_t \beta p_{t+1}^t B_{t+1}$. The term structure of this debt is irrelevant (but see Appendix B). We are free to think of it all as one-period state contingent debt promising to pay off B_{t+1} state-contingent units of consumption at $t + 1$.

The martingale equivalent measure

Equation (27) looks like an asset pricing equation. The value of the asset at time t is B_t and the time t ‘dividend’ is the government surplus $\tau_t \ell_t - g_t$. Because we are working with complete markets, we can coax from (27) a martingale that forms a counterpart to (4) and (5).⁴ The argument proceeds as follows. We would like it if (25), the asset pricing equation, involved so-called risk neutral pricing by collapsing to

$$B_t = E_t \sum_{j=0}^{\infty} R_{tj}^{-1} (\tau_{t+j} \ell_{t+j} - g_{t+j}), \quad (28)$$

where R_{tj} is the risk free j -period gross rate of return from time t to time $t + j$. The j period risk free rate is $E_t \beta^j p_{t+j}^t$, but (25) does not imply (28), at least not under rational expectations (where E is taken with respect to the correct transition probabilities). But by computing the expectation in (28) with respect to another set of transition probabilities, we can make a version of (28) true.

Here is how to find transition probabilities that work. Note that the one-period risk free interest rate R_t satisfies $R_t^{-1} = \beta E_t p_{t+1}^t$. Consider a portfolio of formed by borrowing $(B_t - (\tau_t \ell_t - g_t))$, and using the proceeds to buy the vector of one-period claims B_{t+1} . The one-period profits from that portfolio will be

$$\pi_{t+1} = B_{t+1} - R_t [B_t - (\tau_t \ell_t - g_t)]. \quad (29)$$

This investment costs no money, so that if risk-neutral investors’ evaluations determined prices, the expected value of the payoff should be zero. (Remember that in Barro’s model, the corresponding object is *identically* zero, not just zero in conditional expectation.) But the representative household is risk-averse, and its preferences are reflected in state prices, making risk neutral pricing fail, at least with the correct specification of probabilities. We can induce a risk-neutral pricing formula by suitably respecifying the probabilities. In particular, equations (27)-(29) state that

$$\tilde{E}_t \pi_{t+1} = 0, \quad (30)$$

⁴ See Duffie (1997, chapter 2).

where \tilde{E}_t is the conditional expectation with respect to the *equivalent* transition measure defined as

$$\tilde{f}(x_{t+1}|x_t) = \frac{f(x_{t+1}|x_t)p_{t+1}^t}{E_t p_{t+1}^t}, \quad (31)$$

where $f(x_{t+1}|x_t)$ is the original Markov transition density for x . The transition measure \tilde{f} is equivalent in the sense of putting positive probability on the same events as f . Condition (30) states that π_{t+1} is a martingale difference sequence with respect to the equivalent transition measure.⁵

The martingale characterization of government debt encapsulates features of a variety of examples calculated by Lucas and Stokey (1983) in which surprise increases in government expenditures are associated with low realized returns on government debt, and low government expenditures are associated with high rates of return. We now turn to some examples of our own.

Three examples

All of the examples set $\beta = 1.05^{-1}$, $b = 2.135$, $d = 0$ and initial debt $B_0 = 0$. The first two examples let $w_{g,t+1}$ be a scalar martingale difference sequence, adapted to its own past, with unit variance. The first example uses the linear stochastic difference equation of assumption 1 and sets

$$g_{t+1} - \mu_g = \rho(g_t - \mu_g) + C_g w_{g,t+1}$$

with $\rho = .7$, $\mu_g = .35$ and $C_g = .035\sqrt{1 - \rho^2}$. The second example also uses assumption 1 and sets

$$g_{t+1} - \mu_g = \rho(g_{t-3} - \mu_g) + C_g w_{g,t+1},$$

where $\rho = .95$ and $C_g = .7\sqrt{1 - \rho^2}$.

The third example uses assumption 2 and sets the Markov chain

$$P = \begin{bmatrix} .8 & .8 & 0 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \end{bmatrix},$$

⁵ The profit or gain $\Pi_t = \sum_{s=1}^t \pi_s$ is a martingale with respect to the measure over sequences of x_t induced by the equivalent transition density. We define $\tilde{\Pi}_t = \sum_{s=0}^t \pi_s$.

with $g(x) = [.5 \ .5 \ .25]'$. Here the first state of the Markov chain is war, the second armistice, the third peace. Government expenditures are identical in war and armistice, but the probabilities of transition to peace differ.

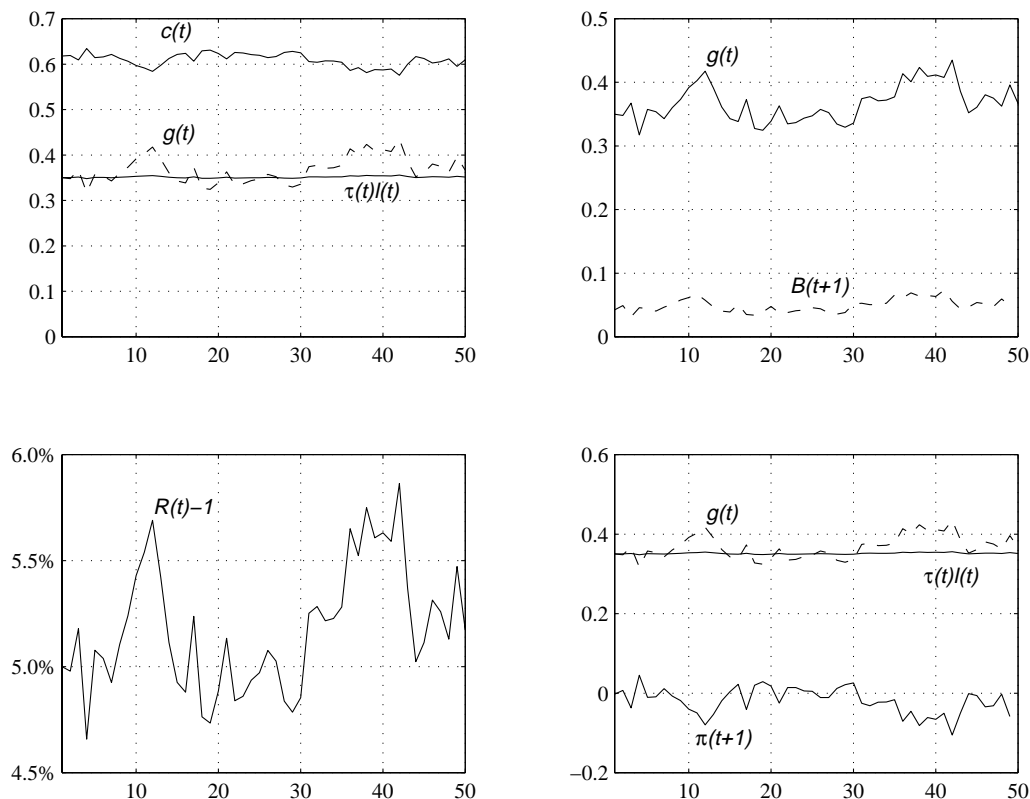


Figure 1: Case of an AR(1) process for g_t .

We calculated Ramsey plans for each of these three economies. We wrote a matlab program `lqramsey` for the assumption 1 economies and `lqramsm` for the assumption 2 economies. Figures 1, 2, and 3 display simulations of outcome paths.

The first panel of each case shows sample paths with tax smoothing, not in the sense of Barro, but in the sense of ‘small variance’. As formula (23) shows, taxes inherit serial correlation properties from the government expenditure process. The second and fourth

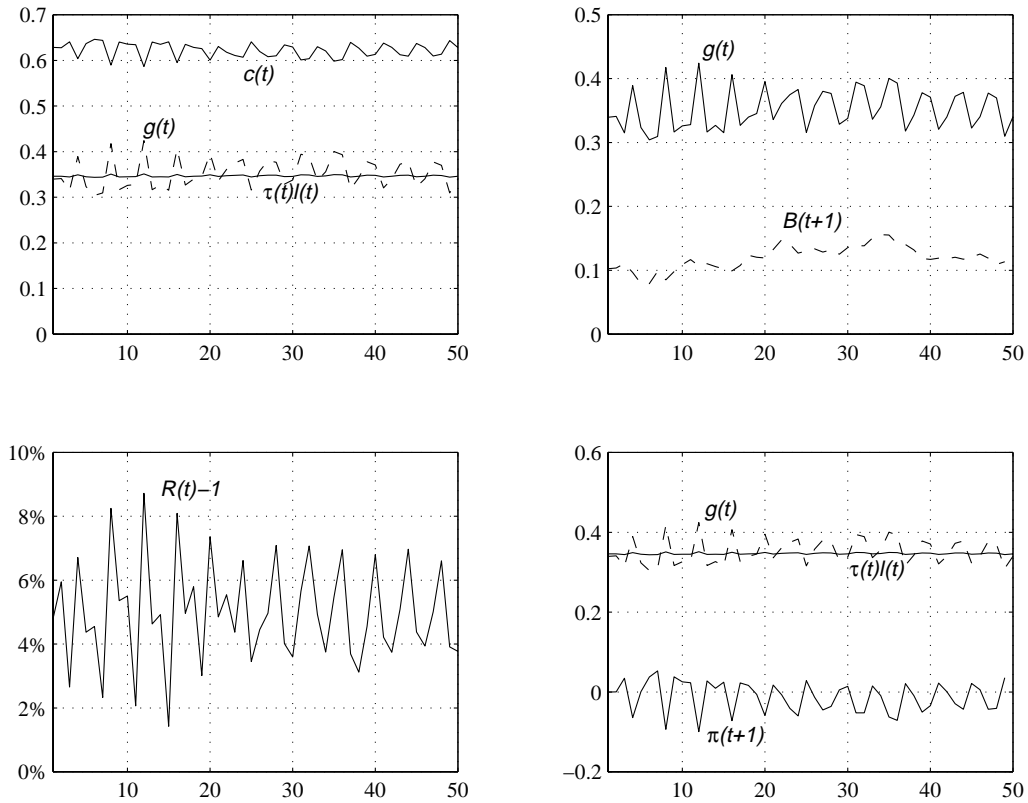


Figure 2: Case of a seasonal AR(4) process for g_t .

panels reveal important differences in the outcomes from Barro's model. The second panel in each case shows how B_{t+1} falls when g_t is above average, and rises when g is below average. This behavior is also reflected in the fourth panel where the payout on the public's portfolio of government debt, π_{t+1} , varies inversely with government expenditures. When government expenditures are high (low) relative to what had been expected, *ex post* government debt pays a low (high) return.

Figures 1 and 2 show linear time series versions of these patterns, example 1 with first order autoregressive government expenditures, example 2 with seasonal government expenditures. The effect of the pattern of government expenditures on the pattern of tax collections are difficult to see from the pictures because the variance of tax collections is so

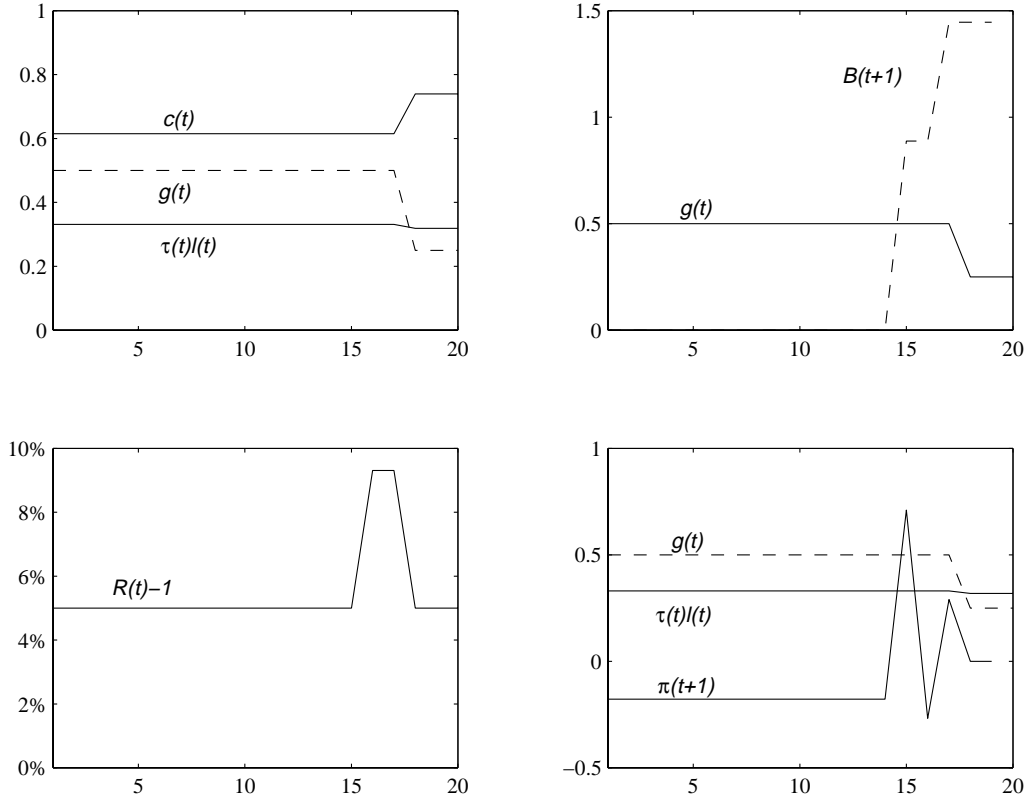


Figure 3: Markov chain g with war, armistice, peace.

small. The contemporaneous correlation of tax collections with government expenditures is .99 in both examples 1 and 2.

Figure 3 shows the Markov example. The economy begins in war, and runs a deficit while war continues. During war, there is no building up of debt: each period of war the government pays zero gross return to its creditors (see the fourth panel). When armistice arrives in period 5, it triggers a big positive payout π_{t+1} , even though government expenditures remain at their war time level. Armistice lingers for another period, causing the payoff on government debt to be negative again. Then peace arrives, causing a large payoff during the first period of peace, to be followed by a permanent string of risk-free positive payouts equal to the permanent government surplus. Only during this period of

permanent peace does formula (4) hold.

Figure 4, Figure 5, and Figure 6 display realizations of $p_{t+1}^t/E_t p_{t+1}^t$ and $\tilde{\Pi}_{t+1}$ for each of our three economies. The term $p_{t+1}^t/E_t p_{t+1}^t$ is the factor that π_t needs to be multiplied to convert Π_t into the martingale $\tilde{\Pi}_t$. The factor is small, meaning that Π_t itself is nearly a martingale. Note how $\tilde{\Pi}_t$ becomes a constant once perpetual peace arrives in Figure 6.

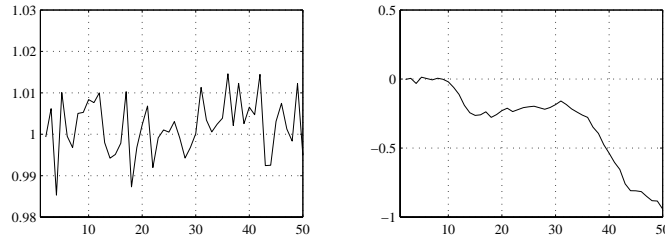


Figure 4: Case of an AR(1) process for g_t . Left panel is $p_{t+1}^t/E_t p_{t+1}^t$, right panel is $\tilde{\Pi}_t$.

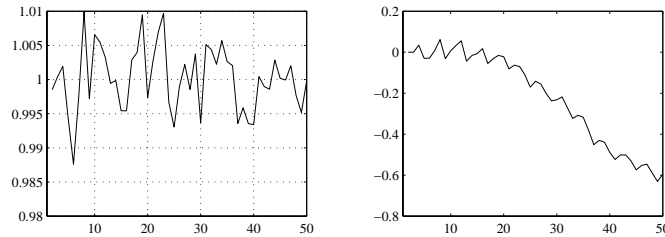


Figure 5: Case of a seasonal AR(4) process for g_t . Left panel is $p_{t+1}^t/E_t p_{t+1}^t$, right panel is $\tilde{\Pi}_t$.

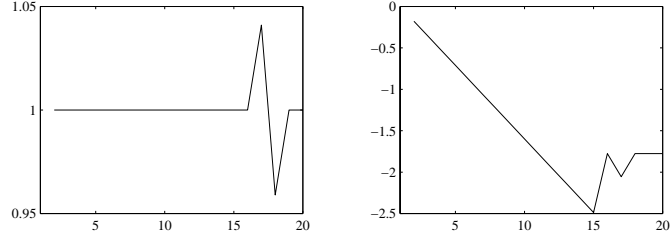


Figure 6: Markov chain g with war, armistice, peace. Left panel is $p_{t+1}^t/E_t p_{t+1}^t$, right panel is $\tilde{\Pi}_t$.

Extensions

Further calculations with linear quadratic economies appear in Hansen (1987) and Hansen and Sargent (1999). Extensions of Lucas and Stokey's analysis to economies with capital appear in Chari, Christiano, and Kehoe (1994, 1996) and Jones, Manuelli, and Rossi (1997). Chari, Christiano, and Kehoe (1994) describe optimal policy in terms of a martingale of a return variable related to Π_t . For an analysis intermediate between Barro's and Lucas and Stokey's, see Marcet, Sargent, and Seppala (1996), who restrict the government in Lucas and Stokey's model to issue only risk-free debt. That limitation puts a large number of additional measurability restrictions on the Ramsey allocation, beyond those incorporated in (14). These restrictions deliver a version of (5) with a time-varying risk free interest rate. The problem with only risk free debt requires computational methods like ones of Marcet and Marimon (1997).

Appendix A: Geometric sums of quadratic forms

The calculations in the text require repeated evaluations of discounted infinite sums of quadratic forms in future values of the state. This appendix gives formulas for these sums under two alternative specifications of stochastic process for the state: (1) the state is governed by a vector first order linear stochastic difference equation; and (2) the state evolves according to an n state Markov chain.

Linear stochastic difference equation

We want a formula for expected discounted sums of quadratic forms

$$q(x_0) = E_0 \sum_{t=0}^{\infty} \beta^t x_t' M x_t \quad (32)$$

where

$$x_{t+1} = Ax_t + Cw_{t+1}$$

and where w_{t+1} is a martingale difference sequence adapted to its own history and to x_0 . The formula is

$$q(x_0) = x_0' Q x_0 + q_0$$

where

$$\begin{aligned} q_0 &= \frac{\beta}{1-\beta} \text{trace} C' Q C \\ Q &= M + \beta A' Q A \end{aligned} \quad (33)$$

The second equation is a Sylvester equation in Q that can be solved by one of a variety of methods, including the doubling algorithm. See Anderson, Hansen, McGrattan, and Sargent (1996) for a review of methods for solving Sylvester equations. The standard Matlab program `dlyap` can be used; so can a homemade one `doubleo` of Hansen and Sargent (1999).

Markov chain

Assume that x_t is the state of an n -state Markov chain with transition matrix P with (i, j) th element $P_{i,j} = \text{Prob}(x_{t+1} = \bar{x}_j | x_t = \bar{x}_i)$. Here \bar{x}_i is the value of x when the

chain is in its i th state. Let $h(\bar{x})$ be a function of the state represented by an $(n \times 1)$ vector h ; the i th component of h denotes the value of h when x is in its i th state. Then we have the following two useful formulas:

$$E[h(x_{t+k}|x_t = \bar{x})] = P^k h$$

$$E\left[\sum_{k=0}^{\infty} \beta^k h(x_{t+k}|x_t = \bar{x})\right] = (I - \beta P)^{-1} h(\bar{x}),$$

where $\beta \in (0, 1)$ guarantees existence of $(I - \beta P)^{-1} = (I + \beta P + \beta^2 P^2 + \dots)$.

Appendix B: Time consistency and the structure of debt

Under complete markets, there are many government debt structures that have the same present value. One of the results of Lucas and Stokey (1983) is that a specific structure is required if the Ramsey plan is to be *time-consistent*; that is, if the Ramsey plan computed at $t = 1$ coincides with the continuation of the Ramsey plan computed at $t = 0$ for all realizations of x_1 . We can compute the debt structure that will induce time consistency.

Assume that the government has solved for the Ramsey plan at $t = 0$, and restructured the debt, that is, chosen a new debt structure of the form ${}_1s_t = {}_1S_s x_t$. We want to find conditions on ${}_1S_s$ such that the Ramsey plan found at time $t = 0$ will be time-consistent. Suppose that, for $t = 1$, we compute the Ramsey plan. Following the same procedure as above, we will need to solve for μ_1 in the equation

$$a_1(x_1) (\mu_1^2 - \mu_1) + b_1(x_1) = 0 \tag{34}$$

where the subscript on a and b indicates the fact that these quadratic forms of x_1 depend on ${}_1S_s$ just as $a_0(x_0)$ and $b_0(x_0)$ depend on ${}_0S_s$ in (19) and (21). Once μ_1 is found, allocations can be computed using (22). Note that the $\tilde{\ell}_t$ and \tilde{c}_t terms in (16) do not depend on the debt structure. Therefore, for the new Ramsey allocation to coincide with the continuation of the Ramsey allocation computed at $t = 0$, all we need is

$$\mu_0 [S_b - S_d - {}_0S_s] = \mu_1 [S_b - S_d - {}_1S_s]$$

or

$$\mu_1 {}_1S_s = \mu_0 {}_0S_s + (\mu_1 - \mu_0) (S_b - S_d). \quad (35)$$

To translate these conditions into conditions on ${}_1S_s$ alone, we use (35) to substitute ${}_1S_s$ in (34), solve for μ_1 as a function of x_1 , and then replace μ_1 in (35).

Rewrite (35) as

$$S_b - S_d - {}_1S_s = \frac{\mu_0}{\mu_1} (S_b - S_d - {}_0S_s). \quad (36)$$

Using (36) in the definition of $a_1(x_1)$:

$$a_1(x_1) = E_1 \sum_{t=0}^{\infty} \beta^t x'_{1+t} \frac{1}{2} [S_b - S_d - {}_1S_s]' [S_b - S_d - {}_1S_s] x_{1+t}$$

we find that

$$a_1(x_1) = \left(\frac{\mu_0}{\mu_1} \right)^2 a_0(x_1).$$

For convenience, write $b_1(x_1) = c(x_1) + d_1(x_1)$ with

$$\begin{aligned} c(x_1) &= E_1 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} x'_{1+t} [S_b - S_d + S_g]' [S_b - S_d + S_g] x_{1+t} d_1(x_1) \\ &= E_1 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} x'_{1+t} [S_b - S_d + S_g]' [-S_b + S_d + {}_1S_s] x_{1+t}. \end{aligned}$$

The term $c(x_1)$ does not depend on ${}_1S_s$, and the term $d_1(x_1)$ can be rewritten, using (36), as:

$$d_1(x_1) = \frac{\mu_0}{\mu_1} d_0(x_1).$$

We now replace a_1 and d_1 in (34) and solve for μ_1 :

$$\mu_1 = \mu_0 \frac{\mu_0 a_0(x_1) - d_0(x_1)}{\mu_0^2 a_0(x_1) + c(x_1)}$$

and substitute μ_1 in (35) to find:

$$\begin{aligned} {}_1S_s &= {}_0S_s \frac{\mu_0^2 a_0(x_1) + c(x_1)}{\mu_0 a_0(x_1) - d_0(x_1)} \\ &\quad + (S_b - S_d) \frac{a_0(x_1) (\mu_0 - \mu_0^2) - c(x_1) - d_0(x_1)}{\mu_0 a_0(x_1) - d_0(x_1)}. \end{aligned} \quad (37)$$

If we examine (37), we see that ${}_1S_s$ will not be independent of x_1 except when the forms a_1 , c and d_1 are, in fact, constants with respect to x_1 . A sufficient condition for this to hold is that b_t , d_t and ${}_0s_t$ be independent of t . When that obtains, the second term in (37) is zero (by (18)) and the time-consistent debt structure will be given by:

$${}_1S_s = {}_0S_s \frac{\mu_0^2 a + c}{\mu_0 a - d}.$$

Appendix C: Matlab programs

Stochastic Difference Equation

```
% LGRAMSEY
%
% Compute Ramsey equilibria in a LQ economy with distorting taxation:
%     case of a stochastic difference equation
% The program computes allocation (consumption, leisure), tax rates,
% revenues and the net present value of the debt.
%     See also LGRAMSM.
% FRV Apr 10 1998
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% customization required %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Gentle user:
%     The only customization required is in part (1).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% (1) Define the parameters of the model
% Misc
T=50;                % length of simulation
beta=1/1.05;        % discount factor
% exogenous state
% specify a stochastic process for the state of the form
% x(t+1)= A * x(t) + C * w(t+1)
% where w is i.i.d. N(0,1)
% ex1: govt spending is AR(1), state = [g ; 1]
%mg=.35;
%rho=.7;
%A=[rho mg*(1-rho); 0 1];
%C=[mg/10*sqrt(1-rho^2);0];
%% These selector matrices are all 1*n_x, where n_x is the size of the
state:
%Sg = [1 0];        % The selector matrix for the government expenditure
%Sd = [0 0];        % The selector matrix for the exogenous endowment
```

```

%Sb = [0 2.135];           % The selector matrix for the bliss point
%Ss = [0 0];             % The selector matrix for the promised coupon payments
% ex2: seasonal
rho=.95;
mg=.35;
A=[0 0 0 rho mg*(1-rho);
  1 0 0 0 0 ;
  0 1 0 0 0 ;
  0 0 1 0 0;
  0 0 0 0 1];
C=[mg/8*sqrt(1-rho^2);0;0;0;0];
Sg = [1 0 0 0 0];
Sd = [0 0 0 0 0];
Sb = [0 0 0 0 2.135];    % this is chosen so that (Sc+Sg)*x0=1
Ss = [0 0 0 0 0];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% This concludes the customization. Let'er rip!
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% (1) generate initial condition
[nx,nx]=size(A);
if ~exist('x0')
    x0=null(eye(nx)-A);
    if (x0(nx)<0)
        x0=-x0;
    end
    x0=x0./x0(nx);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% (2) solve for the Lagrange multiplier on the government BC
% we actually solve for mu=lambda/(1+2*lambda)
% mu is the solution to a quadratic equation a(mu^2-mu)+b=0
% where a and b are expected discounted sums of quadratic forms
% of the state; we use dlyap to compute those sums quickly.
Sm=Sb-Sd-Ss;    % a short-hand
Qa=dlyap(sqrt(beta)*A',-0.5*Sm'*Sm);
qa=trace(C'*Qa*C)*beta/(1-beta);
Qb=dlyap(sqrt(beta)*A',-0.5*(Sb-Sd+Sg)'*(Sg+Ss));
qb=trace(C'*Qb*C)*beta/(1-beta);
a0=x0'*Qa*x0+qa;
b0=x0'*Qb*x0+qb;
if a0<0           % normalize
    a0=-a0; b0=-b0;
end
disc=a0^2-4*a0*b0;

```



```

% there may be no solution:
if ( disc < 0 )
    disp('There is no Ramsey equilibrium.');
```

disp('Hint: you probably set government spending too high;');

disp(' elect a Republican Congress and start over.');

```

    return
end
mu=0.5*(a0-sqrt(disc))/a0;
% or it may be of the wrong sign:
if ( mu*(0.5-mu) < 0 )
    disp('negative multiplier on the government budget constraint!');
```

disp('Hint: you probably set government spending too low;');

disp(' elect a Democratic Congress and start over.');

```

    return
end
%%%%%%%%% (3a) solve for the allocation
Sc=0.5*(Sb+Sd-Sg-mu*Sm);
Sl=0.5*(Sb-Sd+Sg-mu*Sm);
Stau1=2*mu*Sm;
Stau2=Sb-Sd+Sg+mu*Sm;
%%%%%%%%% (4) run the simulation
[nx,nw]=size(C);
x=zeros(nx,T);
w=randn(nw,T);    % draw the shocks
x(:,1)=x0;        % initialize the state
for t=2:T         % compute x recursively
    x(:,t)=(A*x(:,t-1) + C*w(:,t));
end
% exogenous stuff:
g = Sg*x;        % govmt spending
d = Sd*x;        % endowment
b = Sb*x;        % bliss point
s = Ss*x;        % coupon payment on existing debt
% endogenous stuff:
c=Sc*x;          % consumption
l=Sl*x;          % labor
p=(Sb-Sc)*x;     % price
tau=(Stau1*x)./(Stau2*x); % taxes
rev=tau.*l;      % revenues
QB=dlyap(sqrt(beta)*A',-((Sb-Sc)'*(Sl-Sg)-Sl'*Sl));
qB=trace(C'*QB*C)*beta/(1-beta);
B=(x'*QB*x+qB)/p; % debt

```

```

% 1-period risk-free interest rate
R=((beta*(Sb-Sc)*A*x)./p).^(-1);
pi=B(2:T)-R(1:T-1).*(B(1:T-1)-rev(1:T-1)+g(1:T-1));
% risk-adjusted martingale
adjfac=((Sb-Sc)*x(:,2:T))./((Sb-Sc)*A*x(:,1:T-1));
Pitilde=cumsum(pi.*adjfac);
%%%%%%%% (5) plot
set(0,'DefaultAxesColorOrder',[0 0 0],...
    'DefaultAxesLineStyleOrder','-|--|-.')
figure
subplot(2,2,1),
plot([c' g' rev']),grid %cons, g and rev
v=axis; axis([1 T 0 v(4)]);
set(gca,'FontName','Times New Roman');
subplot(2,2,2)
plot((1:T),g,(1:T-1),B(2:T)),grid %g(t) and B(t+1)
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
subplot(2,2,3)
plot((1:T),R-1),grid % interest rate
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
% format the vertical labels for the interest rate plot
yticks=get(gca,'YTick');
set(gca,'YTickLabel',num2str(yticks*100,'%4.1f%%'));
subplot(2,2,4)
plot((1:T),[rev' g'],(1:T-1),pi), grid
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
figure
subplot(2,2,1),
plot((2:T),adjfac),grid
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
subplot(2,2,2),grid
plot((2:T),Pitilde)
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
% th-th-th-that's all, folks!
return
% labels to place on the graphs
gtext('\pi\it(t+1)');

```

```

gtext('\it g(t)');
gtext('\it g(t)');
gtext('\it g(t)');
gtext('\it c(t)');
gtext('\it\tau(t)l(t)');
gtext('\it\tau(t)l(t)');
h=gtext('\it B(t+1)');
h=gtext('\it R(t)-1');

```

Markov Chain

```

% LQRAMSM
%
% Compute Ramsey equilibria in a LQ economy with distorting taxation:
%   case of a Markov chain
% The program computes allocation (consumption, leisure), tax rates,
% revenues, and the net present value of the debt.
%   See also LQRAMSEY.
% FRV Apr 10 1998
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Gentle user:
%   The only customization required is in part (1).
% (1) Define the parameters of the model
% model 1:
% 3 states
P = [0.8 0.2 0;...           % The transition matrix
     0 0.5 0.5;
     0 0   1];
xbar = [ 0.5  0 2.2 0 1;...% Possible states of the world: each column
        0.5  0 2.2 0 1;... %   is a state of the world
        0.25 0 2.2 0 1]'; %   is a state of the world
                                % the rows are: [g d b s 1]
x0=1;                            % index of the initial state
Sg = [1 0 0 0 0];                % The selector matrix for the government
expenditure
Sd = [0 1 0 0 0];                % The selector matrix for the exogenous
endowment
Sb = [0 0 1 0 0];                % The selector matrix for the bliss point
Ss = [0 0 0 1 0];                % The selector matrix for the promised coupon
payments
% Misc

```

```

T=20; % length of simulation
beta=1/1.05; % discount factor
%%%%%%%%%%%%%% This concludes the customization. Let'er rip!
%%%%%%%%%%%%%%
% (2) solve for the Lagrange multiplier on the government BC
% we actually solve for  $\mu = \lambda / (1 + 2\lambda)$ 
%  $\mu$  is the solution to a quadratic equation  $a(\mu^2 - \mu) + b = 0$ 
% where a and b are expected discounted sums of quadratic forms
% of the state; we use dlyap to compute those sums quickly.
m = size(xbar,2); % Number of possible states of the world
Sm=Sb-Sd-Ss; % a short-hand
invP=inv(eye(size(P))-beta*P);
a0=((1:m)==x0)*0.5*invP*((Sm*xbar)'.^2);
b0=((1:m)==x0)*0.5*invP*(((Sb-Sd+Sg)*xbar).*((Sg+Ss)*xbar))';
if a0<0 % normalize
    a0=-a0; b0=-b0;
end
disc=a0^2-4*a0*b0;
% there may be no solution:
if ( disc < 0 )
    disp('There is no Ramsey equilibrium. ');
    disp('(Hint: you probably set government spending too high; ');
    disp(' elect a Republican Congress and start over. ');
    return
end
mu=0.5*(a0-sqrt(disc))/a0;
% or it may be of the wrong sign:
if ( mu*(0.5-mu) < 0 )
    disp('negative multiplier on the government budget constraint! ');
    disp('(Hint: you probably set government spending too low; ');
    disp(' elect a Democratic Congress and start over. ');
    return
end
% (3a) solve for the allocation
Sc=0.5*(Sb+Sd-Sg-mu*Sm);
Sl=0.5*(Sb-Sd+Sg-mu*Sm);
Stau1=2*mu*Sm;
Stau2=Sb-Sd+Sg+mu*Sm;
% (4) run the simulation
epsilon = rand(1,T);
cumP = cumsum(P')'; % The cumulative sum for each row of P
x(:,1) = xbar(:,x0);

```

```

state=zeros(T,1);
state(1)=x0;
for t = 2:T
    state(t) = min(find(cumP(state(t-1),:) >= epsilon(t)));
    x(:,t) = xbar(:,state(t));
end
% exogenous stuff:
g = Sg*x;          % govmint spending
d = Sd*x;          % endowment
b = Sb*x;          % bliss point
s = Ss*x;          % coupon payment on existing debt
% endogenous stuff:
c=Sc*x;            % consumption
l=S1*x;            % labor
p=(Sb-Sc)*x;       % price
tau=(Stau1*x)./(Stau2*x); % taxes
rev=tau.*l;        % revenues
% compute the debt
B=invP*(((Sb-Sc)*xbar).*((S1-Sg)*xbar)-(S1*xbar).^2)';
B=B(state,:)./p';
% 1-period risk-free interest rate
R=(beta*(P(state,:)*((Sb-Sc)*xbar)')./p').^(-1);
pi=B(2:T)-R(1:T-1).*(B(1:T-1)-rev(1:T-1)'+g(1:T-1)');
% risk-adjusted martingale
adjfac=p(2:T)'./(P(state(1:T-1),:)*((Sb-Sc)*xbar)');
Pitilde=cumsum(pi.*adjfac);
%%%%%% (5) plot
set(0,'DefaultAxesColorOrder',[0 0 0],...
    'DefaultAxesLineStyleOrder','-|--|--|.')
figure
subplot(2,2,1),%grid
plot([c' g' rev']) %cons, g and rev
v=axis; axis([1 T 0 v(4)]);
set(gca,'FontName','Times New Roman');
subplot(2,2,2),%grid
plot((1:T),g,(1:T-1),B(2:T)), %g(t) and B(t+1)
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
subplot(2,2,3),%grid
plot((1:T),R-1) % interest rate
v=axis; axis([1 T 0 v(4)]);
set(gca,'FontName','Times New Roman');

```

```

% format the vertical labels for the interest rate plot
set(gca,'YTickLabel',num2str(get(gca,'YTick')'*100','%3.0f%%'));
subplot(2,2,4),%grid
plot((1:T),[rev' g'],(1:T-1),pi),
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
figure
subplot(2,2,1),
plot((2:T),adjfac),grid
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
subplot(2,2,2),grid
plot((2:T),Pitilde)
v=axis; axis([1 T v(3) v(4)]);
set(gca,'FontName','Times New Roman');
% th-th-th-that's all, folks!
return
% labels to place on the graphs
gtext('\pi\it(t+1)');
gtext('\it g(t)');
gtext('\it g(t)');
gtext('\it g(t)');
gtext('\it c(t)');
gtext('\it\tau(t)l(t)');
gtext('\it\tau(t)l(t)');
h=gtext('\it B(t+1)');
h=gtext('\it R(t)-1');

```

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