Efficiency, Insurance, and Redistribution Effects of Government Policies*

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Abstract

We decompose welfare effects of switching from government policy A to policy B into three components: gains in *aggregate efficiency* from changes in total resources; gains in *redistribution* from altered consumption shares that *ex-ante* heterogeneous agents can expect to receive; and gains in *insurance* from changes in individuals' consumption risks. Our decomposition applies to a broad class of multi-person, multi-good, multi-period economies with diverse specifications of preferences, shocks, and sources of heterogeneity. It has several desirable properties. For example, it attributes to the insurance component all welfare effects that arise purely from mean preserving spreads in consumption. We compare our decomposition to earlier ones developed by Benabou (2002) and Floden (2001) and show that those approaches attribute welfare effects from such spreads to insurance only under special conditions.

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1 Introduction

We want to understand sources of changes in social welfare induced by alternative government policies. When households are heterogeneous, welfare depends on how efficiently goods and services are produced as well as how they are allocated across households. Welfare changes from reallocations are influenced by Pareto weights used to weight utilities of different agents, so it is natural to want to isolate the contribution from redistribution.

We propose an approach that imputes a welfare change from moving from policy A to policy B to three components that we call *aggregate efficiency*, *redistribution*, and *insurance*. Aggregate efficiency captures consequences of changes in aggregate resources. Redistribution captures changes in *ex-ante* consumption shares. Insurance captures changes in the *ex-post* consumption risk.

Our decomposition has several desirable properties. It can be applied both to static and to dynamic stochastic economies with multiple goods with arbitrary preferences that can vary across households. It isolates the components of change in welfare in the three components in an intuitive way. For instance, welfare changes that arise from changes in aggregate resources, holding fixed both *ex-ante* expected consumption shares and *ex-post* risks in consumption for all agents are imputed purely to the the aggregate efficiency component. Welfare changes that arise from changes in *ex-ante* consumption shares, holding fixed both the level of aggregate resources and consumption risk for each household are imputed purely to the redistribution component. Similarly, welfare changes from a policy that reduces *ex-post* consumption risk, while not affecting either total resources or expected consumption shares of any household are imputed purely to the insurance component. Furthermore, our decomposition is *reflexive* in the sense that each component of a welfare change from policy A to policy B is equal in magnitude and of opposite sign to its counterpart for moving from policy B to policy A.

We also compare our approach to popular decompositions developed by Benabou (2002) and Floden (2001).[1] Their approaches compute a certainty equivalent consumption for each agent and then average it across agents to compute a measure of total societal risk. We show that their decompositions generally do not satisfy the properties mentioned above. For example, an increase in welfare from a policy that induces a mean preserving reduction in household consumption is interpreted by those decomposition as coming solely from insurance only under special, non-generic conditions. We show that when these conditions are not satisfied,

[1] Examples of papers who use these, or closely related decompositions, include Abbott, Gallipoli, Meghir, and Violante (2019), Cho, Cooley, and Kim (2015), Conesa, Kitao, and Krueger (2009), Dyrda and Pedroni (2015), Guvenen, Kambourov, Kuruscu, Ocampo-Diaz, and Chen (2019), Heathcote, Storesletten, and Violante (2017), Koehne and Kuhn (2015), Nakajima and Takahashi (2020), Seshadri and Yuki (2004).

these decompositions may impute arbitrarily small or arbitrarily large welfare changes to the insurance component.

The rest of this paper is organized as follows. Section 2 describes the environment. Section 3 develops our decomposition. Section 4 provides examples of applications of our decomposition. Section 5 makes comparisons to the decompositions by Benabou and Floden. Section 6 concludes. Proofs and technical details appear in the Appendix.

2 Environment

A unit measure of *ex-ante* heterogeneous households are subject to risk *ex-post*. Households are distributed on a $[0,1]$ interval endowed with Lebesgue measure. A household i derives utility $U_i\left(\{c_{k,i}\}_k\right)$ from a bundle $\{c_{k,i}\}_k$ of goods. The bundle can be either finite or infinite. Each $c_{k,i} > 0$ is stochastic and drawn from distributions that can differ across households. Expected utility of household i is $\mathbb{E}_i U_i\left(\{c_{k,i}\}_k\right)$, where $\mathbb{E}_i$ is a mathematical expectation that uses household i's probability distribution over $\{c_{k,i}\}_k$. We assume that U_i is twice continuously differentiable and denote its first and second derivatives by $U_{k,i}, U_{km,i}$ for goods k, m. We assume that the joint distribution of stochastic processes $\{c_{k,i}\}_{i,k}$ has finite second moments and that expected utilities are well-defined. We use a shorthand $\mathbb{E}$ to denote the average over households. Thus, $\mathbb{E}x_i$ denotes $\int_{[0,1]} \mathbb{E}_i x_i di$ for any random variable x_i.

Welfare is evaluated using Pareto weights $\{\alpha_i\}_i$ that satisfy $\alpha_i \geq 0$ and $\mathbb{E}\alpha_i = 1$. Welfare is denoted by $\mathcal{W}$ and is given by

$$\mathcal{W} \equiv \mathbb{E}\alpha_i U_i\left(\{c_{k,i}\}_k\right). \tag{1}$$

An *allocation* under a government *policy* is a collection of stochastic processes $\{c_{k,i}\}_{k,i}$ that assign consumptions of all goods for all households. Switching from government policy A to policy B results in an altered allocation and consequent welfare change $\mathcal{W}^B - \mathcal{W}^A$. We use superscripts $j \in \{A, B\}$ to denote all variables under the alternative policies. Our goal is to partition a welfare change $\mathcal{W}^B - \mathcal{W}^A$ into economically interpretable components.

3 A decomposition

We start with a single good economy. Section 3.2 extends the decomposition to settings with multiple goods.

3.1 Single good economy

When there is a single good, subscripts k are redundant. We can use U_c, U_{cc} to denote first and second derivatives of the utility function. Without loss of generality, a household's consumption is a product of three terms

$$c_i = \mathbb{E}c_i \times \frac{\mathbb{E}_i c_i}{\mathbb{E}c_i} \times \frac{c_i}{\mathbb{E}_i c_i} \equiv C \times w_i \times (1 + \varepsilon_i) \,. \tag{2}$$

Aggregate consumption C measures the size of an aggregate "pie" to be divided. The fraction w_i is the share of that pie that household i expects to receive. Finally, ε_i captures the uncertainty that household i faces. By construction, we have $\mathbb{E}_i\varepsilon_i = 0$ and $\mathbb{E}w_i = 1$.

Identity (2) motivates us to decompose a welfare change across two consumption allocations into three components that measure *aggregate efficiency, redistribution,* and *insurance*. Before presenting our decomposition, it is useful to describe what we think are desirable properties for these components.

Property a. A welfare change from a policy that affects aggregate consumption C but not $\{w_i, \varepsilon_i\}_i$ should be imputed solely to aggregate efficiency;

Property b. A welfare change from a policy that affects expected shares $\{w_i\}_i$ but not C and $\{\varepsilon_i\}_i$ should be imputed solely to redistribution;

Property c. A welfare change from a policy that affects the stochastic process for $\{\varepsilon_i\}_i$ but not C and $\{w_i\}_i$ should be imputed solely to insurance.

Our notion of aggregate efficiency requires that redistribution and insurance are unaffected if consumption of every household is multiplied by the same positive scalar. This seems consistent with common usages of these terms. More generally, it also implies that redistribution and insurance remain unchanged so long as the distribution of expected consumption shares, $\{w_i\}_i$ and dispersions of consumptions relative to their means, $\{std_i(c_i)/\mathbb{E}_i c_i\}_i$ are both unchanged. Once we accept the Property a of pure aggregate efficiency, the other two properties follow naturally. By redistribution, we capture effects from reshuffling resources across households, that is, changes in expected consumption shares $\{w_i\}_i$. Thus, we attribute to redistribution all changes that keep C and $\{\varepsilon_i\}_i$ constant and affect neither aggregate resources nor dispersions of individual consumptions. By insurance, we capture changes in consumption risk that each household faces. Thus, we attribute to the insurance component the welfare consequences of a policy-induced change in variances of $\{\varepsilon_i\}_i$ that keeps aggregate resources and expected consumption shares constant (i.e., pure mean-preserving spreads in consumption).

Finally, it is desirable that a decomposition does not depend on whether one compares policy B to policy A, or policy A to policy B. We call this property *reflexivity* and formally state it as follows.

Property d. The absolute value of each component of the welfare change from policy A to policy B equals its counterpart in moving from policy B to policy A.

Our approach rests on Taylor expansions of welfare difference $\mathcal{W}^B - \mathcal{W}^A$. Welfare $\mathcal{W}^j$ for $j \in \{A, B\}$ can be represented as a mapping from an allocation—sequences and stochastic processes $\left\{C^j, w_i^j, \varepsilon_i^j\right\}_i$—to a real number. We expand around a "midpoint" $\left\{C^Z, w_i^Z, \mathbf{0}\right\}_i$ defined as

$$C^Z \equiv \sqrt{C^A C^B},\ w_i^Z \equiv \sqrt{w_i^A w_i^B},\ c_i^Z \equiv C^Z w_i^Z. \tag{3}$$

Applying Taylor's theorem, one can show (see the appendix) that

$$\mathcal{W}^B - \mathcal{W}^A \simeq \underbrace{\mathbb{E}\phi_i \Gamma}_{\text{agg. efficiency}} + \underbrace{\mathbb{E}\phi_i \Delta_i}_{\text{redistribution}} + \underbrace{\mathbb{E}\phi_i \gamma_i \Lambda_i}_{\text{insurance}}, \tag{4}$$

where "$\simeq$ " denotes equality up to a third order reminder term in the Taylor expansion, $\phi_i \equiv \alpha_i U_{c,i}\left(c_i^Z\right) c_i^Z$, $\gamma_i \equiv -U_{cc}\left(c_i^Z\right) c_i^Z / U_c\left(c_i^Z\right)$, and

$$\Gamma \equiv \ln C^B - \ln C^A,\ \Delta_i \equiv \ln w_i^B - \ln w_i^A,\ \Lambda_i \equiv -\frac{1}{2}\left[var_i\left(\ln c_i^B\right) - var_i\left(\ln c_i^A\right)\right]. \tag{5}$$

Equation (4) shows that up to the third-order expansion terms, the welfare effect $\mathcal{W}^B - \mathcal{W}^A$ is represented as a sum of three terms. The first term is proportional to the increase in the aggregate resources, Γ. The second term is proportional to a measure that captures changes in households' expected consumption shares, $\{\Delta_i\}_i$. The third term is proportional to changes in the *ex-post* risk and is captured by $\{\Lambda_i\}_i$. Although we consider a second order expansion, there are no interaction terms among Γ, $\{\Delta_i\}_i$ and $\{\varepsilon_i\}_i$. This suggests that a natural way to attribute contributions of aggregate efficiency, redistribution, and insurance to $\mathcal{W}^B - \mathcal{W}^A$ is in terms of the proportions $\frac{\mathbb{E}\phi_i\Gamma}{\mathbb{E}\phi_i[\Gamma+\Delta_i+\gamma_i\Lambda_i]}$, $\frac{\mathbb{E}\phi_i\Delta_i}{\mathbb{E}\phi_i[\Gamma+\Delta_i+\gamma_i\Lambda_i]}$, and $\frac{\mathbb{E}\phi_i\gamma_i\Lambda_i}{\mathbb{E}\phi_i[\Gamma+\Delta_i+\gamma_i\Lambda_i]}$ respectively.[2] These terms must sum to 1:

$$1 = \frac{\mathbb{E}\phi_i \Gamma}{\mathbb{E}\phi_i\left[\Gamma + \Delta_i + \gamma_i \Lambda_i\right]} + \frac{\mathbb{E}\phi_i \Delta_i}{\mathbb{E}\phi_i\left[\Gamma + \Delta_i + \gamma_i \Lambda_i\right]} + \frac{\mathbb{E}\phi_i \gamma_i \Lambda_i}{\mathbb{E}\phi_i\left[\Gamma + \Delta_i + \gamma_i \Lambda_i\right]}. \tag{6}$$

It is easy to verify that the decomposition (6) satisfies Properties a, b, c, and d. Any policy change that affects aggregate resources C but not $\{w_i, \varepsilon_i\}_i$ implies that $\Delta_i = \Lambda_i = 0$ for all

[2] In general, the remainder term in the Taylor expansion (4) depends on higher-order interactions of Γ, $\{\Lambda_i\}_i$ and $\{\varepsilon_i\}_i$ and cannot be partitioned unambiguously way across the three components. Defining the proportions as we do implicitly assumes that the residual is split across these components in the same proportions as the first and second order expansion terms. See the appendix for details.

i; therefore the aggregate efficiency component of such a policy is 1. Similar arguments verify that Properties b and c are satisfied. Property d (i.e., it is reflexive) is verified by noticing that moving from policy B to policy A implies that both numerator and the denominator of each fraction changes only its sign, leaving ratios unchanged.

Terms in decomposition (6) have natural interpretations. Quasi-weights $\{\phi_i\}_i$ convert percent changes $\{\Gamma, \Delta_i, \Lambda_i\}_i$ that summarize how allocations change between policies A and B into a welfare change $\mathcal{W}^B - \mathcal{W}^A$, measured in utils. Aggregate efficiency is simply the percent change in total consumption, Γ, adjusted by $\mathbb{E}\phi_i$. The insurance component depends on the changes in the variance of log consumption and the coefficients of the relative risk aversion $\{\gamma_i\}_i$.

Finally, the redistribution component depends on changes in expected consumption shares $\{w_i\}_i$ and is captured by $\{\Delta_i\}_i$. To help understand this term, it is instructive to write it as

$$\mathbb{E}\phi_i\Delta_i = \sqrt{C^AC^B}\int_{[0,1]} \alpha_i U_{c,i} \ln \frac{w_i^B}{w_i^A}\sqrt{w_i^A w_i^B}di.$$

Note that the integral in this expression resembles the well-known Kullback–Leibler (K-L) divergence, which takes the form $\int \ln \frac{w_i^B}{w_i^A} w_i^A di$ in our context. Just like the K-L divergence, our redistribution component is a measure of differences between distributions $\left\{w_i^A\right\}_i$ and $\left\{w_i^B\right\}_i$. (Note that $\left\{w_i^A\right\}_i$ and $\left\{w^B\right\}_i$ are are positive and sum to one so that they are proper probability distributions.) There are two important differences from the K-L divergence. Firstly, the K-L divergence is not reflexive and violates our desired Property d. By using midpoint between the two distributions, $\sqrt{w_i^A w_i^B}$, rather than w_i^A, we overcome this problem. Secondly, the K-L divergence by construction "weights" resources given to each household i equally. This corresponds to a very specific point on the Pareto frontier. On an arbitrary point of the Pareto frontier, these weights are given by $\alpha_i U_{c,i}$.

We wrote the redistributive component as $\mathbb{E}\phi_i\Delta_i$ in formula (4) to emphasize the common structure underlying the three components and to show the relationship to standard measures of divergence. An alternative way of representing it is by

$$\text{redistribution} \simeq \sqrt{C^AC^B}\mathbb{E}\alpha_i U_{c,i}\left(\bar{c}_i^Z\right)\left(w_i^B - w_i^A\right). \tag{7}$$

From (7) one can immediately see that redistribution is a measure of changes in shares $w_i^B - w_i^A$ weighted with $\alpha_i U_{c,i}$. Evidently, the redistributive component is zero if the planner is utilitarian and agents have linear utility of consumption.

3.1.1 Order of approximation and an alternative decomposition

One might be interested in the size of the omitted third order residual in equation (4) and in understanding how it depends on the difference between allocations under policies A and B. In this section we provide a short summary of these issues, leaving detailed proofs for the appendix.

In the appendix, we define a space that consists of sequences and stochastic processes $\left\{\tilde{\Gamma}, \tilde{\Delta}_i, \tilde{\varepsilon}_i\right\}_i$ endowed with an appropiate norm $||\cdot||$. From (2), any allocation $\{c_i^j\}_i$ can be mapped to a point in that space. We use $||\Gamma, \Delta, \varepsilon^B - \varepsilon^A||$ to measure the distance between allocations under two policies, $\left\{c_i^A\right\}_i$ and $\left\{c_i^B\right\}_i$ and denote it by $||c^B - c^A||$. The residual in equation (4) goes to zero as $||c^B - c^A|| \to 0$ but relatively little can be said theoretically about the relative speeds at which the residual and $||c^B - c^A||$ converge to zero. This is because when we expand around a non-stochastic point $\left\{c_i^Z\right\}_i$, processes $\left\{c_i^B\right\}_i$ and $\left\{c_i^A\right\}_i$ need not to converge to this point as $||c^B - c^A|| \to 0$.

We can modify our decomposition to ensure that the approximation error shrinks to zero at a rate faster than $||c^B - c^A||^2$. Write stochastic process c_i as an explicit mapping from some vector of primitive shocks ξ into consumption $c_i(\xi)$, with distribution of ξ given by $\Pr_i(d\xi)$. Equation (2) can be rewritten as $c_i(\xi) = C \times w_i \times \epsilon_i(\xi)$ where $\epsilon_i(\xi) \equiv c_i(\xi) / \mathbb{E}_i c_i$. Let the point of expansion be $\tilde{c}_i^Z(\xi) \equiv \sqrt{C^A C^B} \times \sqrt{w_i^A w_i^B} \times \sqrt{\epsilon_i^A(\xi)\, \epsilon_i^B(\xi)}$ and let $\delta_i(\xi) \equiv \ln \epsilon_i^B(\xi) - \ln \epsilon_i^A(\xi)$, $\tilde{\phi}_i(\xi) \equiv \alpha_i U_{c,i}\left(\tilde{c}_i^Z(\xi)\right) \tilde{c}_i^Z(\xi)$. Then the welfare decomposition can be written as

$$\mathcal{W}^B - \mathcal{W}^A = \underbrace{\mathbb{E}\tilde{\phi}_i \Gamma}_{\text{agg. efficiency}} + \underbrace{\mathbb{E}\tilde{\phi}_i \Delta_i}_{\text{redistribution}} + \underbrace{\mathbb{E}\tilde{\phi}_i \delta_i}_{\text{insurance}} + o\left(\|\Gamma, \Delta, \delta\|^2\right). \tag{8}$$

This decomposition retains the four properties of decomposition (4) that we discussed in previous section, and also guarantees that the reminder term converges to zero at a faster rate than $\left\|c^B - c^A\right\|^2$.

While decomposition (8) has nicer theoretical properties as the difference between allocations induces by policies A and B becomes small, we did not find any meaningful discrepancies in comparing decompositions (8) and (4) in all examples that we considered. For this reason, for the rest of this paper we focus on decomposition (4), both because it is easier to compute and because $\{\Lambda_i\}_i$ maps directly into statistics that are already routinely reported in quantitative models.

3.2 Decomposition in a multi-good economy

It is straightforward to extend our decomposition to general multi-good settings. To decompose welfare gains into components, first compute points of approximation $\left\{c_{k,i}^Z\right\}_k$ for each good as in equation (3). Then extend definitions of Γ_k and $\Delta_{k,i}$ from equation (5) to every good k and define $\Lambda_{km,i}$ for each pair of goods k, m as

$$\Lambda_{km,i} \equiv -\frac{1}{2}\left[cov_i\left(\ln c^B_{k,i}, \ln c^B_{m,i}\right) - cov_i\left(\ln c^A_{k,i}, \ln c^A_{m,i}\right)\right].$$

Let $U_{k,i}, U_{km,i}$ be first and second derivatives of U_i evaluated at $\left\{c^Z_{k,i}\right\}_k$ and let weights $\left\{\phi_{k,i}\right\}_k$ and cross-elasticities $\left\{\gamma_{km,i}\right\}_{k,m}$ be defined as

$$\phi_{k,i} \equiv \alpha_i U_{k,i} c^Z_i, \quad \gamma_{km,i} \equiv -\frac{U_{km,i} c^Z_{m,i}}{U_{k,i}}.$$

Using the same steps as in the one good economy, we can show that

$$\mathcal{W}^B - \mathcal{W}^A \simeq \text{agg. efficiency} + \text{redistribution} + \text{insurance}, \tag{9}$$

where

$$\begin{aligned} \text{agg. efficiency} &= \mathbb{E}\sum_k \phi_{k,i}\Gamma_k, \\ \text{redistribution} &= \mathbb{E}\sum_k \phi_{k,i}\Delta_{k,i}, \\ \text{insurance} &= \mathbb{E}\sum_k\sum_m \phi_{k,i}\gamma_{km,i}\Lambda_{km,i}. \end{aligned}$$

When utility is separable across all goods, this decomposition amounts to first computing decomposition (4) separately for each good and then summing each respective component across all goods. When utility is not separable, proper accounting for the insurance component requires adding changes in the covariances in dispersions of different goods weighted with cross-elasticities $\gamma_{km,i}$.

This approach applies directly to decomposing welfare gains from switching from policy A to policy B in infinite horizon economies. A typical application in quantitative macro is to find an invariant distribution under some policy τ^A, use that as an initial condition for an economy in which a different policy τ^B is introduced. One then compares welfare in the invariant distribution under policy τ^A to the transition path to the invariant distribution under policy τ^B (see, Guerrieri and Lorenzoni (2017) or Rohrs and Winter (2017) for examples of such applications). The utility function is typically assumed to be of the form

$$\mathbb{E}_i U_i\left(\left\{c_{k,i}\right\}_k\right) = \mathbb{E}_i \sum_{k=1}^{\infty} \beta^{k-1} u\left(c_{k,i}\right),$$

where β is the discount factor, u is the within period utility function, k is time and $\mathbb{E}_i$ is the period 0 conditional expectation.

These applications map directly into the framework developed in this section. Each good k in our multi-good set up corresponds to consumption in period k, tuple $\left(C_k^j, w_{k,i}^j, \varepsilon_{k,i}^j\right)$ consists of aggregate consumption, consumption shares of agent i, and consumption risk for agent i in period k under policies $j \in \{A, B\}$. If the initial condition is the invariant distribution under policy A, then sequence $\left\{C_k^A\right\}_k$ takes the same values for all k, but sequences $\left\{w_{k,i}^A, \varepsilon_{k,i}^A\right\}_k$ change over time due, for example, to idiosyncratic shocks experiences by agents.

3.3 Comparison of invariant distributions

In some applications, it can be difficult to compute a full transition path from changing government policies, so researchers often rely on comparing welfare across steady states. That is, they compare welfare in the invariant distribution under policy B to welfare under invariant distribution under policy A. We provide a natural way to extend our framework to such settings.

Suppose that an invariant distribution under policy j is characterized by the probability density f^j on a space of household characteristics $S \subset \mathbb{R}^n$. In common applications (e.g., Aiyagari (1995), Floden (2001), Conesa, Kitao, and Krueger (2009)) welfare is defined as an integral over agents' expected utilities,

$$\mathcal{W}^j = \int_S \mathbb{E}_s U\left(\left\{c_{k,s}^j\right\}_k\right) f^j(s)\, ds,$$

where $\mathbb{E}_s$ is the expected utility of agent $s \in S$.

It is easiest to consider first a case in which the space of characteristics is unidimensional, so that $n = 1$. Let F^j be the cdf of f^j. For any c_s^j that is function of $s \in S$, define $\bar{c}_i$ by $\bar{c}_{F(s)}^j = c_s^j$. By construction, $\bar{c}_i^j$ is defined for $i \in [0,1]$. Standard "integration by substitution" arguments (see, e.g. Corollary 3.7.2 in Bogachev (2007)) imply that

$$\mathcal{W}^j = \int_S \mathbb{E}_s U\left(\left\{c_{k,s}^j\right\}_k\right) f^j(s)\, ds = \int_{[0,1]} \mathbb{E}_i U\left(\left\{\bar{c}_{k,i}^j\right\}_k\right) di.$$

The last term is a special case of our definition of welfare (1), so all the steps from section 3.2 apply directly.

This approach generalizes to any n via induction. We verify this for $n = 2$. Suppose the $S = \mathcal{A} \times \Theta \subset \mathbb{R}^2$ probability density $f^j(a, \theta)$ under policy j. Let $f^j(\theta) \equiv \int_{\mathcal{A}} f^j(a, \theta)\, da$. We can write welfare as

$$\mathcal{W}^j = \int_\Theta \left[\int_{\mathcal{A}} \mathbb{E}_{(a,\theta)} U\left(\left\{c_{k,(a,\theta)}^j\right\}_k\right) \frac{f^j(a,\theta)}{f^j(\theta)} da\right] f^j(\theta)\, d\theta.$$

Applying the same procedure that we described in the $n = 1$ case twice, first to the inner integral, and then to the outer one, we can represent welfare

$$\mathcal{W}^j = \int_{[0,1]^2} \mathbb{E}_{(i,\iota)} U\left(\left\{\bar{c}^j_{k,(i,\iota)}\right\}_k\right) di d\iota.$$

The application of the decomposition in section 3.2 is now straightforward, except that now households are distributed over $[0,1]^2$ rather than $[0,1]$.

4 Applications

In this section we consider examples of application of our decomposition

4.1 Benabou (2002) economy

Benabou (2002) used a tractable framework analytically to study a heterogeneous agents economy with taxes. Agents are infinitely lived, have preferences over consumption good $c_{t,i}$ and labor $l_{t,i}$ ordered by

$$\sum_{t=0}^{\infty} \beta^t \left[\ln c_{t,i} - \frac{1}{1+\eta} l_{t,i}^{1+\eta}\right].$$

Pre-tax earnings of household i in period t are $y_{t,i} = \theta_{t,i} l_{t,i}$, where $\theta_{t,i}$ is labor productivity that satisfies $\theta_{t,i} = \exp\left(e_i + \xi_{t,i}\right)$. The first component of productivity $e_i \sim \mathcal{N}\left(-\frac{\upsilon_e^2}{2}, \upsilon_e^2\right)$, captures heterogeneity in initial skill endowments across households. The second component $\xi_{t,i} \sim \mathcal{N}\left(-\frac{\upsilon_{\xi,t}^2}{2}, \upsilon_{\xi,t}^2\right)$, is driven by idiosyncratic productivity shocks experienced by households.[3] Households hold no assets and consume their after-tax labor incomes each period. After-tax labor income is $\bar{\tau}_t y_{t,i}^{1-\tau}$, where τ is the degree of tax progressivity and $\bar{\tau}_t$ is chosen so that the net tax revenues are zero in each period t.

It is easy to solve for an equilibrium as a function of the progressivity parameter τ. Logarithmic utility in consumption and absence of non-labor income imply that all households choose the same labor supply in all periods, $l_{t,i}(\tau) = (1-\tau)^{\frac{1}{1+\eta}}$. This implies that both the aggregate labor $L(\tau)$ and the aggregate consumption $C(\tau)$ are

$$C(\tau) = L(\tau) = (1-\tau)^{\frac{1}{1+\eta}}.$$

Heterogeneity in productivity translates one for one into heterogeneity in household consumption

$$c_{t,i}(\tau) = C(\tau) \times \exp\left((1-\tau) e_i + \tau(1-\tau)\frac{\upsilon_e^2}{2}\right) \times \exp\left((1-\tau)\xi_{t,i} + \tau(1-\tau)\frac{\upsilon_{\xi,t}^2}{2}\right). \quad (10)$$

[3] Benabou (2002) represents idiosyncratic shocks slightly differently. He assumes that each period households are subject to multiplicative random walk shocks drawn from a fixed log-normal distribution. It can be shown that this is equivalent to our specification of shocks $\xi_{t,i}$, where $\upsilon_{\xi,t}^2$ grows linearly with t.

Utilitarian welfare is $\mathcal{W}(\tau) = \sum_{t=0}^{\infty} \beta^t \mathcal{W}_t(\tau)$, where

$$\mathcal{W}_t(\tau) = \ln C(\tau) - \frac{1}{1+\eta} L(\tau)^{1+\eta} - (1-\tau)^2 \frac{\upsilon_e^2}{2} - (1-\tau)^2 \frac{\upsilon_{\xi,t}^2}{2}. \tag{11}$$

We now apply our section 3.2 decomposition. Explicitly computing each component in that decomposition for two different tax progressivity parameters τ^A and τ^B, we have

$$\begin{aligned} \mathcal{W}_t^B - \mathcal{W}_t^A \quad \simeq \quad & \underbrace{\left[\ln \frac{C(\tau^B)}{C(\tau^A)} - \sqrt{(1-\tau^A)(1-\tau^B)} \ln \frac{L(\tau^B)}{L(\tau^A)}\right]}_{\text{agg. efficiency}} \\ & + \underbrace{\frac{\upsilon_e^2}{2}\left[\left(1-\tau^A\right)^2 - \left(1-\tau^B\right)^2\right]}_{\text{redistribution}} + \underbrace{\frac{\upsilon_{\xi,t}^2}{2}\left[\left(1-\tau^A\right)^2 - \left(1-\tau^B\right)^2\right]}_{\text{insurance}}. \end{aligned} \tag{12}$$

Changes in tax progressivity affect all components of the decomposition. Higher taxes reduce total output, aggregate consumption and labor. That changes aggregate efficiency. Higher taxes also reduce dispersion in outputs $\{y_{t,i}\}_i$. That dispersion is driven in part by permanent initial heterogeneity, captured by parameter υ_e^2, and in part by *ex-post* risk, captured by $\upsilon_{\xi,t}^2$. Thus, higher taxes provoke both redistribution and insurance. Equation (12) shows that our decomposition imputes all changes in aggregate resources to aggregate efficiency, all changes proportional to *ex-ante* heterogeneity υ_e^2 to redistribution, and all changes proportional to *ex-post* risk $\upsilon_{\xi,t}^2$ to insurance.

By comparing equations (11) and (12), we can also evaluate the size of the omitted residual in our decomposition and study how it depends on τ^A and τ^B. Denoting this residual by R, we have

$$\begin{aligned} R \quad &= \quad \sqrt{(1-\tau^A)(1-\tau^B)} \frac{\ln\left(1-\tau^B\right) - \ln\left(1-\tau^A\right)}{1+\eta} - \frac{\left(1-\tau^B\right) - \left(1-\tau^A\right)}{1+\eta} \\ &= \quad O\left(\left|\ln \frac{1-\tau^B}{1-\tau^A}\right|^3\right). \end{aligned} \tag{13}$$

The residual emerges because we approximate non-linear function $\frac{1}{1+\eta}\left[L\left(\tau^B\right)^{1+\eta} - L\left(\tau^A\right)^{1+\eta}\right]$ with a linear term $\left[\ln L\left(\tau^B\right) - \ln L\left(\tau^A\right)\right]$. This residual goes to zero with the rate $O\left(\left|\ln \frac{1-\tau^B}{1-\tau^A}\right|^3\right)$ as $\tau^B \to \tau^A$. To put this into perspective, Feenberg, Ferriere, and Navarro (2017) estimate parameter τ for the U.S. for various years and find that between 1980 and 2010 it varied between 0.07 and 0.1. For tax policies in this range, we have $\left|\ln \frac{1-\tau^B}{1-\tau^A}\right|^3 \leq \left(\ln \frac{0.93}{0.9}\right)^3 \approx 0.000035$.

5 Comparison to decompositions by Benabou and Floden

Motivated by the same questions as we are, Benabou (2002) and Floden (2001) developed alternative decompositions of a policy-induced welfare change into analogues of our efficiency, insurance, and redistribution components. A substantial literature uses their decomposition to study consequences of alternative public policies. In this section, we compare our approach to theirs. We focus in this section on a single good economy since it allows us to make our points transparently.

5.1 Description of decompositions

The decompositions developed by Benabou (2002) and by Floden (2001) share several common features. They require that all households have identical preferences U and start by computing a certainty equivalent $c_i^{ce,j}$ level of consumption for each household i under any policy j as follows

$$U\left(c_i^{ce,j}\right) = \mathbb{E}_i U\left(c_i^j\right). \tag{14}$$

Then they define an aggregate certainty equivalent $C^{ce,j}$ as $C^{ce,j} \equiv \mathbb{E}c_i^{ce,j}$.

Benabou's decomposition of welfare change between policies A and B is based on the identity

$$\begin{aligned} 1 \;=\; & \frac{U\left(C^B\right) - U\left(C^A\right)}{\mathcal{W}^B - \mathcal{W}^A} + \frac{\left\{\mathcal{W}^B - \mathcal{W}^A\right\} - \left\{U\left(C^{ce,B}\right) - U\left(C^{ce,A}\right)\right\}}{\mathcal{W}^B - \mathcal{W}^A} \\ & + \frac{\left\{U\left(C^{ce,B}\right) - U\left(C^{ce,A}\right)\right\} - \left\{U\left(C^B\right) - U\left(C^A\right)\right\}}{\mathcal{W}^B - \mathcal{W}^A}. \end{aligned}$$

The three fractions on the right side correspond to our notions of aggregate efficiency, redistribution, and insurance.[4]

Benabou created his decomposition to study implications of changes in tax progressivity in the economy that we described in section 4.1. Since preferences over consumption are logarithmic in that model, changes in utils corresponds to changes in log points of consumption, so that all terms on in the decomposition have natural economically interpretable units. This is no longer the case when preferences are not logarithmic.

Floden (2001) extended Benabou's approach to more general settings by converting each component of the decomposition into consumption units. As a first step, he computes numbers p_{insur}^j and p_{redis}^j using

$$U\left(\left(1 - p_{insur}^j\right) C^j\right) = U\left(C^{ce,j}\right), \quad U\left(\left(1 - p_{redis}^j\right) C^{ce,j}\right) = \mathbb{E}\alpha_i U\left(c_i^{ce,j}\right). \tag{15}$$

[4] Benabou and Floden use slightly different terminology when they refer to their decomposition terms, both from each other and from our paper. To avoid confusion, we use our terminology throughout.

Then he constructs contributions of aggregate efficiency, redistribution and insurance as

$$1+\omega_{eff} \equiv \frac{C^B}{C^A},\ 1+\omega_{redis} \equiv \frac{1-p^B_{redis}}{1-p^A_{redis}},\ 1+\omega_{insur} \equiv \frac{1-p^B_{insur}}{1-p^A_{insur}}.$$

Similarly, the total welfare change from policy B is also computed in consumption units as

$$\mathbb{E}\alpha_i U\left(c_i^B\right) = \mathbb{E}\alpha_i U\left(\left(1+\omega\right)c_i^A\right). \tag{16}$$

Floden shows that if U has the CRRA form $U\left(c\right)=\frac{c^{1-\gamma}}{1-\gamma}$ if $\gamma>0,\ \gamma\neq 1$ and $U\left(c\right)=\ln c$ for $\gamma=1$, then the following relationship holds

$$1=\frac{\ln\left(1+\omega_{eff}\right)}{\ln\left(1+\omega\right)}+\frac{\ln\left(1+\omega_{redis}\right)}{\ln\left(1+\omega\right)}+\frac{\ln\left(1+\omega_{insur}\right)}{\ln\left(1+\omega\right)}, \tag{17}$$

with the three terms on the right hand side corresponding to our notions of aggregate efficiency, redistribution, and insurance.

5.2 An example

In this section, we start with a simple example that illustrates limitations of the Benabou-Floden approach. We assume that U is logarithmic, so that the decompositions by Benabou and Floden coincide, and that the planner is utilitarian. We assume that consumptions of households under policies A and B are

$$\begin{aligned} c_i^A &= a_i + l_i\left(1+\tilde{\varepsilon}_i\right), \\ c_i^B &= a_i + l_i, \end{aligned}$$

where $\{a_i, l_i\}_i$ are some non-stochastic variables and $\tilde{\varepsilon}_i$ is a non-trivial stochastic process. We assume that $\mathbb{E}_i\tilde{\varepsilon}_i=0$ and that $var_i\left(\tilde{\varepsilon}_i\right)=var\left(\tilde{\varepsilon}\right)$ is the same for all i.

This highly stylized example captures some key features of richer heterogeneous agents models like Aiyagari (1995). In such models, household consumption comes from asset income (captured here by a_i) and labor income (l_i), which is also subject to idiosyncratic shocks ($\tilde{\varepsilon}_i$ here). Policy B is a simple example of a social insurance program that removes all the uncertainty that households face about their earnings without reallocating resources across households or changing the aggregate amount of resources. Since agents are risk-averse, policy B improves welfare. All improvement comes from better insurance.

We apply first our decomposition (6) to this economy. Let $\bar{a}\equiv\mathbb{E}a_i$, $\bar{l}\equiv\mathbb{E}l_i$, and write c_i^A and c_i^B as

$$\begin{aligned} c_i^A &= \left(\bar{a}+\bar{l}\right)\times\frac{a_i+l_i}{\bar{a}+\bar{l}}\times\left(1+\frac{l_i}{a_i+l_i}\tilde{\varepsilon}_i\right), \\ c_i^B &= \left(\bar{a}+\bar{l}\right)\times\frac{a_i+l_i}{\bar{a}+\bar{l}}\times 1. \end{aligned}$$

It is evident from these expression that $\Gamma = \Delta_i = 0$ for all i, and therefore our decomposition attributes 100% of welfare gains to the insurance component.

We now apply the Benabou-Floden decomposition. Since it is not available in closed form in this example, we consider approximations of their expressions for small values of idiosyncratic shocks, $||\varepsilon||$. It is easy to show (see the appendix) that a certainty equivalent of consumption for household i satisfies

$$\begin{aligned} c_i^{ce,A} &= c_i^Z - \frac{1}{2}\bar{c}_i^Z \left(\frac{l_i}{a_i + l_i}\right)^2 var\left(\varepsilon\right) + o\left(||\varepsilon||^2\right), \\ c_i^{ce,B} &= c_i^Z , \end{aligned} \tag{18}$$

where $c_i^Z = a_i + l_i$. Naturally, the certainty equivalent of consumption under policy B coincides with expected consumption c_i^Z, since households face no uncertainty. Under policy A, the certainty equivalent is equal to the expected consumptions adjusted by the coefficient of risk aversion (which is equal to one with logarithmic preferences) and the variance of consumption $\left(\frac{l_i}{a_i+l_i}\right)^2 var\left(\tilde{\varepsilon}\right)$. Note that even though all households face the same uncertainty about labor earnings $\ln\left[l_i\left(1+\tilde{\varepsilon}_i\right)\right]$, their consumption risk varies due to heterogeneity in their asset holdings.

Using expressions (18), it is easy to show that

$$\begin{aligned} \ln\left(1+\omega_{insur}\right) &= \ln \mathbb{E} c_i^{ce,B} - \ln \mathbb{E} c_i^{ce,A} = \frac{1}{2} var\left(\tilde{\varepsilon}\right) \frac{\mathbb{E} c_i^Z \left(\frac{l_i}{a_i+l_i}\right)^2}{\mathbb{E} c_i^Z} + o\left(||\varepsilon||^2\right), \\ \ln\left(1+\omega\right) &= \mathbb{E} \ln c_i^{ce,B} - \mathbb{E} \ln c_i^{ce,A} = \frac{1}{2} var\left(\tilde{\varepsilon}\right) \mathbb{E}\left(\frac{l_i}{a_i+l_i}\right)^2 + o\left(||\varepsilon||^2\right). \end{aligned}$$

Combining these two expressions, we get

$$\frac{\ln\left(1+\omega_{insur}\right)}{\ln\left(1+\omega\right)} = \frac{\mathbb{E} c_i^Z \left(\frac{l_i}{a_i+l_i}\right)^2}{\mathbb{E} c_i^Z \times \mathbb{E}\left(\frac{l_i}{a_i+l_i}\right)^2} + o\left(1\right), \tag{19}$$

where $o(1) \to 0$ as $||\varepsilon|| \to 0$.

Equation (19) shows that the Benabou-Floden decomposition would assign 100% of welfare gains to insurance only if $c_i^Z = a_i + l_i$ is uncorrelated with $\left(\frac{l_i}{a_i+l_i}\right)^2$ across households. This condition will not be generally satisfied unless $a_i = 0$ for all i, so that agents have no asset income.

It is easy to see why the Benabou-Floden decomposition generally fails to assign all welfare gains to the insurance component. Welfare is equal to $\mathbb{E} \ln c_i^{ce}$, while they measure the insurance component as $\ln \mathbb{E} c_i^{ce}$. There is no reason to expect that alternative insurance policies will affect

$\mathbb{E} \ln c_i^{ce}$ and $\ln \mathbb{E} c_i^{ce}$ identically. The general problem is that while $U\left(c_i^{ce}\right)$ is a good measure of how risk affects the utility of individual household, there is no reason to think that $U\left(\mathbb{E} c_i^{ce}\right)$ is a good measure of how risk affects social welfare.

The insurance component in equation (19) is always positive, but it is easy to see that it can be arbitrarily large or small. Suppose that the relationship between a_i and l_i is

$$a_i + l_i = l_i^{\kappa} \text{ for all } i,$$

where κ reflects covariance between assets and labor income. Furthermore, assume that l_i is distributed according to a Pareto distribution with shape parameter ρ. It is easy to calculate explicitly the expressions in (19) to show that

$$\frac{\ln\left(1+\omega_{insur}\right)}{\ln\left(1+\omega\right)} = \frac{\left(\rho-\kappa\right)\left(\rho-\left(2-2\kappa\right)\right)}{\rho\left(\rho-\left(2-\kappa\right)\right)} + o\left(1\right).$$

The first term on the right hand side is well defined as long as $\rho > \max\left\{2-\kappa, \kappa, 2-2\kappa\right\}$. By varying ρ and κ, the left side of the above equation can take any value in the $(0, \infty)$ interval. The residual term $o\left(1\right)$ can be made arbitrarily small by choosing small enough idiosyncratic shocks. This implies that the Benabou-Floden decomposition might assign any value in $(0, \infty)$ to the insurance component in our simple social insurance example. Since the aggregate consumption C is unchanged, and the three components always sum to one, this also implies that the Benabou-Floden decomposition might assign any value in $(-\infty, 1)$ to the redistribution component. We summarize this discussion in

Lemma 1 *In the constructed example,* $\frac{\ln(1+\omega_{insur})}{\ln(1+\omega)}$ *can take any value in* $(0, \infty)$*, and* $\frac{\ln(1+\omega_{redis})}{\ln(1+\omega)}$ *can take any value in* $(-\infty, 1)$*, depending on* κ*,* ρ*, and the shock process* ε*.*

5.3 Sufficient conditions

Previous section shows that Benabou's and Floden's decompositions will typically violate Property c, i.e. it will attribute mean-preserving changes in the stochastic process in consumption to components other than insurance. Yet it is easy to construct examples where their decompositions do satisfy desirable Properties a, b, and c. In particular, Benabou developed his decomposition to study welfare effect of changes in the progressivity of tax system in an economy that we described in section 4.1. We show in the appendix that Benabou's decomposition satisfies Properties a, b, and c in his economy, and coincides with our decomposition to the order of approximation we consider. The following lemma provides a more systematic analysis of conditions under which Benabou's and Floden's decompositions satisfy Properties a, b, and c, and their relationship to our decomposition.

Lemma 2 *In the single good economy, the decompositions of Benabou and Floden generically fail Properties a, b, and c. If U is CRRA (logarithmic) and there exists some Λ such that*

$$var_i\left(\ln c_i^A\right) - var_i\left(\ln c_i^B\right) = 2\Lambda \text{ for all } i, \tag{20}$$

then Floden's (Benabou's) decomposition satisfies Properties a, b, and c and coincides with our decomposition, up to a residual term that goes to zero as $\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\| \to 0$ for $j \in \{A, B\}$.

A key conclusion from this lemma is that in addition to restricting attention to CRRA preferences, Benabou's and Floden's decompositions require that policy B changes the variance of log consumption of all agents by the same amount. To see why, observe that with CRRA preferences welfare is given by $\frac{1}{1-\gamma}\mathbb{E}\left(c_i^{ce}\right)^{1-\gamma}$, while their decompositions identify the insurance component with $\frac{1}{1-\gamma}\left(\mathbb{E}c_i^{ce}\right)^{1-\gamma}$. Changes in risk affect $\frac{1}{1-\gamma}\mathbb{E}\left(c_i^{ce}\right)^{1-\gamma}$ and $\frac{1}{1-\gamma}\left(\mathbb{E}c_i^{ce}\right)^{1-\gamma}$ in the same way only if risk for each household i changes proportionally to c_i^{ce}. This happens if condition (20) is satisfied.

It is easy to verify that condition (20) indeed holds in Benabou's economy for consumption good. Using equation (10) we have

$$var_i\left(\ln c_i^A\right) - var_i\left(\ln c_i^B\right) = \frac{\upsilon_{\xi,t}^2}{2}\left[\left(1-\tau^A\right)^2 - \left(1-\tau^B\right)^2\right] \text{ for all } i.$$

This condition relies critically on the assumption that agents hold no assets. As we illustrated in the previous section, relaxing the no-asset assumption can have important consequences for inferences from this decomposition in heterogeneous agents economies. Since asset dispersion plays an important role in many heterogenous agents economies (including the one considered by Floden (2001)), condition (20) is unlikely to hold in those models.

In calibrated economies, violation of condition (20) and its multi-good generalizations can have substantial impacts on decompositions. For example, in Bhandari, Evans, Golosov, and Sargent (2021) we considered a standard incomplete market heterogeneous agent (HA) New Keynesian economy. Labor productivity shocks in our calibration were similar to those used by Benabou (2002), but our agents also hold financial assets calibrated to U.S. data. In our baseline setting, we showed that switching from an optimal policy prescribed by a textbook representative agent economy to an optimal policy in the HA economy increased welfare, and that 158% of that welfare gain could be attributed to insurance, -66% to aggregate efficiency and 8% to redistribution. Our decomposition was also robust to changes in assumptions about preferences, redistributive objectives of the government, and other parameters of the model. That finding was also consistent with the observation that an optimal policy was governed by a desire to replicate missing Arrow-Debreu markets, i.e., to provide insurance against heterogeneous consequences of aggregate shocks. In contrast, when we applied directly Floden's decomposition, it attributed +800% of welfare gains to both the insurance and redistribution component. Moreover, computed values varied wildly even with modest changes in parameters such as degrees of agents' risk aversion and the planner's inequality aversion even though neither optimal policies nor allocations were very sensitive to them. The sources of these findings traces back to the one isolated in the example summarized in Lemma 1. Dyrda and Pedroni (2015) report closely related issues with the sensitivity of Floden's decomposition in their application.

6 Conclusion

We developed a decomposition of welfare changes into three components: aggregate efficiency, which captures effects from changes in the aggregate quantity of resources; redistribution, which captures effects from changes in shares of resources that *ex-ante* heterogeneous agents expect to receive; and insurance, which captures effects of changes in the uncertainties that agents face. Our decomposition applies to a large class of multi-person, multi-good, multi-period economies with general specifications of preferences and shocks and sources of heterogeneity.

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A Appendix

A.1 Taylor series in abstract spaces

Recall some properties of Taylor series in general spaces. Let $f : X \to \mathbb{R}$ be a mapping from a normed space X (with some norm $||\cdot||$) into $\mathbb{R}$. The first-order Frechet derivative at a point $x \in X$ is a linear mapping $f'(x) : X \to \mathbb{R}$ such that for each $h \in X$, we have $\lim_{||h||\to 0} \frac{|f(x+h)-f(x)-f'(x)\cdot h|}{||h||} = 0$. The second-order Frechet derivative is the Frechet derivative of $f'(x)$. It is a bilinear map. Any function f that is twice Frechet differentiable satisfies

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{1}{2}f''(x)\cdot(h)^2 + R^f(x,h), \tag{21}$$

where $R^f(x,h)$ is a residual that is of the order $o\left(||h||^2\right)$ (see Cartan (1971), Theorem 5.6.3). When $X \subset \mathbb{R}^n$, functionals $f'(x)$ and $f''(x)$ are simply the Jacobian and Hessian of f, respectively.

For any given $(x,h) \in X \times X$, define function $g : \mathbb{R} \to \mathbb{R}$ by $g(\sigma) \equiv f(x+\sigma h)$. Take Taylor expansion of g around $\sigma = 0$ to get

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) + R^g(1). \tag{22}$$

Since we have $g'(0) = f'(x)\cdot h$ and $g''(0) = f''(x)\cdot(h)^2$, we have

$$R^g(1) = R^f(x,h) = o\left(||h||^2\right). \tag{23}$$

Finally, we use "$\simeq$" to denote that two relationships are equal up to any term of order $o\left(||h||^2\right)$. In this notation, relationship (21) and (22) can be rewritten as

$$\begin{aligned} f(x+h) &\simeq f(x) + f'(x)\cdot h + \frac{1}{2}f''(x)\cdot(h)^2, \\ g(1) &\simeq g(0) + g'(0) + \frac{1}{2}g''(0). \end{aligned}$$

A.2 Conventions and terminology

We define some conventions to be used throughout our proofs. For expressions that take the same value for policies $j = A$ and $j = B$ we occasionally use a shorthand "t.i.p.", meaning "terms independent of policy". By the Law of Iterated Expectations (LIE), we mean the property that for any deterministic function x_i of i and random variable ε_i with $\mathbb{E}_i\varepsilon_i = 0$, we have $\mathbb{E}x_i\varepsilon_i = \mathbb{E}\mathbb{E}_i x_i\varepsilon_i = \mathbb{E}x_i\mathbb{E}_i\varepsilon_i = 0$, and that, by analogous arguments, $\mathbb{E}x_i(\varepsilon_i)^2 = \mathbb{E}x_i var_i(\varepsilon_i)$. We use shorthands $U_i, U_{c,i}, U_{cc,i}$ for $U_i\left(c_i^Z\right), U_{c,i}\left(c_i^Z\right), U_{cc,i}\left(c_i^Z\right)$ in a one-good economy, and $U_i, U_{k,i}, U_{km,i}$ for $U_i\left(\left\{c_{k,i}^Z\right\}_k\right), U_{k,i}\left(\left\{c_{k,i}^Z\right\}_k\right), U_{km,i}\left(\left\{c_{k,i}^Z\right\}_k\right)$ in multi-good economies.

A.3 Derivations of equations (4) and (7)

Let $\Gamma^j \equiv \ln C^j - \ln C^Z$ and $\Delta_i^j \equiv \ln w_i^j - \ln w_i^Z$. Observe that by construction we have

$$\Gamma^B = -\Gamma^A = \frac{1}{2}\Gamma,\ \Delta_i^B = -\Delta_i^A = \frac{1}{2}\Delta_i, \tag{24}$$

and therefore

$$\left(\Gamma^j\right)^2, \left(\Delta_i^j\right)^2, \Gamma^j\Delta_i^j \text{ are t.i.p. for all } i.$$

Let $\mathcal{W}^Z \equiv \mathbb{E}\alpha_i U_i\left(c_i^Z\right)$ and write

$$\mathcal{W}^B - \mathcal{W}^A = \left(\mathcal{W}^B - \mathcal{W}^Z\right) - \left(\mathcal{W}^A - \mathcal{W}^Z\right). \tag{25}$$

We now apply Taylor series from section A.1 to $\left(\mathcal{W}^B - \mathcal{W}^Z\right)$ and $\left(\mathcal{W}^A - \mathcal{W}^Z\right)$. We can write

$$\mathcal{W}^j - \mathcal{W}^Z = \mathbb{E}\alpha_i U\left(\exp\left(\Gamma^j + \Delta_i^j\right)\left(1+\varepsilon_i^j\right)c_i^Z\right) - \mathbb{E}\alpha_i U\left(c_i^Z\right).$$

In the language of section A.1, the space X consists of sequences and stochastic processes $\left\{\tilde{\Gamma}, \tilde{\Delta}_i, \tilde{\varepsilon}_i\right\}_i$. We have that $\tilde{\Gamma} \in \mathbb{R}$ and $\left\{\tilde{\Delta}_i\right\}_i$ is a mapping from $[0,1]$ to $\mathbb{R}$. We can represent stochastic processes $\tilde{\varepsilon}_i$ as mappings $\tilde{\varepsilon}_i(\xi)$ with distribution $\Pr_i(d\xi)$ where without loss of generality $\xi \in [0,1]$. Thus, $\{\tilde{\varepsilon}_i\}_i$ maps from $[0,1]^2$ to $\mathbb{R}$. Any $x \in X$ can be represented as $x = \left(\tilde{\Gamma}, \tilde{\Delta}, \tilde{\varepsilon}\right) \in \mathbb{R} \times L^2([0,1]) \times L^2\left([0,1]^2\right)$. We endow X with a corresponding norm. We define function $f: X \to R$ by $\mathbb{E}\alpha_i U\left(\exp\left(\tilde{\Gamma}^j + \tilde{\Delta}_i^j\right)\left(1+\tilde{\varepsilon}_i^j\right)c_i^Z\right) - \mathbb{E}\alpha_i U\left(c_i^Z\right)$. The analogue to $g(\sigma)$ is

$$\mathcal{W}^j(\sigma) = \mathbb{E}\alpha_i U\left(\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)\left(1+\sigma\varepsilon_i^j\right)c_i^Z\right).$$

To apply (21) and (22), we set $x = (0, 0, \mathbf{0})$ and $h = \left(\Gamma^j, \Delta^j, \varepsilon^j\right)$. Applying (22) and (23) we get

$$\begin{aligned} \mathcal{W}^j - \mathcal{W}^Z &= \mathbb{E}\alpha_i U_{c,i} c_i^Z\left(\Gamma^j + \Delta_i^j\right) + \frac{1}{2}\mathbb{E}\alpha_i U_{c,i} c_i^Z\left(\Gamma^j + \Delta_i^j\right)^2 \\ &\quad + \frac{1}{2}\mathbb{E}\alpha_i U_{cc,i} \times \left(c_i^Z\right)^2\left[\left(\Gamma^j + \Delta_i^j\right)^2 + \left(\varepsilon_i^j\right)^2\right] + o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right) \\ &\simeq \mathbb{E}\alpha_i U_{c,i} c_i^Z\left(\Gamma^j + \Delta_i^j\right) + \frac{1}{2}\mathbb{E}\alpha_i U_{cc,i} \times \left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right)^2 + \text{t.i.p.} \end{aligned} \tag{26}$$

Here the expression in the last line is obtained by applying the LIE and dropping $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ terms from the first expression on the right side of (26).

Substitute (26) into (25) to get

$$\mathcal{W}^B - \mathcal{W}^A \simeq \mathbb{E}\underbrace{\alpha_i U_{c,i} c_i^Z}_{\equiv\phi_i}\Gamma + \mathbb{E}\underbrace{\alpha_i U_{c,i} c_i^Z}_{\equiv\phi_i}\Delta_i + \frac{1}{2}\mathbb{E}\underbrace{\alpha_i U_{c,i} c_i^Z}_{\equiv\phi_i}\underbrace{\frac{U_{cc,i} c_i^Z}{U_{c,i}}}_{\equiv -\gamma_i}\left[\mathbb{E}_i\left(\varepsilon_i^B\right)^2 - \mathbb{E}_i\left(\varepsilon_i^A\right)^2\right]. \tag{27}$$

Finally, observe that

$$var_i\left(\ln c_i^j\right)=\mathbb{E}_i\left(\ln\left(1+\varepsilon_i^j\right)-\mathbb{E}_i\ln\left(1+\varepsilon_i^j\right)\right)^2\simeq\mathbb{E}_i\left(\varepsilon_i^j\right)^2. \tag{28}$$

Substitute this relationship and the definitions of γ_i, ϕ_i into (27) to obtain (4).

We now verify equation (7). As a preliminary step, observe that

$$\frac{w_i^j}{w_i^Z}=\exp\left(\Delta_i^j\right)\simeq\Delta_i^j+\text{t.i.p.}=\ln\frac{w_i^j}{w_i^Z}+\text{t.i.p.} \tag{29}$$

Therefore,

$$\begin{aligned}\text{redistribution} &= \sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i}\ln\frac{w_i^B}{w_i^A}\sqrt{w_i^A w_i^B}=\sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i}\left(\ln\frac{w_i^B}{w_i^Z}-\ln\frac{w_i^A}{w_i^Z}\right)w_i^Z\\ &\simeq \sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i}\left(\frac{w_i^B}{w_i^Z}-\frac{w_i^A}{w_i^Z}\right)w_i^Z=\sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i}\left(w_i^B-w_i^A\right),\end{aligned}$$

which verifies equation (7).

A.4 Details for section 3.1.1

We can write the welfare difference as

$$\mathcal{W}^B-\mathcal{W}^A=\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]+R,$$

where R is a residual in decomposition (4). Let $R_\Gamma, R_\Delta, R_\Lambda$ with $R_\Gamma+R_\Delta+R_\Lambda=R$ be parts of R attributed to the aggregate efficiency, redistribution, and insurance components so that the "true" contribution, for example, of aggregate efficiency component is $\frac{\mathbb{E}\phi_i\Gamma+R_\Gamma}{\mathcal{W}^B-\mathcal{W}^A}$. We have

$$\frac{\mathbb{E}\phi_i\Gamma+R_\Gamma}{\mathcal{W}^B-\mathcal{W}^A}=\frac{\mathbb{E}\phi_i\Gamma+R_\Gamma}{\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]+R}=\frac{\mathbb{E}\phi_i\Gamma}{\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]}\times\frac{1+\frac{R_\Gamma}{\mathbb{E}\phi_i\Gamma}}{1+\frac{R}{\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]}}.$$

Therefore, the aggregate efficiency under the true decomposition and the one given in equation (4) coincide if the second term in the second equality is equal to 1 or, equivalently,

$$\frac{R_\Gamma}{R}=\frac{\mathbb{E}\phi_i\Gamma}{\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]}.$$

Analogous arguments apply to the redistribution and insurance components.

The third-order residual in decomposition (4) can be written as

$$R\left(\left(\ln C^Z,\ln w^Z,\mathbf{0}\right),\left(\ln C^B,\ln w^B,\varepsilon^B\right)\right)-R\left(\left(\ln C^Z,\ln w^Z,\mathbf{0}\right),\left(C^A,w^A,\varepsilon^A\right)\right).$$

It converges to zero as $||c^B-c^A||\to 0$. The speed of this convergence is

$$\max\left\{o\left(||\Gamma^A,\Delta^A,\varepsilon^A||^2\right),o\left(||\Gamma^B,\Delta^B,\varepsilon^B||^2\right)\right\}.$$

Since $\Gamma^B = -\Gamma^A = \frac{1}{2}\Gamma$ and $\Delta_i^B = -\Delta_i^A = \frac{1}{2}\Delta_i$, using the properties of a norm, we have

$$\left\|\Gamma^A, \Delta^A, \varepsilon^A\right\|^2 = \left\|-\frac{1}{2}\Gamma, -\frac{1}{2}\Delta, \varepsilon^A\right\|^2 = \frac{1}{4}\left\|\Gamma, \Delta, -2\varepsilon^A\right\|^2,$$

and $o\left(\frac{1}{4}||\Gamma, \Delta, -2\varepsilon^A||^2\right) = o\left(||\Gamma, \Delta, -2\varepsilon^A||^2\right)$. This and the analogous argument for $||\Gamma^B, \Delta^B, \varepsilon^B||^2$ provide the approximation errors in decomposition (4).

The construction of composition (3.1.1) is identical to that of (4), with appropriate modification of the space X to consist of sequences $\left\{\tilde{\Gamma}, \tilde{\Delta}_i, \tilde{\delta}_i\right\}_i$, but now $\delta_i^B(\xi) = -\delta_i^A(\xi) = \frac{1}{2}\delta_i(\xi)$ and, therefore,

$$\left\|\Gamma^A, \Delta^A, \varepsilon^A\right\|^2 = \left\|\Gamma^B, \Delta^B, \varepsilon^B\right\|^2 = \frac{1}{4}\left\|\Gamma, \Delta, \varepsilon\right\|^2,$$

and $\max\left\{o\left(||\Gamma^A, \Delta^A, \delta^A||^2\right), o\left(||\Gamma^B, \Delta^B, \delta^B||^2\right)\right\} = o\left(\|\Gamma, \Delta, \delta\|^2\right)$.

A.5 Derivations of equation (9)

As in section A.3, we define $\Gamma_k^j \equiv \ln C_k^j - \ln C_k^Z$ and $\Delta_{k,i}^j \equiv \ln w_{k,i}^j - \ln w_{k,i}^Z$, and observe that they satisfy $\Gamma_k^A = -\Gamma_k^B$, $\Delta_{k,i}^A = -\Delta_{k,i}^B$ and, therefore,

$$\Gamma_k^j \Gamma_m^j, \Gamma_k^j \Delta_{m,i}^j, \Delta_{k,i}^j \Delta_{m,i}^j \text{ are t.i.p. for all } k, m, i.$$

We have

$$\mathcal{W}^j(\sigma) = \mathbb{E}\alpha_i U\left(\left\{\exp\left(\sigma\left(\Gamma_k^j + \Delta_{k.i}^j\right)\right)\left(1 + \sigma\varepsilon_{k,i}^j\right) c_{k,i}^Z\right\}_k\right).$$

Therefore, its second-order expansion can be written, using the LIE, as

$$\mathcal{W}^j(1) - \mathcal{W}^j(0) \simeq \mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z \left(\Gamma^j + \Delta_{k,i}^j\right) + \frac{1}{2}\mathbb{E}\sum_k\sum_m \alpha_i U_{km,i} c_{k,i}^Z c_{m,i}^Z \mathbb{E}_i\left(\varepsilon_{k,i}^j \varepsilon_{m,i}^j\right) + \text{t.i.p.}$$

Since $\mathcal{W}^j = \mathcal{W}^j(1)$, we have

$$\begin{aligned}\mathcal{W}^B - \mathcal{W}^A \quad \simeq \quad & \mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z \Gamma_k + \mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z \Delta_{k,i} \\ & + \frac{1}{2}\mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z \sum_m \frac{U_{km,i} c_{m,i}^Z}{U_{k,i}}\left[\mathbb{E}_i\left(\varepsilon_{k,i}^B \varepsilon_{m,i}^B\right) - \mathbb{E}_i\left(\varepsilon_{k,i}^A \varepsilon_{m,i}^A\right)\right].\end{aligned}$$

Use the approximation

$$cov_i\left(\ln c_{k,i}^j, \ln c_{m,i}^j\right) \simeq \mathbb{E}_i\left(\varepsilon_{k,i}^j \varepsilon_{m,i}^j\right),$$

together with the definitions of $\phi_{k,i}, \gamma_{km,i}$, we obtain (9).

A.6 Proofs for section 4.1

Properties of normal distributions imply that

$$\begin{aligned}
\mathbb{E}_i \exp\left((1-\tau)\,\xi_{t,i} + \tau(1-\tau)\frac{\upsilon_{\xi,t}^2}{2}\right) &= \mathbb{E}\exp\left((1-\tau)\,e_i + \tau(1-\tau)\frac{\upsilon_e^2}{2}\right) = 1, \\
\mathbb{E}\ln\left\{\exp\left((1-\tau)\,e_i + \tau(1-\tau)\frac{\upsilon_e^2}{2}\right)\right\} &= -(1-\tau)^2\frac{\upsilon_e^2}{2}, \\
\mathbb{E}\ln\left\{\exp\left((1-\tau)\,\xi_{t,i} + \tau(1-\tau)\frac{\upsilon_{\xi,t}^2}{2}\right)\right\} &= -(1-\tau)^2\frac{\upsilon_{\xi,t}^2}{2}.
\end{aligned}$$

Welfare in period t is $\mathcal{W}_t(\tau) = \mathbb{E}\ln c_{t,i}(\tau) - \frac{1}{1-\eta}L(\tau)^{1-\eta}$. Substitute (10) and the expressions above into it to obtain (11). Since utility is separable, we can apply our decomposition separately for each good. Consider good c first. From (10) and properties of normal distributions, we have

$$w_{c,i} = \exp\left((1-\tau)\,e_i + \tau(1-\tau)\frac{\upsilon_e^2}{2}\right), \quad 1+\varepsilon_{c,i} = \exp\left((1-\tau)\,\xi_{t,i} + \tau(1-\tau)\frac{\upsilon_{\xi,t}^2}{2}\right).$$

Therefore,

$$\begin{aligned}
\Gamma_c &= \ln C\left(\tau^B\right) - \ln C\left(\tau^A\right), \\
\Delta_{c,i} &= \left[\left(1-\tau^B\right) - \left(1-\tau^A\right)\right] e_i + \left[\tau^B\left(1-\tau^B\right) - \tau^A\left(1-\tau^A\right)\right]\frac{\upsilon_e^2}{2}, \\
\Lambda_{c,i} &= -\left[\left(1-\tau^B\right)^2 - \left(1-\tau^A\right)^2\right]\frac{\upsilon_{\xi,t}^2}{2}.
\end{aligned}$$

Given logarithmic preferences and utilitarian weights, we have $\phi_{c,i} = \gamma_{c,i} = 1$ and therefore

$$\begin{aligned}
\text{agg. efficiency}_c &= \ln C\left(\tau^B\right) - \ln C\left(\tau^A\right), \\
\text{redistribution}_c &= \mathbb{E}\Delta_{c,i} = -\left[\left(1-\tau^B\right)^2 - \left(1-\tau^A\right)^2\right]\frac{\upsilon_{e,t}^2}{2}, \\
\text{insurance}_c &= \mathbb{E}\Lambda_{c,i} = -\left[\left(1-\tau^B\right)^2 - \left(1-\tau^A\right)^2\right]\frac{\upsilon_{\xi,t}^2}{2}.
\end{aligned}$$

We now apply this decomposition to labor. Since there is no heterogeneity in hours, we immediately have $\Delta_{l,i} = \Lambda_{l,i} = 0$, $l_i^Z = \sqrt{L\left(\tau^A\right)L\left(\tau^B\right)} = \left[\left(1-\tau^A\right)\left(1-\tau^B\right)\right]^{1/2(1+\eta)}$, and $\phi_{l,i} = \left[\left(1-\tau^A\right)\left(1-\tau^B\right)\right]^{1/2}$. This gives

$$\begin{aligned}
\text{agg. efficiency}_l &= -\sqrt{\left(1-\tau^A\right)\left(1-\tau^B\right)}\left(\ln L\left(\tau^B\right) - \ln L\left(\tau^A\right)\right), \\
\text{redistribution}_l &= \text{insurance}_l = 0.
\end{aligned}$$

Combine the decompositions for consumption and labor to get (12).

Finally, consider the residual term in equation (13). Let $1+e \equiv \ln \frac{1-\tau^B}{1-\tau^A}$. We have

$$\frac{\left(1-\tau^B\right)-\left(1-\tau^A\right)}{1+\eta} = \frac{1-\tau^A}{1+\eta}\left[\exp e - 1\right] = \frac{1-\tau^A}{1+\eta}\left[e+\frac{1}{2}e^2+O\left(e^3\right)\right]$$

and

$$\sqrt{\left(1-\tau^A\right)\left(1-\tau^B\right)}\frac{\ln\left(1-\tau^B\right)-\ln\left(1-\tau^A\right)}{1+\eta} = \frac{1-\tau^A}{1+\eta}\exp\left(\frac{1}{2}e\right)e = \frac{1-\tau^A}{1+\eta}\left[e+\frac{1}{2}e^2+O\left(e^3\right)\right].$$

This implies that $R = O\left(e^3\right)$.

A.7 Proofs for section 5

A.7.1 Derivations of equation (17)

Floden derives (17) for the case of utilitarian planner. We show here that it holds more generally. We have

$$\begin{aligned}
\mathbb{E}\alpha_i U\left(c_i^B\right) &= \mathbb{E}\alpha_i U\left(c_i^{ce,B}\right) = \left(1-p_{redis}^B\right)^{1-\gamma}U\left(C^{ce,B}\right) = \left(1-p_{redis}^B\right)^{1-\gamma}\left(1-p_{insur}^B\right)^{1-\gamma}U\left(C^B\right) \\
&= \left(1-p_{redis}^B\right)^{1-\gamma}\left(1-p_{insur}^B\right)^{1-\gamma}\left(1+\omega_{eff}\right)^{1-\gamma}U\left(C^A\right)
\end{aligned}$$

and

$$\mathbb{E}\alpha_i U\left(\left(1+\omega\right)c_i^A\right) = \left(1+\omega\right)^{1-\gamma}\left(1-p_{redis}^A\right)^{1-\gamma}\left(1-p_{insur}^A\right)^{1-\gamma}U\left(C^A\right).$$

Therefore

$$\left(1+\omega\right)^{1-\gamma} = \left(1+\omega_{eff}\right)^{1-\gamma}\left(\frac{1-p_{redis}^B}{1-p_{redis}^A}\right)^{1-\gamma}\left(\frac{1-p_{insur}^B}{1-p_{insur}^A}\right)^{1-\gamma},$$

or

$$\left(1+\omega\right) = \left(1+\omega_{eff}\right)\left(1+\omega_{redis}\right)\left(1+\omega_{insur}\right).$$

Take logs to get (17).

A.7.2 Derivations of equations in section 5.2

Most of the expressions used in this section are special cases of the more general formulas that we derived to prove lemma 2. In particular, the approximation for $c_i^{ce,j}$ used in equation (18) is a special case of the expression provided in claim 1 shown below, adapted to policies c_i^j in the example constructed in section 5.2. Similarly, the approximations for $\ln\left(1+\omega_{insur}\right)$ and $\ln\left(1+\omega\right)$ follow from claims 5 and 6.

A.7.3 Proof of Lemma 2

We focus in the proof only on Floden decomposition, since the proof for Benabou decomposition follows the same steps but is simpler.

It is easy to see that generically Properties a, b, and c will be violated in the Floden decomposition. Consider, for example, Property a. Take any allocation $\left\{c_i^A\right\}_i$, where consumptions of households are non-trivial stochastic process, and construct $\left\{c_i^B\right\}_i$ by $c_i^B = Dc_i^A$ for all i for some $D > 0$. Property a is satisfied if all welfare changes from this policy are attributed to the aggregate efficiency component. This would require that $1 - p_{insur}^B = 1 - p_{insur}^A$. Since $C^B = DC^A$, this would be the case only if $C^{ce,B} = DC^{ce,A}$. But for arbitrary U function, there is no reason to expect that $\mathbb{E}c_i^{ce}$ scales with D under policy B. On the other hand, if U is CRRA, using equation (14) it is easy to verify that $c_i^{ce,B} = Dc_i^{ce,A}$ for all i, and therefore $C^B = DC^A$. Failure of other properties follow from analogous arguments.

We now prove the second part of the lemma. As a first step, we want to characterize approximations of consumption certainty equivalent $c_i^{ce,j}$. Define a function $c_i^{ce,j}(\sigma)$ by

$$U\left(c_i^{ce,j}(\sigma)\right) = \mathbb{E}_i U\left(\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)\left(1 + \sigma\varepsilon_i^j\right)c_i^Z\right). \tag{30}$$

While $c_i^{ce,j}(1) = c_i^{ce,j}$, it is more convenient to work with an arbitrary σ first. We will prove several intermediate claims first about $c_i^{ce,j}(\sigma)$ and $C_i^{ce,j}(\sigma) = \mathbb{E}c_i^{ce,j}(\sigma)$. Throughout these proofs, unless noted otherwise, U, U_c have arguments $\mathbb{E}c_i^Z$, while $U_{c,i}, U_{cc,i}$ have arguments c_i^Z.

Claim 1 $c_i^{ce,j}(\sigma) = c_i^Z + \sigma c_i^Z\left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2}c_i^Z\left[\left(\Gamma^j + \Delta_i^j\right)^2 - var_i\left(\varepsilon_i^j\right)\right] + o\left(\sigma^2\right).$

Proof. The right and left sides of (30), respectively, are

$$\begin{aligned} RHS(\sigma) &= U_i + \sigma U_{c,i}c_i^Z\left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2}U_{cc,i} \times \left(c_i^Z\right)^2\left[\left(\Gamma^j + \Delta_i^j\right)^2 + var_i\left(\varepsilon_i^j\right)\right] \\ &\quad + \frac{\sigma^2}{2}U_{c,i}c_i^Z\left(\Gamma^j + \Delta_i^j\right)^2 + o\left(\sigma^2\right), \end{aligned}$$

and

$$LHS(\sigma) = U\left(\bar{c}_i^{ce,j}\right) + \sigma U_c\left(\bar{c}_i^{ce,j}\right)c_{i,\sigma}^{ce,j} + \frac{\sigma^2}{2}\left[U_{cc}\left(\bar{c}_i^{ce,j}\right)\left(c_{i,\sigma}^{ce,j}\right)^2 + U_c\left(\bar{c}_i^{ce,j}\right) \times c_{i,\sigma\sigma}^{ce,j}\right] + o\left(\sigma^2\right),$$

where $\bar{c}_i^{ce,j} \equiv c_i^{ce,j}(0)$, $c_{i,\sigma}^{ce,j} \equiv \left.\frac{\partial c_i^{ce,j}(\sigma)}{\partial\sigma}\right|_{\sigma=0}$, $c_{i,\sigma\sigma}^{ce,j} \equiv \left.\frac{\partial^2 c_i^{ce,j}(\sigma)}{\partial\sigma^2}\right|_{\sigma=0}$. Since $RHS(\sigma) = LHS(\sigma)$ for all σ, it follows that

$$c_{i,0}^{ce,j} = c_i^Z, \; c_{i,\sigma}^{ce,j} = c_i^Z\left(\Gamma^j + \Delta_i^j\right), \quad c_{i,\sigma\sigma}^{ce,j} = \frac{U_{cc,i}}{U_{c,i}}\left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right) + c_i^Z\left(\Gamma^j + \Delta_i^j\right)^2.$$

We have

$$c_i^{ce,j}(\sigma) = c_{i,0}^{ce,j} + \sigma c_{i,\sigma}^{ce,j} + \frac{\sigma^2}{2} c_{i,\sigma\sigma}^{ce,j} + o(\sigma^2).$$

Substitute previous expression and use the fact that U is CRRA to prove the claim. ■

Claim 2 $U\left(C^{ce,j}(\sigma)\right) = \sigma \frac{\mathbb{E} c_i^Z \left(\Gamma^j + \Delta_i^j\right)}{\mathbb{E} c_i^Z} - \frac{\sigma^2}{2} \gamma \frac{\mathbb{E} c_i^Z var_i\left(\varepsilon_i^j\right)}{\mathbb{E} c_i^Z} + t.i.p. + o(\sigma^2).$

Proof. By claim 1,

$$\begin{aligned} U\left(C^{ce,j}(\sigma)\right) &= U + \sigma U_c \times \mathbb{E} c_i^Z \left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2} U_{cc} \times \left(\mathbb{E} c_i^Z \left(\Gamma^j + \Delta_i^j\right)\right)^2 \\ &\quad + \frac{\sigma^2}{2} U_c \times \left[\mathbb{E} \frac{U_{cc,i}}{U_{c,i}} \left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right) + \mathbb{E} c_i^Z \left(\Gamma^j + \Delta_i^j\right)^2\right] + o(\sigma^2) \\ &= \sigma U_c \mathbb{E} c_i^Z \left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2} U_c \mathbb{E} \frac{U_{cc,i}}{U_{c,i}} \left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right) + \text{t.i.p.} + o(\sigma^2). \end{aligned} \tag{31}$$

Use the fact that U is CRRA and evaluate this expression at $\sigma = 1$ to prove the claim. ■

Claim 3 *Let $x(\sigma)$ be a twice differentiable function, with $\bar{x} = x(0)$, $x_\sigma = x'(0)$, $x_{\sigma\sigma} = x''(0)$, so that*

$$x(\sigma) = \bar{x} + \sigma x_\sigma + \frac{\sigma^2}{2} x_{\sigma\sigma} + o(\sigma^2).$$

Then

$$\ln x(\sigma) = \ln \bar{x} + \sigma \frac{x_\sigma}{\bar{x}} + \frac{\sigma^2}{2}\left[\frac{x_{\sigma\sigma}}{\bar{x}} - \left(\frac{x_\sigma}{\bar{x}}\right)^2\right] + o(\sigma^2). \tag{32}$$

Proof. This follows from a routine application of a Taylor expansion of $\ln x(\sigma)$ around $\ln x(0)$. ■

Claim 4

$$\begin{aligned} \ln \mathcal{W}^j &\simeq (1-\gamma) \frac{\mathbb{E}\alpha_i \left(c_i^Z\right)^{1-\gamma}\left(\Gamma^j + \Delta_i^j\right) - \frac{\gamma}{2} \mathbb{E}\alpha_i \left(c_i^Z\right)^{1-\gamma} var_i\left(\varepsilon_i^j\right)}{\mathbb{E}\alpha_i \left(c_i^Z\right)^{1-\gamma}} + t.i.p., \\ \ln U\left(C^{ce,j}\right) &\simeq (1-\gamma) \frac{\mathbb{E} c_i^Z \left(\Gamma^j + \Delta_i^j\right) - \frac{\gamma}{2} \mathbb{E} c_i^Z var_i\left(\varepsilon_i^j\right)}{\mathbb{E} c_i^Z} + t.i.p., \\ \ln U\left(C^j\right) &\simeq (1-\gamma) \frac{\mathbb{E} c_i^Z \left(\Gamma^j + \Delta_i^j\right)}{\mathbb{E} c_i^Z} + t.i.p. \end{aligned}$$

Proof. (*first equation*). From (26), we can write

$$\begin{aligned} \mathcal{W}^j(\sigma) &= \mathcal{W}^Z + \sigma \mathbb{E}\alpha_i U_{c,i} c_i^Z \left(\Gamma^j + \Delta_i^j\right) \\ &\quad + \frac{\sigma^2}{2}\left\{\mathbb{E}\alpha_i U_{c,i} c_i^Z \left(\Gamma^j + \Delta_i^j\right)^2 + \mathbb{E}\alpha_i U_{cc,i} \times \left(c_i^Z\right)^2 \left[\left(\Gamma^j + \Delta_i^j\right)^2 + \left(\varepsilon_i^j\right)^2\right]\right\} + o(\sigma^2). \end{aligned}$$

Apply claim 3 to get

$$\ln \mathcal{W}^j(\sigma) = \sigma \frac{\mathbb{E}\alpha_i U_{c,i} c_i^Z \left(\Gamma^j + \Delta_i^j\right)}{\mathcal{W}^Z} + \frac{\sigma^2}{2} \frac{\mathbb{E}\alpha_i U_{cc,i} \times \left(c_i^Z\right)^2 \left(\varepsilon_i^j\right)^2}{\mathcal{W}^Z} + t.i.p. + o\left(\sigma^2\right).$$

Use the fact that U is CRRA, $\mathcal{W}^Z = (1-\gamma)^{-1} \mathbb{E}\alpha_i \left(c_i^Z\right)^{1-\gamma}$, and that $o\left(\sigma^2\right)$ at $\sigma = 1$ is of order $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ by equation (23) to show the first equation of the claim.

(*second equation*). Combine claims 2 and 3 to get

$$\ln U\left(C^{ce,j}(\sigma)\right) = \sigma \frac{U_c \times \mathbb{E}c_i^Z \left(\Gamma^j + \Delta_i^j\right)}{U} + \frac{\sigma^2}{2} \frac{U_c \times \left[\mathbb{E}\frac{U_{cc,i}}{U_{c,i}} \left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right)\right]}{U} + t.i.p. + o\left(\sigma^2\right).$$

Use the fact that U is CRRA and that $o\left(\sigma^2\right)$ at $\sigma = 1$ is of order $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ by equation (23), to show the second equation of the claim.

(*third equation*). Let $c_i^j(\sigma) \equiv \exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)\left(1 + \sigma\varepsilon_i^j\right) c_i^Z$, $C^j(\sigma) \equiv \mathbb{E}c_i^j(\sigma) = \mathbb{E}\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right) c_i^Z$, where the last equation follows due to the LIE. Therefore,

$$\begin{aligned} U\left(C^j(\sigma)\right) &= U\left(\mathbb{E}\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right) c_i^Z\right) \\ &= U + \sigma U_c \mathbb{E}c_i^Z \left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2}\left\{U_{cc}\left[\mathbb{E}c_i^Z\left(\Gamma^j + \Delta_i^j\right)\right]^2 + U_c\mathbb{E}\left[c_i^Z\left(\Gamma^j + \Delta_i^j\right)\right]^2\right\} + o\left(\sigma^2\right). \end{aligned}$$

Apply claim 3 to get

$$\ln U\left(C^j(\sigma)\right) = \sigma \frac{U_c \mathbb{E}c_i^Z \left(\Gamma^j + \Delta_i^j\right)}{U} + t.i.p. + o\left(\sigma^2\right).$$

Use the fact that U is CRRA and that $o\left(\sigma^2\right)$ at $\sigma = 1$ is of order $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ by equation (23), to show the third equation of the claim. ■

Claim 5 $\ln(1+\omega) \simeq \frac{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i) + \frac{\gamma}{2}\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}\left[var_i\left(\varepsilon_i^A\right) - var_i\left(\varepsilon_i^B\right)\right]}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}}$.

Proof. From equation (16), the term $\ln(1+\omega)$ in Floden decomposition satisfies

$$(1-\gamma)\ln(1+\omega) = \ln \mathcal{W}^B - \ln \mathcal{W}^A. \tag{33}$$

Substitute the first equation from claim 4 to prove this claim. ■

Claim 6 $\ln(1+\omega_{insur}) \simeq \frac{\gamma}{2} \frac{\mathbb{E}c_i^Z\left[var_i\left(\varepsilon_i^A\right) - var_i\left(\varepsilon_i^B\right)\right]}{\mathbb{E}c_i^Z}$.

Proof. Using its definition, observe that $\ln(1+\omega_{insur})$ can written as

$$(1-\gamma)\ln(1+\omega_{insur}) = \left[\ln U\left(C^{ce,B}\right) - \ln U\left(C^{ce,A}\right)\right] - \left[\ln U\left(C^B\right) - \ln U\left(C^A\right)\right].$$

Apply the second and third equations from claim 4 and simplify. ■

Claim 7 $\ln(1+\omega_{eff}) = \Gamma$.

Proof. This follows from the definitions of $1+\omega_{eff}$ and Γ. ■

With these claims we can now prove the second part of the lemma. Suppose that condition (20) is satisfied. Then $2\left[var_i\left(\varepsilon_i^A\right) - var_i\left(\varepsilon_i^B\right)\right] \simeq \Lambda$ for all i and therefore

$$\begin{aligned}
\frac{\ln(1+\omega_{insur})}{\ln(1+\omega)} &= \frac{\gamma\Lambda\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i+\gamma\Lambda)} + o(1), \\
\frac{\ln(1+\omega_{eff})}{\ln(1+\omega)} &= \frac{\Gamma\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i+\gamma\Lambda)} + o(1),
\end{aligned}$$

and, since equation (17) holds,

$$\frac{\ln(1+\omega_{redis})}{\ln(1+\omega)} = \frac{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}\Delta_i}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i+\gamma\Lambda)} + o(1).$$

The first terms on the right sides of these equations are the very same terms that we obtained using our decomposition (6) under the assumptions of the lemma. Thus, the two decompositions coincide up to $o(1)$, meaning that $o(1) \to 0$ as $\|\Gamma, \Delta, \varepsilon\| \to 0$. Since our decomposition satisfies Properties a, b, and c, so does Floden's, to the order $o(1)$.