Debt as a Blessing: A Capital Screening Mechanism*

Feng Dong† Thomas J. Sargent‡ Pengfei Wang§ Yizhen Wang¶

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Abstract

We challenge a recently popular view that a negative interest-growth rate gap ($r < g$) offers a "free lunch" for debt-financed government spending by formulating a model in which r and g are endogenous variables shaped by fiscal policy through its effects on equilibrium multiplicity and capital allocation. Observing $r < g$ can signal that sustained government deficits have generated multiple steady states, and the economy has converged to a stable low-efficiency equilibrium. With its heterogeneous entrepreneurs, the model's real interest rate serves as a screening device for investment efficiency. Causation runs from the fiscal regime to equilibrium selection and outcomes: fiscal surpluses eliminate equilibrium multiplicity and anchor expectations that sustain a unique, high-productivity equilibrium, thereby rationalizing Alexander Hamilton's characterization of "debt as a blessing." Persistent deficits can push the economy into a "misallocation trap" characterized by scarce safe assets, low interest rates, survival of inefficient firms, depressed aggregate productivity, and self-validating low growth. Thus, costs of debt-financed fiscal deficits consist not only of deferred taxes, but also of permanently lower national productive capacity.

Keywords: $r < g$; government debt; heterogeneous agents; capital misallocation; fiscal policy
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†School of Economics and Management, Tsinghua University. Email: dongfeng@sem.tsinghua.edu.cn.
‡New York University. Email: thomas.sargent@nyu.edu. Corresponding author.
§Peking University HSBC Business School. Email: pfwang@phbs.pku.edu.cn.
¶School of Finance, Central University of Finance and Economics. Email: wyz_thu@foxmail.com.

1 Introduction

Disagreements about whether government debt is a blessing or a curse are longstanding. Historians including Dorfman (1946) and McDonald (1985) have documented that, in arranging a fiscal framework for the United States, the Founders confronted a tension between Steuart (1767)'s view that public debt could mobilize capital and warnings from Hume (1752) and Smith (1776) about the adverse effects of large public debts. Alexander Hamilton, the first U.S. Treasury Secretary, balanced these considerations when he concluded that a national debt, "if it is not excessive, will be to us a national blessing" (Hamilton, 1790). This paper reevaluates Hamilton's recommendation in light of a modern macro-finance model.

The mechanical intertemporal government budget arithmetic used by Darby (1984) and others indicates that real interest rates (r) that fall below economic growth rates (g) expand a government's "fiscal space" by making risk-free bond-financed persistent deficits a cheap source of public funding (also see Blanchard (2019, 2024); Mian et al. (2025)). One form of that claim takes r and g as exogenous parameters that are not affected by alternative government debt policies being evaluated (for example, see Darby (1984)), a view criticized by Leeper (2024).

Our paper shares with Miller and Sargent (1984) the view that r and g should be treated as endogenous. We model g, the growth rate of aggregate capital, in a way that makes it depend partly on the fiscal deficit, through the deficit's effect on average capital productivity $\tilde{y}$ (the output–capital ratio). We thereby move beyond the concern expressed by Diamond (1965) that government debt might crowd out the *quantity* of private physical capital, to how it can degrade allocative efficiency and hence the aggregate *quality* of capital.

A striking feature of our model is that chronic primary deficits are associated with low real interest rates and low debt, while surpluses are associated with high interest rates and high debt. The mechanism operates through an **interest-rate screening channel**: to absorb more government bonds, asset-market clearing requires a higher investment threshold z^*, which screens out the least productive projects. A higher threshold raises the bar for all entrepreneurs, excluding the least productive projects within each regime and raising aggregate capital productivity $\tilde{y}$. Higher productivity increases both r and g, but r rises faster because the no-arbitrage condition ties r directly to the threshold z^*, while g responds more slowly through the resource constraint and is partially offset by higher debt-service drag. Hence, a higher level of government debt implies a larger interest-growth gap $r - g$. The steady-state government budget constraint then selects among these equilibria via the required sign of $r - g$: a primary surplus requires $r > g$, pinning down a unique high-debt, high-productivity equilibrium. In contrast, chronic primary deficits allow for multiple steady states; the economy converges to the unique stable equilibrium — a misallocation trap characterized by scarce safe assets, suppressed interest rates, and widespread inefficient investment. This capital misallocation degrades aggregate productivity, depressing both growth and interest rates. Thus, deficits do not directly cause low interest rates; rather, persistent deficits make a low-r, low-g

trap the only stable outcome. The observed $r < g$ is therefore a symptom of fiscal profligacy, not an opportunity for cheap borrowing.

In this light, we modify the view—formalized in Barro (2023)—of "modern monetary theory" that $r < g$ makes debt-financed deficit spending inexpensive. We raise the possibility of a "reverse causality" in which observing an $r < g$ condition indicates that the economy is trapped in an equilibrium in which low interest rates and low growth rates are both consequences of excessive government deficits.

Our work complements but differs from Reis (2021), who analyzes a financial "bubble premium" constraint on government debt even when $r < g < m$, where m denotes the marginal product of capital. Although Reis focuses on the gap between m and r as a necessary condition for sustainable bubbles, we model how fiscal policy endogenously determines m. We show that persistent deficits can reduce m by eroding productive capacity, thereby shrinking the available fiscal space. In our model, just as Hamilton (1790) said, a well-managed and safe public debt helps allocate capital to more productive industrial and commercial prospects, while a profligate fiscal policy degrades that allocation.[1]

In this sense, the familiar "fiscal space" logic associated with $r < g$ does not imply that persistent deficits are harmless. Sustaining deficits in equilibrium requires continued rollover of safe public debt, but in our model that very policy suppresses the interest-rate screening channel, degrades capital allocation efficiency, and permanently lowers productive capacity.

2 Model Structure

We consider an economy with a continuum of heterogeneous entrepreneurs. Each entrepreneur operates a private firm with linear production technology and faces an exogenous leverage constraint. We adopt a discrete-time formulation to make explicit the information timing of the model: at the start of period t, an entrepreneur's persistent regime $\mathcal{A}^j_{t-\Delta}$ and idiosyncratic shock $z^j_{t-\Delta}$ are already realized and observed before the capital choice k^j_t is made. We then take the continuous-time limit $\Delta \to 0$ to obtain the equilibrium ODE system used in the analysis below.

2.1 Preferences and Technology

Time is indexed by $t = 0, \Delta, 2\Delta, \ldots$, with discount factor $e^{-\rho\Delta}$. Entrepreneurs maximize expected lifetime utility with logarithmic preferences:

$$\sum_t e^{-\rho t} \log(c^j_t)\Delta,$$

[1]Hamilton seems to have been influenced by the measured advocacy of Steuart (1767) for commercial and industrial social benefits that would flow from a fully-serviced public debt, while Thomas Jefferson was more persuaded by warnings from Hume (1752) and Smith (1776) about the adverse consequences of large public debts. The $r - g$ budget arithmetic underlying Blanchard (2019, 2024) did not appear in the Hamilton-Jefferson controversy. See (Dorfman, 1946, ch. XVII) and (McDonald, 1985, ch. 4).

where c_t^j is consumption. The production of entrepreneur j is given by

$$y_t^j \Delta = \mathcal{A}_{t-\Delta}^j z_{t-\Delta}^j k_t^j \Delta.$$

Here $\mathcal{A}_t^j \in \{A_H, A_L\}$ is a persistent productivity regime (high or low) following a continuous-time two-state Markov chain: in any short interval Δ, an L-type switches to H with probability $\eta\Delta$ and an H-type switches to L with probability $\chi\eta\Delta$ ($\eta > 0$, $\chi > 0$), yielding unconditional population shares $\pi_H = 1/(1+\chi)$ and $\pi_L = \chi/(1+\chi)$, so η controls the turnover speed while χ pins down the steady-state composition. The idiosyncratic shock z_t^j is drawn independently from a continuous distribution $F(z)$ with finite mean, and k_t^j is capital chosen at t given the period-t information set $\{\mathcal{A}_{t-\Delta}^j, z_{t-\Delta}^j\}$.

2.2 Individual Investment Decisions

Entrepreneurs allocate wealth w_t^j between capital k_t^j and real government bonds b_t^j, subject to a leverage constraint λ. The real return on capital net of tax τ and depreciation δ is $(1-\tau)\mathcal{A}_{t-\Delta}^j z_{t-\Delta}^j - \delta$; the real risk-free rate is r.

Investment is profitable only when the expected return exceeds the risk-free rate. This defines a productivity threshold for each regime. For high-productivity entrepreneurs, the threshold z_H^* satisfies

$$(1-\tau)A_H z_H^* - \delta = r.$$

For low-productivity entrepreneurs, the corresponding threshold is $z_L^* = (A_H/A_L)z_H^*$. Entrepreneurs with $z \geq z^*$ invest all their wealth in capital (leveraging up to $1+\lambda$), while those with $z < z^*$ hold only bonds.

2.3 Aggregate Dynamics

The aggregate state consists of two variables: the government debt-to-capital ratio $\tilde{b}_t = B_t/K_t$ and the wealth share Σ_t of entrepreneurs in the high-productivity regime.

The law of motion for $\tilde{b}_t$ is derived from the government budget constraint and market clearing:

$$\dot{\tilde{b}}_t = \left[\xi - \tau - \tilde{b}_t(1-\xi)\right]\tilde{y}_t + \tilde{b}_t\left[\rho + \rho\tilde{b}_t + (1-\tau)A_H z_H^*\right], \tag{1}$$

where ξ is the government spending-to-output ratio and $\tilde{y}_t \equiv Y_t/K_t$ is the *average capital productivity* of the economy (the output–capital ratio).

The dynamics of the wealth share Σ_t are governed by the return differential Δ_r between regimes:

$$\dot{\Sigma}_t = \eta + \left(\Delta_r - (\chi+1)\eta\right)\Sigma_t - \Delta_r \Sigma_t^2, \tag{2}$$

where $\Delta_r \equiv (1+\lambda)(1-\tau)\left[A_H\Omega(z_H^*) - A_L\Omega(z_L^*)\right]$ with $\Omega(z^*) \equiv \int_{z^*}^{\infty}(z-z^*)dF(z)$ denoting the expected excess productivity above the threshold.

The system is closed by the asset-market clearing condition and the expression for the average capital productivity $\tilde{y}_t$:

$$1 + \tilde{b}_t = \frac{1}{(1+\lambda)\left[\bar{F}(z_H^*)\Sigma_t + \bar{F}(z_L^*)(1-\Sigma_t)\right]}, \tag{3}$$

$$\tilde{y}_t = \frac{A_H \Sigma_t \int_{z_H^*}^{\infty} z\, dF(z) + A_L(1-\Sigma_t)\int_{z_L^*}^{\infty} z\, dF(z)}{\bar{F}(z_H^*)\Sigma_t + \bar{F}(z_L^*)(1-\Sigma_t)}. \tag{4}$$

The following sections characterize the steady-state equilibria of this system.

3 Steady-State Implications

3.1 Distributional Assumption

We collect here the single technical condition on the productivity distribution F that powers all the steady-state monotonicity results below.

Assumption 3.1 (Strictly increasing failure rate)**.** $F(z)$ is continuous with finite mean and admits a continuous density $f(z)$ that is strictly positive on a connected support that contains the relevant active region. Its hazard rate $h(z) \equiv f(z)/[1-F(z)]$ is *strictly increasing* in z.

This implies that the expected excess productivity $e(z^*) \equiv \mathbb{E}[Z - z^* \mid Z \geq z^*]$ is strictly decreasing in the threshold z^* (equivalently, $h(z^*)\, e(z^*) < 1$ for all z^* in the active region).

3.2 Debt and Investment Selection

Proposition 3.1 (Debt raises investment thresholds)**.** *Under Assumption 3.1, the steady-state productivity thresholds z_H^* and z_L^* are strictly increasing in the government debt-to-capital ratio $\tilde{b}$.*

Intuition. The asset-market clearing condition (3) shows that, for a fixed wealth share Σ, a higher $\tilde{b}$ requires a lower aggregate demand for capital. This is achieved by raising the thresholds z_H^* and z_L^*, which exclude marginal, low-productivity projects. Although the wealth share Σ adjusts endogenously (it falls when z_H^* rises because higher thresholds truncate the excess-productivity integrals $\Omega(z_i^*)$ in both regimes, narrowing the return differential Δ_r), this secondary effect does not overturn the direct crowding-out channel. Hence, higher debt unambiguously tightens the selection of investment projects. □

3.3 Debt and Aggregate Productivity

Proposition 3.2 (Debt raises average capital productivity)**.** *Under Assumption 3.1, in the steady state, an increase in $\tilde{b}$ strictly raises the average capital productivity $\tilde{y}$.*

Intuition. Average capital productivity is a weighted average of productivity within each regime. Two forces act when debt increases:

1. **Selection effect:** Higher thresholds truncate the lower tail of productivity in each regime, raising the average productivity of projects that are undertaken.

2. **Reallocation effect:** Higher thresholds reduce the return advantage of high-productivity entrepreneurs, shifting wealth share Σ toward the low-productivity regime, which drags down average capital productivity.

Under Assumption 3.1, the gains from tighter selection strictly outweigh the losses from reallocation. Consequently, a larger supply of safe government debt crowds out inefficient capital and improves overall investment efficiency. □

4 Fiscal Policy and Equilibrium Determinacy

4.1 The Reverse Causality Mechanism: From Fiscal Regime to Equilibrium Selection

The following causal chain runs from the fiscal stance to equilibrium selection:

1. **Asset Market Clearing:** For the market to absorb a higher supply of government bonds $\tilde{b}$, the aggregate demand for private capital must fall, which necessitates an increase in the investment threshold z^*.

2. **Selection Effect:** A higher z^* screens out the least productive projects, reallocating capital toward higher-productivity entrepreneurs on average, thereby driving up the economy's average capital productivity $\tilde{y}$.

3. **Interest-Growth Gap:** Although the real interest rate r and the growth rate g both rise with $\tilde{y}$, by Proposition 4.2, r rises faster than g. Consequently, the interest-growth differential $\mathcal{D}(\tilde{b}) \equiv r - g$ is strictly increasing in the debt-to-capital ratio $\tilde{b}$.

4. **Equilibrium Selection via Government Budget Constraint:** The steady-state government budget constraint requires that $\tilde{b}(r-g) = (\tau - \xi)\tilde{y}$, where $\tau - \xi$ is the primary surplus. A primary surplus requires $r > g$, which anchors the economy in the high-$\tilde{b}$, high-productivity equilibrium. Conversely, chronic primary deficits require $r < g$, forcing the economy into the low-$\tilde{b}$, low-productivity state.

Definition (Misallocation Trap) Throughout the paper we use "*misallocation trap*" to refer to the stable steady-state equilibrium that arises under a persistent primary deficit ($\tau < \xi$) when the economy is dynamically inefficient in autarky. It is characterized by (i) a low debt-to-capital ratio $\tilde{b}_{\text{def}}$, (ii) a low average capital productivity $\tilde{y}_{\text{def}}$, (iii) a low real interest rate r_{def}, and (iv) $r < g$, with deficits financed indefinitely at suppressed interest rates while inefficient firms remain active. The equilibrium is self-confirming: low $\tilde{b}$ keeps the screening threshold z^* low, which keeps $\tilde{y}$ and r low, which in turn validates the low-$\tilde{b}$ equilibrium via the steady-state government budget identity.

4.2 Interest-Growth Differential

Fiscal sustainability is governed by the gap between the real interest rate r and the steady-state growth rate g of aggregate capital, denoted by $\mathcal{D}(\tilde{b}) \equiv r(\tilde{b}) - g(\tilde{b})$. In steady state,

$$g \;=\; (1-\xi)\tilde{y} \;-\; \delta \;-\; \rho(1+\tilde{b}). \tag{5}$$

This growth identity is derived in Appendix A; it follows from the aggregate resource constraint $\dot{K}_t = (1-\xi)Y_t - C_t - \delta K_t$ together with the optimal aggregate consumption rule $C_t = \rho W_t = \rho(1+\tilde{b})K_t$ implied by log utility.

This growth equation makes the growth implications of deficits explicit. A deficit equilibrium features (i) a lower average capital productivity $\tilde{y}$ due to misallocation—this suppresses growth via the term $(1-\xi)\tilde{y}$; and (ii) a lower safe-asset supply (lower $\tilde{b}$), which mechanically reduces the debt-service drag $\rho(1+\tilde{b})$ and thus partially offsets the productivity effect. If higher deficits are implemented through higher ξ, growth is further reduced through the first term, independently of any change in $\tilde{y}$. Hence the net effect on growth is not predetermined by an accounting identity alone; in Proposition 4.8 we characterize the ranking of g and r across fiscal regimes.

Proposition 4.1 (Financial Frictions and Dynamic Efficiency)**.** *There exists a critical level of financial development λ^* such that the autarky economy ($\tilde{b} = 0$) is dynamically efficient ($r > g$) if and only if $\lambda > \lambda^*$.*

Intuition. When financial markets are not developed (low λ), capital cannot flow efficiently to high-productivity agents. This results in a low aggregate investment threshold and a depressed equilibrium interest rate that is potentially below the growth rate ($r < g$). As λ increases, the improved allocation of resources increases the marginal product of capital and thus the interest rate, eventually restoring dynamic efficiency.

Proposition 4.2 (Monotonicity of $r - g$ under general fiscal policy)**.** *Under Assumption 3.1, the interest-growth differential $\mathcal{D}(\tilde{b}) \equiv r(\tilde{b}) - g(\tilde{b})$ is a strictly increasing function of $\tilde{b}$ for any fixed fiscal parameters (τ, ξ).*

Intuition. Using the no-arbitrage condition, the growth equation, and the steady-state government budget identity, $\mathcal{D}(\tilde{b})$ can be written in a representation that is valid for any fiscal regime. Under Assumption 3.1, higher $\tilde{b}$ raises selection thresholds and tightens project screening; this implies that the derivative of $\mathcal{D}(\tilde{b})$ with respect to $\tilde{b}$ is strictly positive. A formal proof is provided in the Appendix B. □

4.3 Multiple Equilibria under Balanced Budgets

When the government runs a balanced budget ($\tau = \xi$), the dynamics of debt follow $\dot{\tilde{b}} = \tilde{b}(r-g)$.

Proposition 4.3 (Two steady states)**.** *Under Assumption 3.1 and a balanced budget, if the economy is dynamically inefficient in autarky ($r < g$ at $\tilde{b} = 0$), there exist exactly two steady-state equilibria:*

1. *A fundamental equilibrium at $\tilde{b} = 0$,*

2. *A high-debt "bubble" equilibrium at $\tilde{b}^* > 0$ where $r = g$.*

Intuition. A steady state requires $\tilde{b}(r - g) = 0$. The autarky equilibrium $\tilde{b} = 0$ always exists. Because $r - g$ is strictly increasing (Proposition 4.2) and negative at $\tilde{b} = 0$ but becomes positive for large $\tilde{b}$, it can cross zero at most once, so there exists a unique positive $\tilde{b}^*$ where $r = g$. At this point debt can be rolled over indefinitely because the interest rate equals the growth rate. □

4.4 Unique Equilibrium under Primary Surpluses

If the government runs a primary surplus ($\tau > \xi$), the steady-state condition becomes $r - g = s\,(\tilde{y}/\tilde{b})$ with $s \equiv \tau - \xi > 0$.

Proposition 4.4 (Unique equilibrium with surplus)**.** *Under Assumption 3.1 and a primary surplus ($s > 0$), there exists a unique steady-state equilibrium with positive debt $\tilde{b}^* > 0$.*

Intuition. The left-hand side $r - g$ is increasing in $\tilde{b}$ under Assumption 3.1 (via the same monotonicity argument used in Proposition 4.2). The right-hand side $s\tilde{y}/\tilde{b}$ is decreasing because the elasticity of average capital productivity with respect to the threshold is smaller than the elasticity of debt (a consequence of strict IFR). Hence the two curves intersect exactly once. □

4.5 Impossibility of Deficits in Dynamically Efficient Economies

Proposition 4.5 (No deficit equilibrium under dynamic efficiency)**.** *Under Assumption 3.1, if the economy is dynamically efficient in autarky ($r(0) > g(0)$), there is no steady-state equilibrium with positive public debt under a primary deficit ($\tau < \xi$).*

Intuition. A primary deficit requires $r < g$ to sustain debt. But under dynamic efficiency $r(0) > g(0)$, and because $r - g$ is increasing in $\tilde{b}$, $r > g$ for all $\tilde{b} > 0$. Hence, the necessary condition $r < g$ cannot be met. □

4.6 Limits to Deficit Finance

When the economy is dynamically inefficient in autarky, deficits may be sustainable, but only up to a limit.

Proposition 4.6 (Deficit Laffer curve)**.** *Assume the economy is dynamically inefficient in autarky ($r(0) < g(0)$). Define the primary deficit per unit of average capital productivity, $d \equiv \xi - \tau > 0$. Then:*

1. *There exists a maximum sustainable primary deficit $d_{\max} > 0$. If the primary deficit d exceeds $d_{\max}$, no steady-state equilibrium exists.*

2. *For any feasible primary deficit $d < d_{\max}$, there are at least two steady-state equilibria: a lower-debt and a higher-debt equilibrium. The value $d_{\max}$ is the peak of the Laffer-type debt-service schedule $L(\tilde{b}) \equiv -\tilde{b}\,\mathcal{D}(\tilde{b})/\tilde{y}(\tilde{b})$ characterized in Appendix B.*

Intuition. The steady-state condition with deficit d is $\tilde{b}(g - r) = d\tilde{y}$. The left-hand side, viewed as a function of $\tilde{b}$, starts at zero, rises to a maximum $d_{\max}$, and eventually declines. For $d < d_{\max}$ the horizontal line at d intersects the curve twice, giving two possible debt levels consistent with the same deficit. □

4.7 Hierarchy of Debt, Interest Rates, and Growth Across Fiscal Regimes

Before stating the cross-regime ranking, we record the joint existence of equilibria across the three fiscal regimes under our maintained assumptions.

Lemma 4.1 (Joint existence across fiscal regimes)**.** *Suppose Assumption 3.1 holds and the economy is dynamically inefficient in autarky, $\mathcal{D}(0) = r(0) - g(0) < 0$. Then for any feasible primary deficit $d = \xi - \tau \in (0, d_{\max})$, with $d_{\max}$ characterized in Appendix B:*

1. ***Surplus regime** ($\tau > \xi$): a unique positive-debt steady state $\tilde{b}_{sur} > 0$ exists, by Proposition 4.4.*

2. ***Balanced regime** ($\tau = \xi$): two steady states exist, $\tilde{b} = 0$ and $\tilde{b}_{bal} > 0$, by Proposition 4.3; the positive root is the relevant one.*

3. ***Deficit regime** ($\tau < \xi$, $d < d_{\max}$): two steady states exist, $\tilde{b}_{D1} < \tilde{b}_{D2}$, by Proposition 4.6; the lower root $\tilde{b}_{def} \equiv \tilde{b}_{D1}$ is the stable one.*

Hence the comparison ($\tilde{b}_{sur}$, $\tilde{b}_{bal}$, $\tilde{b}_{def}$) in Proposition 4.7 below is well posed.

Proof. The three claims are immediate corollaries of Propositions 4.4, 4.3, and 4.6, respectively, all of which hold under Assumption 3.1. The deficit existence requires only $d < d_{\max}$, which is the content of part (2) of Proposition 4.6. □

Proposition 4.7 (Ranking of debt and average capital productivity)**.** *Under Assumption 3.1, dynamic inefficiency in autarky, and a feasible primary deficit ($d < d_{\max}$), so that the joint-existence Lemma 4.1 applies, let $\tilde{b}_{sur}$, $\tilde{b}_{bal}$, $\tilde{b}_{def}$ denote the stable steady-state debt levels, and $\tilde{y}_{sur}$, $\tilde{y}_{bal}$, $\tilde{y}_{def}$ the corresponding average capital productivity levels. Then*

$$\tilde{b}_{sur} > \tilde{b}_{bal} > \tilde{b}_{def} \quad \text{and} \quad \tilde{y}_{sur} > \tilde{y}_{bal} > \tilde{y}_{def}.$$

Intuition. The steady-state budget constraint $\tilde{b}(r - g) = (\tau - \xi)\tilde{y}$ must hold in equilibrium. A primary surplus ($\tau > \xi$) raises the right-hand side, requiring a higher product $\tilde{b}(r - g)$. Since $r - g$ is strictly increasing in $\tilde{b}$, this is achieved by a higher equilibrium debt level. Conversely, a primary deficit lowers the right-hand side, yielding lower equilibrium debt. Propositions 3.1 and 3.2 then imply that average capital productivity ranks in the same order as debt across fiscal regimes. □

Thus, running a permanent deficit constrains the economy's safe-asset capacity, leading to lower debt and, through the selection channel, lower average capital productivity.

Proposition 4.8 (Absolute ranking of steady-state r and g)**.** *Assume the economy is dynamically inefficient in autarky and that a feasible deficit policy admits two steady states $\tilde{b}_{D1} < \tilde{b}_{D2}$. Let $\tilde{b}_{def} \equiv \tilde{b}_{D1}$ denote the stable deficit steady state. Then:*

$$r_{D1} < r_{D2} < r_{bal} < r_{sur}, \quad g_{D1} < g_{D2} < g_{bal} < g_{sur}.$$

Hence, across stable fiscal regimes,

$$r_{def} < r_{bal} < r_{sur}, \quad g_{def} < g_{bal} < g_{sur}.$$

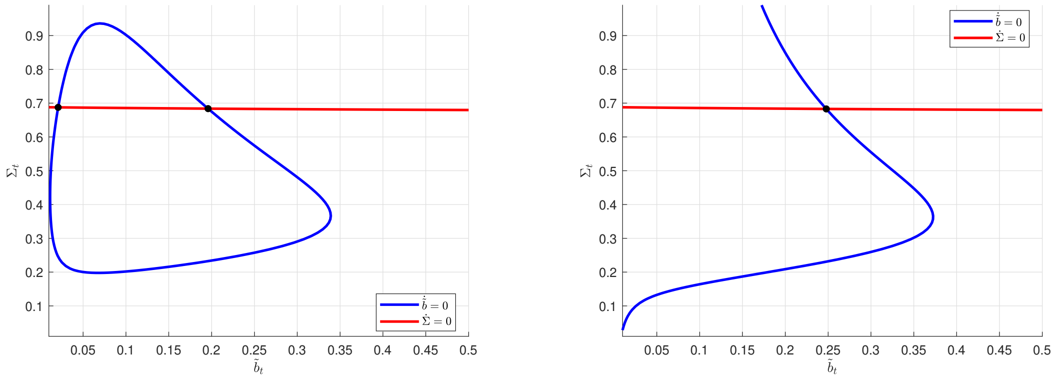
Intuition. By Proposition 4.2, $\mathcal{D}(\tilde{b}) = r - g$ is strictly increasing in $\tilde{b}$. The deficit steady-state condition requires $\mathcal{D}(\tilde{b}) < 0$, so both deficit roots satisfy $\tilde{b}_{D1} < \tilde{b}_{D2} < \tilde{b}_{\text{bal}}$, while surplus requires $\tilde{b}_{\text{sur}} > \tilde{b}_{\text{bal}}$. The no-arbitrage condition implies r increases with thresholds and thus with $\tilde{b}$. Finally, $g = r - \mathcal{D}(\tilde{b})$, and the derivative argument in the Appendix B shows g is also strictly increasing in $\tilde{b}$. Therefore r and g inherit the same strict ranking as debt. □

5 Numerical Examples

We conduct numerical simulations to demonstrate our theoretical results across different productivity distributions. These distributions are selected to span a range of hazard rate properties, including both distributions that satisfy and those that violate the Increasing Failure Rate (IFR) condition (Assumption 3.1). This exercise serves two key purposes: first, to verify that the central predictions hold under the IFR assumption; second, to show that our results are not artifacts of this specific technical condition but reflect broader economic mechanisms.

We examine four canonical cases: (1) Uniform (strictly increasing hazard rate, satisfying IFR); (2) Exponential (constant hazard rate, borderline IFR); (3) Log-Normal (non-monotonic hazard rate); and (4) Pareto (strictly decreasing hazard rate, violating IFR). In the main text, we present results for the Log-Normal distribution as our benchmark, since it represents an empirically relevant specification with a non-monotonic hazard rate. Results for all other distributions are qualitatively identical and are presented in Appendix C.

Figure 1 displays the phase diagrams for the Log-Normal case. Under a primary deficit ($\tau < \zeta$), the system exhibits two steady-state equilibria: a stable low-debt equilibrium and an unstable high-debt equilibrium. Conversely, under a primary surplus ($\tau > \zeta$), the economy converges to a unique stable high-debt, high-productivity equilibrium.



(a) Primary Deficit Regime ($\tau < \xi$) (b) Primary Surplus Regime ($\tau > \xi$)

Figure 1: Equilibrium Determinacy under Log-Normal Distribution. $\ln z \sim \mathcal{N}(0, \sigma^2)$. Panel (a) shows the phase diagram under a slight primary deficit ($\tau = 0.2$, $\xi = 0.201$), where the system exhibits two steady states. Panel (b) shows the phase diagram under a slight primary surplus ($\tau = 0.2$, $\xi = 0.199$), where only one stable steady state exists. Common parameters: $\sigma = 0.3$, $\eta = 0.5$, $\rho = 0.04$, $\delta = 0.1$, $\lambda = 1.0$, $\chi = 0.5$, $A_H = 0.2$, $A_L = 0.1$.

The fact that all distributionsincluding the Pareto distribution, which explicitly violates the IFR conditiongenerate identical qualitative dynamics (shown in Appendix C) confirms that the IFR condition is sufficient but not necessary. The coordination failure mechanism driven by the feedback between safe-asset scarcity and average capital productivity is robust across all four distributional specifications.

5.1 Numerical Welfare Comparison Across Steady States

We complement the equilibrium-determinacy results above with a numerical welfare comparison across the three relevant steady states identified by Lemma 4.1: the unique surplus equilibrium ($\tilde{b}_{\text{sur}}$), and the two deficit roots ($\tilde{b}_{D1} < \tilde{b}_{D2}$, with the lower root $\tilde{b}_{\text{def}} \equiv \tilde{b}_{D1}$ being the stable one). The aim is to translate the productivity ranking of Proposition 4.7 into a welfare statement; a fully analytical welfare characterization (including transition dynamics, type-specific weights, and decompositions) is the subject of a companion paper.

Welfare metric in the baseline economy The baseline economy is populated entirely by entrepreneurs, each maximizing $\sum_t e^{-\rho t} \log(c_t^j)\Delta \to \int_0^\infty e^{-\rho t} \log c_t^j \, dt$. Aggregating over the unit measure of entrepreneurs, we define the BGP utilitarian welfare as

$$V \equiv \int_0^\infty e^{-\rho t} \, \overline{\log c}_t \, dt, \qquad \overline{\log c}_t \equiv \int_0^1 \log c_t^j \, dj. \tag{6}$$

Under log utility we have $c_t^j = \rho w_t^j$, so $\overline{\log c}_t = \log \rho + \overline{\log w}_t$, where $\overline{\log w}_t \equiv \int \log w_t^j \, dj$. Along a BGP, individual wealth grows on average at rate g and the cross-sectional dispersion of $\log w_t^j$ is stationary, so $\overline{\log w}_t = \overline{\log w}_0 + gt$. Substituting into (6) yields the closed-form BGP welfare expression

$$V_{\text{ss}} \;=\; \frac{1}{\rho}\Big[\log\rho + \overline{\log w_0}\Big] \;+\; \frac{g}{\rho^2}. \tag{7}$$

Normalizing the initial aggregate capital stock at $K_0 = 1$, total entrepreneurial wealth at $t = 0$ equals $W_0 = (1+\tilde{b})K_0 = 1+\tilde{b}$, so $\overline{\log w_0}$ depends on the cross-sectional distribution of wealth. With log utility, type-conditional wealth distributions are stationary along the BGP and a summary measure for $\overline{\log w_0}$ is the aggregate per-capita wealth $\log W_0 = \log(1+\tilde{b})$ adjusted for the (stationary) within-type log-dispersion $\sigma_w^2/2$, which is identical across the three steady states for fixed structural parameters. We therefore use the following welfare metric across regimes:

$$\widetilde{V}_{\text{ss}} \;\equiv\; \frac{1}{\rho}\Big[\log\rho + \log(1+\tilde{b})\Big] \;+\; \frac{g}{\rho^2}. \tag{8}$$

Differences in $\widetilde{V}_{\text{ss}}$ across regimes thus combine (i) a level effect through the safe-asset stock $\log(1+\tilde{b})$, which scales aggregate wealth and hence consumption, and (ii) a growth effect through g/ρ^2. By Propositions 4.7–4.8, both terms are largest in the surplus regime and smallest at the misallocation-trap deficit steady state $\tilde{b}_{D1}$.

Numerical comparison Table 1 reports steady-state values of $(\tilde{b}, \tilde{y}, r, g, r-g)$ together with the welfare metric $\widetilde{V}_{\text{ss}}$ for the three regimes, using the calibration of Figure 1.

Table 1: Steady-state outcomes and BGP welfare across fiscal regimes

	Deficit, $\tilde{b}_{D1}$ (stable trap)	**Deficit, $\tilde{b}_{D2}$** (unstable)	**Surplus, $\tilde{b}_{\text{sur}}$** (unique)
$\tilde{b}$	0.0209	0.1959	0.2478
$\tilde{y}$	0.2370	0.2467	0.2493
r	0.0372	0.0480	0.0507
g	0.0485	0.0493	0.0497
$r-g$	-0.0114	-0.0013	0.0010
$\widetilde{V}_{\text{ss}}$	-49.6159	-45.1869	-43.8491
$\Delta\widetilde{V}_{\text{ss}}$ (vs. surplus)	-5.7668	-1.3378	0

Note: Baseline model, Log-Normal calibration. The deficit regime admits two roots $\tilde{b}_{D1} < \tilde{b}_{D2}$, with $\tilde{b}_{D1} = \tilde{b}_{\text{def}}$ being the stable misallocation-trap equilibrium; the surplus regime admits a unique positive-debt root. The welfare metric is $\widetilde{V}_{\text{ss}}$ in equation (8); $\Delta\widetilde{V}$ measures the welfare gap relative to the surplus equilibrium (negative numbers indicate welfare losses).

The structural prediction is unambiguous: by Propositions 4.7 and 4.8, $\tilde{b}_{D1} < \tilde{b}_{D2} < \tilde{b}_{\text{sur}}$ and $g_{D1} < g_{D2} < g_{\text{sur}}$, so the welfare metric (8) satisfies $\widetilde{V}_{D1} < \widetilde{V}_{D2} < \widetilde{V}_{\text{sur}}$. Persistent deficits are not free: although the deficit regime can be supported in equilibrium, it entails a welfare loss relative to the surplus regime, with the stable misallocation trap the worst of the three. A complete welfare analysis that incorporates type-specific welfare weights is left to future work.

6 Robustness: Extensive and Intensive Margins, and Convenience Yields

Our baseline model abstracts from convenience yields on government bonds and adopts a bang-bang portfolio rule to isolate a distinct macroeconomic mechanism: the allocative gains from public debt stem from its role as a screening device that improves capital allocation efficiency, as an alternative to traditional liquidity premium theories (e.g., Krishnamurthy and Vissing-Jorgensen (2012)). In this section, we establish the robustness of our core mechanism along two nested dimensions. First, we introduce convex portfolio adjustment costs to relax the bang-bang portfolio choice, endogenizing leverage decisions and explicitly incorporating an intensive margin of investment. Second, we build on this framework to develop a full general equilibrium extension that integrates a representative household sector with inelastic labor supply, liquidity convenience yields on government bonds, Cobb-Douglas production with aggregate capital externalities, and the same convex adjustment costs. Both extensions confirm that our core results—fiscal policy shapes equilibrium selection via a capital allocation screening channel, and the selection effect dominates the classic Diamond (1965) quantity crowding-out effect—are robust to these perturbations.

To orient the reader, Table 2 summarizes the three nested model specifications used throughout the paper, indicating which features are added at each step and which baseline propositions are preserved.

Table 2: Overview of the three nested model specifications.

	Model A (Baseline)	**Model B** (Adj. costs)	**Model C** (Full GE)
Features			
Bang-bang portfolio rule	yes	no	no
Convex adjustment cost ($\psi > 0$)	no	yes	yes
Household sector	no	no	yes
Inelastic labor supply	no	no	yes
Production technology	linear	linear	C-D
Convenience yield on bonds ($\varphi > 0$)	no	no	yes
Key propositions			
$dz_i^*/d\tilde{b} > 0$ (Prop. 3.1)	✓	✓	✓
$d\tilde{y}/d\tilde{b} > 0$ (Prop. 3.2)	✓	✓	✓
$d\mathcal{D}/d\tilde{b} > 0$ (Prop. 4.2)	✓	✓	✓
Unique surplus equilibrium (Prop. 4.4)	✓	✓	✓
Multiple deficit equilibria (Prop. 4.6)	✓	✓	✓
Cross-regime ranking (Prop. 4.7)	✓	✓	✓

Note: Model A is the baseline; Models B and C add features sequentially.

6.1 Model with Convex Portfolio Adjustment Costs

We generalize the baseline model by introducing a convex portfolio adjustment cost per unit of wealth, which captures financial frictions or monitoring costs associated with adjusting the capital share of an entrepreneur's portfolio. We specify the quadratic form:

$$\Psi(\omega_t^j) = \frac{\psi}{2}(\omega_t^j - 1)^2, \quad \psi \geq 0, \tag{9}$$

where $\omega_t^j \equiv k_t^j / w_t^j$ is the portfolio share invested in private capital, and ψ governs the magnitude of adjustment costs. When $\psi = 0$, the model collapses to the baseline bang-bang portfolio rule. Entrepreneurs also face an exogenous leverage constraint $\omega_t^j \leq 1 + \lambda$, consistent with the baseline model.

6.1.1 Optimal Portfolio Choice and Core Mechanisms

With logarithmic utility, entrepreneurs maximize the expected growth rate of wealth, yielding an optimal portfolio rule that nests the baseline model (full derivation in Appendix D):

$$\omega^*(z) = \max\left\{0,\ \min\left\{1+\lambda,\ 1+\frac{1}{\psi}\left[(1-\tau)\mathcal{A}_t^j z - \delta - r_t\right]\right\}\right\}. \tag{10}$$

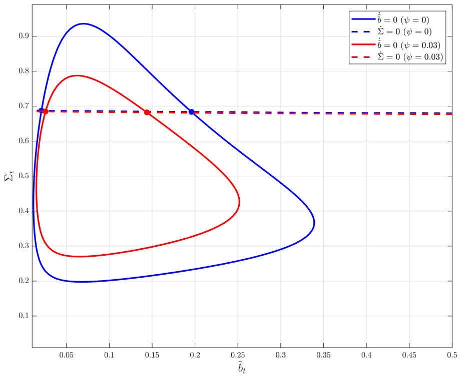
This rule divides entrepreneurs into three regions, preserving the baseline extensive margin while introducing an intensive margin of investment:

- **Selection Region ($z < z^*$):** $\omega^*(z) = 0$. Entrepreneurs with productivity below the threshold z^* hold only risk-free bonds, preserving the core extensive margin screening mechanism of the baseline model. The threshold z^* is strictly increasing in the risk-free rate r_t.
- **Intensive Margin Region ($z^* \leq z < z^{**}$):** $0 < \omega^*(z) < 1 + \lambda$. Leverage is strictly increasing in productivity and strictly decreasing in the risk-free rate. A higher r_t reduces the scale of investment for all active firms in this region, introducing the classic quantity crowding-out effect.
- **Constrained Region ($z \geq z^{**}$):** $\omega^*(z) = 1 + \lambda$. Entrepreneurs at the leverage constraint invest at maximum capacity, consistent with the baseline model.

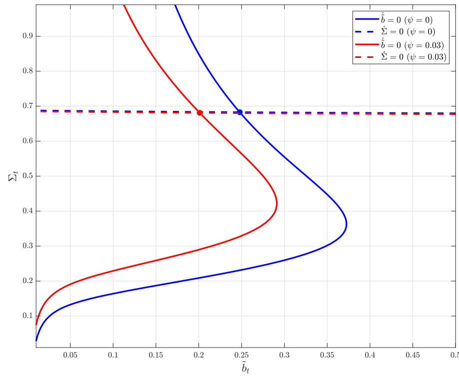
6.1.2 Aggregation and Equilibrium Implications

Aggregation yields a generalized asset market clearing condition, where the aggregate leverage ratio $\Gamma(r_t) \equiv \int \omega^*(z)dF(z)$ is strictly decreasing in r_t. The core price mechanism of the baseline model is preserved: a higher supply of government debt raises the equilibrium risk-free rate r_t to clear the asset market.

The intensive margin reinforces rather than offsets the baseline selection effect. While a higher r_t reduces the aggregate quantity of capital, it shifts the relative share of capital toward the most productive firms: a uniform absolute reduction in leverage represents a smaller percentage decline in capital for high-productivity firms with higher initial leverage. As we prove formally in Appendix D.3, this implies that the average capital productivity is strictly increasing in the risk-free rate r_t, even in the presence of the intensive margin.



(a) Primary Deficit Regime ($\tau < \xi$)



(b) Primary Surplus Regime ($\tau > \xi$)

Figure 2: Comparison of Equilibrium Dynamics: Baseline vs Adjustment Costs. We plot phase diagrams for two models with identical state spaces: Model A (solid lines, filled markers): Baseline model with bang-bang portfolio choice; Model B (dashed lines, open markers): Extended model with convex portfolio adjustment costs. All structural parameters are held constant across the two models.

All core propositions of the baseline model hold in this extended framework: higher debt raises investment thresholds, raises average capital productivity, and generates a strictly increasing $r - g$ schedule. Multiple equilibria under persistent deficits and a unique equilibrium under primary surpluses are preserved.

To directly verify that the intensive margin of investment does not alter our core equilibrium properties, we plot the phase diagrams for the baseline model (Model A) and the model with convex portfolio adjustment costs (Model B) side by side. The two models share identical state variables ($\tilde{b}$, the debt-to-capital ratio, and Σ, the wealth share of high-productivity entrepreneurs), enabling direct comparison of dynamics.

Figure 2 shows the results. The phase trajectories and steady-state configurations are nearly indistinguishable across the two models: under a primary deficit, both exhibit a stable low-debt steady state and an unstable high-debt steady state; under a primary surplus, both converge to a unique stable high-debt steady state. The core mechanism survives relaxation of the bang-bang portfolio rule.

6.2 Extended Model with Labor, Households, and Convenience Yields

We build on the adjustment cost framework to develop a full general equilibrium extension that addresses standard macro-finance frictions, aligns with empirical evidence on $r < g$, and retains the screening mechanism. This extension adds four features: (1) a representative household sector that supplies labor inelastically, earns wage income, and derives liquidity convenience yields from government bond holdings; (2) a Cobb-Douglas production technology with aggregate capital externalities to endogenize labor demand and wage setting; (3) convex portfolio adjustment costs to retain the intensive margin of investment; and (4) a closed labor market and aggregate resource constraint.

6.2.1 Economic Environment

We retain the continuous-time infinite-horizon structure of the baseline model. The economy is populated by two unit-mass agent types: (i) a representative household (savers with no access to productive capital, who supply labor and hold risk-free government bonds); (ii) a continuum of heterogeneous entrepreneurs (the only producers, who hold private capital, hire labor, and allocate wealth between capital and bonds). We maintain the baseline two-type persistent productivity regime $\mathcal{A}_t^j \in \{A_H, A_L\}$ and idiosyncratic productivity shocks $z_t^j \sim F(z)$.

6.2.2 Household Sector

The representative household maximizes lifetime utility over consumption c_{ht} and real government bond holdings b_{ht}, which deliver a convenience yield:

$$\max_{\{c_{ht}, b_{ht}\}} \int_0^\infty e^{-\rho t} \left[\log(c_{ht}) + \varphi \log(b_{ht})\right] dt, \tag{11}$$

where $\rho > 0$ is the subjective discount rate (identical to entrepreneurs) and $\varphi \geq 0$ governs the strength of the convenience yield. The household faces a flow budget constraint that incorporates labor income, interest income, and consumption:

$$\dot{b}_{ht} = r_t b_{ht} + w_t - c_{ht}, \tag{12}$$

where w_t is the competitive real wage rate, and labor supply is normalized to 1.

Solving the household's dynamic optimization problem yields a consumption Euler equation that modifies the standard Ramsey condition to account for the convenience yield (full derivation in Appendix E.1). Along a balanced growth path (BGP) where all aggregate variables grow at rate g, the steady-state Euler equation is:

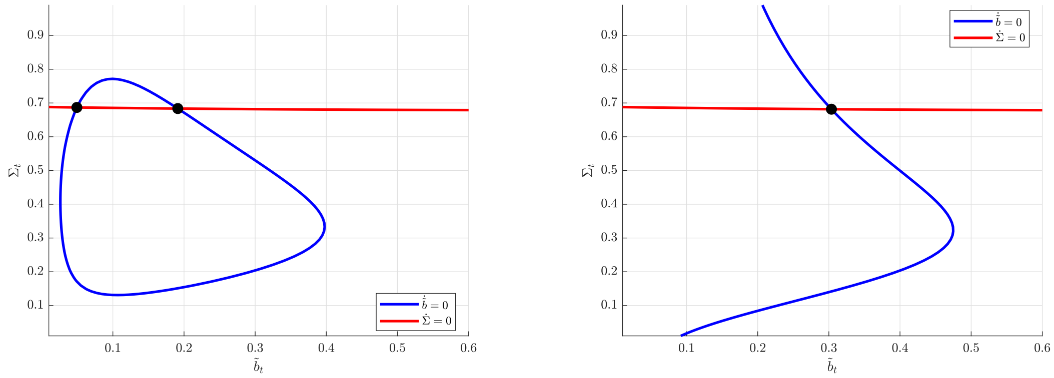
$$r = \rho + g - \varphi \frac{\tilde{c}_h}{\tilde{b}_h}, \tag{13}$$

where $\tilde{c}_h \equiv c_{ht}/K_t$ and $\tilde{b}_h \equiv b_{ht}/K_t$ are household consumption and bond holdings normalized by the aggregate capital stock. The convenience yield term $\varphi > 0$ reduces the pecuniary risk-free rate r required by households to hold bonds, providing a micro-foundation for why $r < g$ can emerge endogenously in equilibrium, consistent with cross-country empirical evidence.

6.2.3 Entrepreneur Sector

Each entrepreneur operates a Cobb-Douglas production technology with labor-augmenting aggregate capital externalities to sustain endogenous growth:

$$y_t^j = \left(\mathcal{A}_t^j z_t^j k_t^j\right)^\alpha \left(K_t n_t^j\right)^{1-\alpha}, \tag{14}$$



(a) Primary Deficit Regime ($\tau < \xi$) (b) Primary Surplus Regime ($\tau > \xi$)

Figure 3: Equilibrium Dynamics in the Full Model with Households and Convenience Yields. We plot phase diagrams for Model C in the $(\tilde{b}, \Sigma)$ subspace (holding household bond holdings $\tilde{b}_h$ and other variables fixed at their steady-state values). Model C extends the framework to include inelastic labor supply, Cobb-Douglas production, convex portfolio adjustment costs, and bond convenience yields. *Common parameters*: Log-normal productivity distribution ($\sigma = 0.3$), $\eta = 0.5$, $\tau = 0.2$, $\rho = 0.04$, $\lambda = 1.0$, $\chi = 0.5$, $A_H = 0.2$, $A_L = 0.1$, $\delta = 0.1$, capital share $\alpha = 0.33$, adjustment cost $\psi = 0.01$, convenience yield weight $\varphi = 0.02$. *Fiscal regime distinction*: Panel (a) uses $\xi = 0.201$; Panel (b) uses $\xi = 0.199$.

where $\alpha \in (0, 1)$ is the private capital share, n_t^j is labor hired by the entrepreneur, and K_t is the aggregate capital stock (taken as exogenous by individual firms). The government levies a proportional tax τ on output, and private capital depreciates at rate δ.

Given capital and the market wage, entrepreneurs choose labor to maximize short-run after-tax profits. Labor market clearing yields an equilibrium wage rate that follows the standard Cobb-Douglas factor share property (full derivation in Appendix E.2):

$$\tilde{w} \equiv \frac{w_t}{K_t} = (1 - \tau)(1 - \alpha)\tilde{y}, \tag{15}$$

where $\tilde{y} \equiv Y_t / K_t$ is aggregate output per unit of capital (the average capital productivity, in the same sense as the baseline). Substituting optimal labor demand into the profit function yields a linear net return to private capital:

$$\mathcal{R}(\mathcal{A}_i z; \tilde{y}) = \alpha(1 - \tau)\tilde{y}^{-(1-\alpha)/\alpha}\mathcal{A}_i z - \delta, \tag{16}$$

which nests the linear return structure of the baseline model (recovering it when $\alpha \to 1$), preserving the screening mechanism.

Entrepreneurs face the same convex portfolio adjustment costs and leverage constraint as in Section 6.1, yielding an optimal portfolio rule identical in structure to Equation (10), with the modified capital return function. The law of motion for the wealth share of high-productivity entrepreneurs Σ_t retains the same quadratic form as in the baseline model.

6.2.4 General Equilibrium and Core Robustness Results

The model's BGP equilibrium is fully characterized by a closed system of 8 equations in 8 unknowns, which pin down the endogenous variables $\{r, g, \tilde{y}, \tilde{b}, \tilde{b}_h, \tilde{c}_h, \tilde{w}, \Sigma\}$ (full system in Appendix E.4). The core asset market clearing condition generalizes to:

$$1 + \tilde{b} = \tilde{b}_h + \frac{1}{\Gamma(r, \Sigma, \tilde{y})}, \tag{17}$$

where $\Gamma(r, \Sigma, \tilde{y})$ is the aggregate leverage ratio of entrepreneurs, which remains strictly decreasing in r_t. The steady-state government budget constraint is identical to the baseline model:

$$\tilde{b}(r - g) = (\tau - \xi)\tilde{y}, \tag{18}$$

preserving the core causal link between fiscal regime and equilibrium selection.

Proposition 6.1 (Core properties are preserved in the household extension)**.** *Under Assumption 3.1, the full model with households, labor, convenience yields, and adjustment costs preserves the baseline equilibrium properties:*

1. *the investment thresholds are increasing in debt, $dz_i^*/d\tilde{b} > 0$;*
2. *average capital productivity is increasing in debt, $d\tilde{y}/d\tilde{b} > 0$;*
3. *the differential $\mathcal{D}(\tilde{b}) \equiv r - g$ is strictly increasing in debt;*
4. *surplus policy admits a unique positive-debt steady state, while deficit policy can generate multiple steady states.*

Sketch. The household block changes bond demand and introduces a convenience-yield wedge in the Euler equation, but the entrepreneur screening block is preserved. Crucially, the household Euler equation (13) and the entrepreneur no-arbitrage condition together pin down r and $\tilde{b}$ jointly in a consistent system; neither condition overidentifies the other. The generalized asset-market-clearing equation implies that a higher debt supply requires a higher equilibrium r, which raises investment thresholds. Under Assumption 3.1, tighter thresholds increase capital-quality selection, and with convex adjustment costs the intensive margin reinforces this selection force. Consequently, $d\tilde{y}/d\tilde{b} > 0$ and $d(r - g)/d\tilde{b} > 0$ continue to hold. Combining this monotonicity with the unchanged government budget identity yields the same determinacy logic as in the baseline model. Appendix E.5 provides the formal derivations. □

The bond convenience yield expands the parameter space for endogenous $r < g$, but does not alter the core deficit-misallocation feedback loop or the central thesis about fiscal prudence and Hamilton's "debt as a blessing" claim.

Figure 3 plots the extended model's phase dynamics in the $(\tilde{b}, \Sigma)$ subspace. The equilibrium structure is qualitatively identical to the baseline: multiple steady states under deficits, and a unique stable high-productivity equilibrium under surpluses.

6.3 Summary of Robustness

Both extensions preserve the core mechanism. The intensive margin introduced by convex adjustment costs reinforces rather than offsets the baseline selection effect, since higher interest rates shift capital toward the most productive firms. The full general equilibrium extension provides a micro-foundation for $r < g$ while preserving the causal link between fiscal policy, capital allocation, and equilibrium selection. In both cases, fiscal policy drives equilibrium outcomes via the screening channel, and only fiscally prudent regimes sustain the high-productivity equilibrium that rationalizes Hamilton's "debt as a blessing" claim.

7 Conclusion

Our model challenges the view that $r < g$ makes deficit-financed spending inexpensive. Observing $r < g$ can signal that persistent deficits have trapped the economy in a self-fulfilling "misallocation trap" of low productivity and low growth. Our results formalize Hamilton (1790)'s vision: a well-funded national debt is a blessing because it provides safe assets that screen out inefficient investment — but only when backed by primary surpluses that sustain the high-productivity equilibrium by keeping $r > g$. Fiscal prudence is the necessary condition for government debt to be a blessing. By contrast, deficits sustained by continued rollover of safe debt at suppressed interest rates do not merely defer taxes: by keeping the economy in the low-efficiency equilibrium, they permanently erode national productive capacity.

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A Model Derivation

Individual Wealth Dynamics

Entrepreneurs maximize $\sum_t e^{-\rho t} \log(c_t^j)\Delta$ subject to the budget constraint:

$$w_{t+\Delta}^j - w_t^j = \left[(1-\tau)\mathcal{A}_{t-\Delta}^j z_{t-\Delta}^j - \delta\right] \omega_t^j w_t^j \Delta + r(1-\omega_t^j) w_t^j \Delta - c_t^j \Delta,$$

where $\omega_t^j = k_t^j / w_t^j$ is the portfolio share in capital. Optimal consumption follows from logarithmic utility: $c_t^j = \rho w_t^j$. In continuous time ($\Delta \to 0$), this becomes $dw_t^j = w_t^j [r(\mathcal{A}_t^j, z_t^j) - \rho] dt$, where $r(\mathcal{A}_t^j, z_t^j) = \left[(1-\tau)\mathcal{A}_t^j z_t^j - \delta\right] \omega_t^j + r(1-\omega_t^j)$.

Investment Threshold

Capital investment is profitable iff $(1-\tau)\mathcal{A}_t^j z_t^j - \delta > r$. Thus the optimal portfolio rule is $\omega_t^j = 1 + \lambda$ if the condition holds, and 0 otherwise. For high-productivity agents ($\mathcal{A}_t^j = A_H$) the threshold z_H^* satisfies $(1-\tau)A_H z_H^* - \delta = r$. For low-productivity agents ($\mathcal{A}_t^j = A_L$) the corresponding threshold is $z_L^* = \frac{A_H z_H^*}{A_L}$.

Aggregation

Let $W_{Ht} = \int_{\mathcal{A}_t^j = A_H} w_t^j \, dj$, $W_{Lt} = \int_{\mathcal{A}_t^j = A_L} w_t^j \, dj$, and $W_t = W_{Ht} + W_{Lt}$. Define $\Sigma_t = W_{Ht}/W_t$. Aggregate capital is $K_t = \int k_t^j \, dj = (1+\lambda) \left[\bar{F}(z_H^*) W_{Ht} + \bar{F}(z_L^*) W_{Lt}\right]$, where $\bar{F}(z) = 1 - F(z)$. Aggregate real bond holdings are $B_t = \int (1-\omega_t^j) w_t^j \, dj = W_t - K_t$. Hence $1 + \tilde{b}_t = \frac{W_t}{K_t} = \frac{1}{(1+\lambda)\left[\bar{F}(z_H^*)\Sigma_t + \bar{F}(z_L^*)(1-\Sigma_t)\right]}$.

Aggregate output is $Y_t = \int \mathcal{A}_{t-\Delta}^j z_{t-\Delta}^j k_t^j \, dj = (1+\lambda) \left[A_H W_{Ht} \int_{z_H^*}^\infty z \, dF(z) + A_L W_{Lt} \int_{z_L^*}^\infty z \, dF(z)\right]$. Therefore, $\tilde{y}_t = \frac{Y_t}{K_t}$ as given in the main text.

Wealth Share Dynamics

The expected return for an entrepreneur in regime $i \in \{H, L\}$ is $\mathbb{E}[r \mid i] = r + (1+\lambda)(1-\tau)A_i \Omega(z_i^*)$, where $\Omega(z^*) = \int_{z^*}^\infty (z - z^*) dF(z)$. The differential between regimes is $\Delta_r \equiv \mathbb{E}[r \mid H] - \mathbb{E}[r \mid L]$. Wealth of high-productivity entrepreneurs evolves as

$$dW_{Ht} = \left[(-\chi\eta + \mathbb{E}[r \mid H] - \rho) W_{Ht} + \eta W_{Lt}\right] dt.$$

Using $W_t = W_{Ht} + W_{Lt}$ and $\Sigma_t = W_{Ht}/W_t$, we obtain $\dot{\Sigma}_t = \eta + (\Delta_r - (\chi+1)\eta)\Sigma_t - \Delta_r \Sigma_t^2$.

Steady-State Growth Equation

We derive the growth identity (5) cited in the main text. The aggregate resource constraint of the baseline economy is

$$\dot{K}_t = (1-\xi)Y_t - C_t - \delta K_t,$$

where ξY_t is government consumption (which we treat as not directly entering household utility in the baseline), C_t is aggregate private consumption, and δK_t is depreciation. With logarithmic preferences and the entrepreneur block above, optimal individual consumption is $c_t^j = \rho w_t^j$, so by aggregation $C_t = \rho W_t$. Combining $W_t = K_t + B_t = (1 + \tilde{b}_t) K_t$ gives $C_t = \rho(1 + \tilde{b}_t) K_t$. Substituting into the resource constraint and dividing by K_t,

$$\frac{\dot{K}_t}{K_t} = (1 - \xi)\,\tilde{y}_t - \delta - \rho(1 + \tilde{b}_t).$$

Along the BGP, $\dot{K}_t / K_t = g$, $\tilde{y}_t = \tilde{y}$, and $\tilde{b}_t = \tilde{b}$, so

$$g = (1 - \xi)\tilde{y} - \delta - \rho(1 + \tilde{b}),$$

which is exactly (5).

B Proofs of Propositions

Proof of Proposition 3.1

We analyze the steady state of the system. The asset market clearing condition is $1 + \tilde{b} = \frac{1}{(1+\lambda)\left[\bar{F}(z_H^*)\Sigma + \bar{F}(z_L^*)(1-\Sigma)\right]}$. Since $\bar{F}(z)$ is strictly decreasing, for fixed Σ, the right-hand side is increasing in z_H^*. The steady-state wealth share Σ satisfies $\Delta_r \Sigma^2 - (\Delta_r - (\chi + 1)\eta)\,\Sigma - \eta = 0$. One can show that Δ_r decreases with z_H^*, and consequently Σ is a decreasing function of z_H^*. An increase in $\tilde{b}$ shifts the asset market curve upward while the wealth distribution curve is unchanged. The new intersection necessarily has a higher z_H^*.

Proof of Proposition 3.2

Average capital productivity is $\tilde{y} = \nu A_H m(z_H^*) + (1 - \nu) A_L m(z_L^*)$, where ν is the effective capital share of high-type agents. The total derivative of $\tilde{y}$ with respect to z_H^* has a selection effect (positive, as $m' > 0$) and a reallocation effect (negative, as ν shifts to low types). Under Assumption 3.1, $m'(z^*) = h(z^*)\,[m(z^*) - z^*] > 0$ since $h(z^*) > 0$ and $m(z^*) > z^*$, while the conditional excess $e(z^*) = m(z^*) - z^*$ is strictly decreasing (the defining contraction property of strict IFR). The selection effect therefore strictly dominates the reallocation effect. Since z_H^* increases with $\tilde{b}$, we conclude $d\tilde{y}/d\tilde{b} > 0$.

Proof of Proposition 4.1

In autarky ($\tilde{b} = 0$), the threshold z_H^* is determined by the market clearing condition with zero debt. The equilibrium threshold $z_H^*(\lambda)$ is strictly increasing in λ. The autarky interest rate $r = (1 - \tau) A_H z_H^* - \delta$ is thus strictly increasing in λ. Let $\mathcal{D}_{\text{aut}}(\lambda) \equiv r(\lambda) - g(\lambda)$ denote the autarky interest–growth differential as a function of financial development. Since $\mathcal{D}_{\text{aut}}(\lambda)$ is strictly increasing, there exists a unique λ^* such that $\mathcal{D}_{\text{aut}}(\lambda^*) = 0$.

Proof of Proposition 4.2

Define $\mathcal{D}(\tilde{b}) \equiv r - g$. In steady state:

$$r = (1-\tau)A_H z_H^* - \delta, \quad g = (1-\xi)\tilde{y} - \delta - \rho(1+\tilde{b}), \quad \tilde{b}\mathcal{D}(\tilde{b}) = (\tau - \xi)\tilde{y}.$$

Eliminating ξ using the budget identity yields a representation valid for any fiscal regime:

$$\mathcal{D}(\tilde{b}) = \rho + \frac{(1-\tau)\big(A_H z_H^* - \tilde{y}\big)}{1+\tilde{b}}.$$

Differentiate with respect to $\tilde{b}$:

$$\frac{d\mathcal{D}}{d\tilde{b}} = (1-\tau)\,\frac{\frac{d(A_H z_H^* - \tilde{y})}{d\tilde{b}}(1+\tilde{b}) - \big(A_H z_H^* - \tilde{y}\big)}{(1+\tilde{b})^2}.$$

We now establish the two facts that drive the sign of $d\mathcal{D}/d\tilde{b}$:

(i) Active-region average productivity strictly exceeds the cutoff. Active H-types invest only when $z \geq z_H^*$, so $A_H z \geq A_H z_H^*$; active L-types satisfy $z \geq z_L^* = (A_H/A_L)z_H^*$, so $A_L z \geq A_H z_H^*$. By Assumption 3.1, F has positive density on $z > z_i^*$, so the inequality is strict on a set of positive measure, hence

$$\tilde{y} \;>\; A_H z_H^*, \qquad A_H z_H^* - \tilde{y} \;<\; 0.$$

(ii) Debt strictly tightens the selection wedge. By Proposition 3.1, $dz_H^*/d\tilde{b} > 0$. Under strict IFR, the conditional excess $e(z^*) = m(z^*) - z^*$ is strictly decreasing, equivalently $\partial m/\partial z^* = h(z^*)\,e(z^*) < 1$. Hence as z_H^* rises with $\tilde{b}$, $A_H z_H^*$ rises strictly faster than the truncation moments $A_H m(z_H^*)$ and $A_L m(z_L^*)$ that combine into $\tilde{y}$. Therefore

$$\frac{d(A_H z_H^* - \tilde{y})}{d\tilde{b}} \;>\; 0.$$

Combining (i) and (ii), the numerator in the displayed expression is strictly positive, so

$$\frac{d\mathcal{D}}{d\tilde{b}} > 0.$$

Hence $\mathcal{D}(\tilde{b})$ is strictly increasing for any fixed (τ, ξ), not only under $\tau = \xi$.

Proof of Proposition 4.3

Steady state requires $\dot{\tilde{b}} = \tilde{b}(r-g) = 0$. One solution is $\tilde{b} = 0$. For a positive solution we need $r = g$. Since $r - g$ is strictly increasing in $\tilde{b}$ and assumed negative at $\tilde{b} = 0$, there exists a unique $\tilde{b}^* > 0$ where $r(\tilde{b}^*) = g(\tilde{b}^*)$. To see that $r - g$ eventually becomes positive, use the representation $\mathcal{D}(\tilde{b}) = \rho + (1-\tau)(A_H z_H^* - \tilde{y})/(1+\tilde{b})$ derived in the proof of Proposition 4.2. Since the second term has bounded numerator (established in part (i) of that proof) and denominator growing without bound, $\mathcal{D}(\tilde{b}) \to \rho > 0$ as $\tilde{b} \to \infty$.

Proof of Proposition 4.4

With primary surplus $s > 0$, condition is $r - g = s\tilde{y}/\tilde{b}$. The LHS is increasing in $\tilde{b}$. The RHS is strictly decreasing because the elasticity of debt to threshold is greater than that of average capital productivity. Hence a unique intersection exists.

Proof of Proposition 4.5

With primary deficit $d > 0$, condition is $r - g = -d\tilde{y}/\tilde{b}$. This requires $r < g$. If autarky is dynamically efficient ($r(0) > g(0)$), then $r > g$ for all $\tilde{b} > 0$ by monotonicity. No solution exists.

Proof of Proposition 4.6

With primary deficit $d \equiv \xi - \tau > 0$, the steady-state government budget identity gives

$$L(\tilde{b}) \;\equiv\; \frac{\tilde{b}(g-r)}{\tilde{y}} \;=\; -\frac{\tilde{b}\,\mathcal{D}(\tilde{b})}{\tilde{y}(\tilde{b})} \;=\; d.$$

Under Assumption 3.1, we have shown $\mathcal{D}(\tilde{b})$ is strictly increasing in $\tilde{b}$ (Proposition 4.2) and starts strictly negative at $\tilde{b} = 0$ (dynamic inefficiency in autarky). Therefore $L(0) = 0$, $L(\tilde{b}) > 0$ for small $\tilde{b} > 0$, L attains an interior maximum $d_{\max}$, and $L(\tilde{b}) < 0$ once $\tilde{b}$ is large enough that $\mathcal{D}(\tilde{b})$ turns positive. By continuity, the equation $L(\tilde{b}) = d$ admits two interior roots whenever $0 < d < d_{\max}$ (one on the rising branch, one on the falling branch).

Characterization of the Maximum Sustainable Deficit

Differentiating $L(\tilde{b}) = -\tilde{b}\,\mathcal{D}(\tilde{b})/\tilde{y}(\tilde{b})$ with respect to $\tilde{b}$,

$$L'(\tilde{b}) \;=\; -\frac{1}{\tilde{y}(\tilde{b})}\Big[\mathcal{D}(\tilde{b}) + \tilde{b}\,\mathcal{D}'(\tilde{b})\Big] \;+\; \frac{\tilde{b}\,\mathcal{D}(\tilde{b})\,\tilde{y}'(\tilde{b})}{\tilde{y}(\tilde{b})^2}.$$

At an interior maximum $\tilde{b}^\star$, $L'(\tilde{b}^\star) = 0$, so multiplying through by $\tilde{y}(\tilde{b}^\star)$:

$$\mathcal{D}(\tilde{b}^\star)\Big[1 - \epsilon_{\tilde{y},\tilde{b}}(\tilde{b}^\star)\Big] + \tilde{b}^\star\,\mathcal{D}'(\tilde{b}^\star) \;=\; 0, \tag{19}$$

where $\epsilon_{\tilde{y},\tilde{b}} \equiv (\tilde{b}/\tilde{y})\,d\tilde{y}/d\tilde{b}$ is the elasticity of the average capital productivity with respect to debt. Equation (19) is a first-order condition that pins down $\tilde{b}^\star$ implicitly as a function of $(\tau, \rho, \delta, \lambda, \chi, \eta, A_H, A_L, F)$. The maximum sustainable primary deficit is then

$$d_{\max} \;=\; L(\tilde{b}^\star) \;=\; -\frac{\tilde{b}^\star\,\mathcal{D}(\tilde{b}^\star)}{\tilde{y}(\tilde{b}^\star)}.$$

On the deficit branch ($\mathcal{D} < 0$) and under Assumption 3.1, $\mathcal{D}' > 0$ (Proposition 4.2) and one can show $\epsilon_{\tilde{y},\tilde{b}} < 1$; these two facts together guarantee the FOC (19) has a unique interior root, and the second-order condition $L''(\tilde{b}^\star) < 0$ holds at that root. Comparative statics on $d_{\max}$ follow from totally differentiating (19); in particular, increases in λ (financial development) raise $\mathcal{D}$ pointwise and therefore weakly lower $d_{\max}$.

Proof of Proposition 4.7

From the steady-state budget identity,

$$\tilde{b}(r-g) = (\tau - \xi)\tilde{y}.$$

Under a primary surplus, the right-hand side is positive; under balance it is zero; under a primary deficit it is negative. By Proposition 4.2, $\mathcal{D}(\tilde{b}) = r - g$ is strictly increasing in $\tilde{b}$. Therefore the equilibrium debt level that solves the identity satisfies

$$\tilde{b}_{\text{sur}} > \tilde{b}_{\text{bal}} > \tilde{b}_{\text{def}}.$$

By Proposition 3.2, $\tilde{y}$ is increasing in $\tilde{b}$ under IFR-dominant selection, so

$$\tilde{y}_{\text{sur}} > \tilde{y}_{\text{bal}} > \tilde{y}_{\text{def}}.$$

Proof of Proposition 4.8

First, by Proposition 4.2, $\mathcal{D}(\tilde{b})$ is strictly increasing. The balanced-budget steady state satisfies $\mathcal{D}(\tilde{b}_{\text{bal}}) = 0$. A deficit steady state must satisfy $\mathcal{D}(\tilde{b}) < 0$, so both deficit roots obey $\tilde{b}_{D1} < \tilde{b}_{D2} < \tilde{b}_{\text{bal}}$. Surplus requires $\mathcal{D}(\tilde{b}) > 0$, hence $\tilde{b}_{\text{sur}} > \tilde{b}_{\text{bal}}$.

Second, from no-arbitrage,

$$r = (1-\tau)A_H z_H^* - \delta,$$

and Proposition 3.1 implies $dz_H^*/d\tilde{b} > 0$. Therefore

$$\frac{dr}{d\tilde{b}} = (1-\tau)A_H \frac{dz_H^*}{d\tilde{b}} > 0,$$

so r is strictly increasing in $\tilde{b}$, yielding

$$r_{D1} < r_{D2} < r_{\text{bal}} < r_{\text{sur}}.$$

Third, $g = r - \mathcal{D}(\tilde{b})$, so

$$\frac{dg}{d\tilde{b}} = \frac{dr}{d\tilde{b}} - \frac{d\mathcal{D}}{d\tilde{b}}.$$

Using

$$\mathcal{D}(\tilde{b}) = \rho + \frac{(1-\tau)(A_H z_H^* - \tilde{y})}{1+\tilde{b}},$$

differentiation and rearrangement imply

$$\frac{dr}{d\tilde{b}} = (1+\tilde{b})\frac{d\mathcal{D}}{d\tilde{b}} + (1-\tau)\frac{d\tilde{y}}{d\tilde{b}} + \frac{(1-\tau)(\tilde{y} - A_H z_H^*)}{1+\tilde{b}}.$$

Each term on the right-hand side is strictly positive under Assumption 3.1 and Propositions 3.1–3.2: the first by the strict monotonicity $d\mathcal{D}/d\tilde{b} > 0$ of Proposition 4.2; the second by Proposition 3.2; the third because $\tilde{y} - A_H z_H^* > 0$, as established in part (i) of the proof of Proposition 4.2. Hence $\frac{dr}{d\tilde{b}} > \frac{d\mathcal{D}}{d\tilde{b}}$, which gives $\frac{dg}{d\tilde{b}} > 0$. Therefore

$$g_{D1} < g_{D2} < g_{\text{bal}} < g_{\text{sur}}.$$

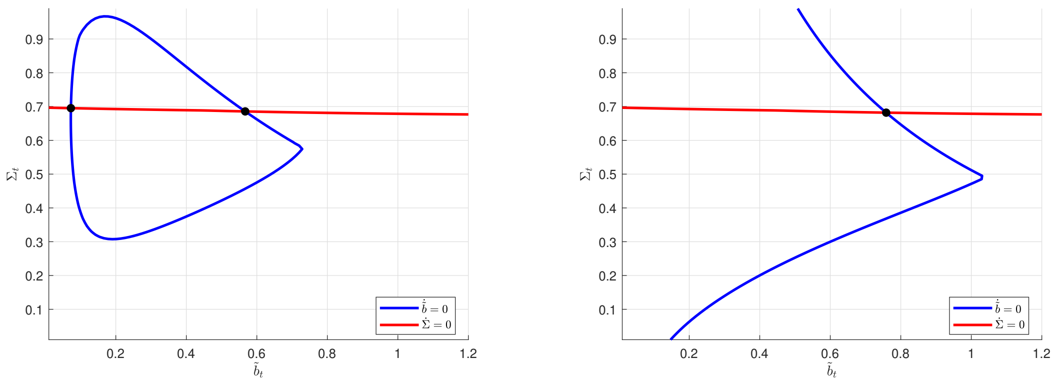
The stable cross-regime ranking follows by setting $\tilde{b}_{\text{def}} = \tilde{b}_{D1}$.

C Additional Numerical Results for Other Productivity Distributions

This appendix presents complete phase diagrams for the three remaining productivity distributions considered in Section 5: Uniform, Exponential, and Pareto. All qualitative results are identical to the Log-Normal case presented in the main text, confirming the robustness of our core mechanism across a wide range of hazard rate specifications.

C.1 Uniform Distribution

The Uniform distribution $z \sim U[0, \bar{z}]$ exhibits a strictly increasing hazard rate, satisfying the IFR assumption most rigorously. Figure 4 presents the phase diagrams.

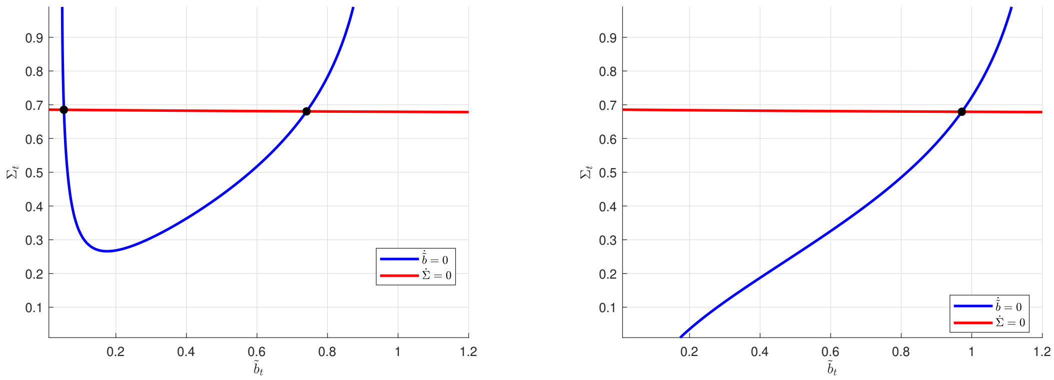


(a) Primary Deficit Regime ($\tau < \xi$)   (b) Primary Surplus Regime ($\tau > \xi$)

Figure 4: Equilibrium Determinacy under Uniform Distribution (Appendix). $z \sim U[0, \bar{z}]$. Panel (a) shows the phase diagram under a primary deficit ($\tau = 0.2$, $\xi = 0.21$), where the system exhibits two steady states. Panel (b) shows the phase diagram under a primary surplus ($\tau = 0.2$, $\xi = 0.19$), where only one stable steady state exists. Common parameters: $\bar{z} = 1.9$, $\eta = 0.5$, $\rho = 0.04$, $\delta = 0.1$, $\lambda = 1.0$, $\chi = 0.5$, $A_H = 0.2$, $A_L = 0.1$.

C.2 Exponential Distribution

The Exponential distribution $z \sim \text{Exp}(\beta)$ features a constant hazard rate, representing the borderline case of the IFR assumption. Figure 5 presents the phase diagrams.



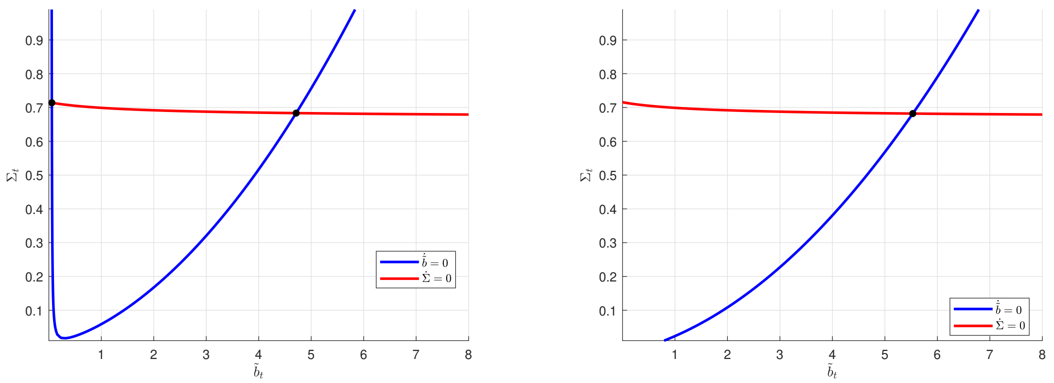
(a) Primary Deficit Regime ($\tau < \xi$)

(b) Primary Surplus Regime ($\tau > \xi$)

Figure 5: Equilibrium Determinacy under Exponential Distribution (Appendix). $z \sim \text{Exp}(\beta)$. Panel (a) shows the phase diagram under a primary deficit ($\tau = 0.2$, $\xi = 0.21$), where the system exhibits two steady states. Panel (b) shows the phase diagram under a primary surplus ($\tau = 0.2$, $\xi = 0.19$), where only one stable steady state exists. Common parameters: $\beta = 2$, $\eta = 0.5$, $\rho = 0.04$, $\delta = 0.1$, $\lambda = 1.0$, $\chi = 0.5$, $A_H = 0.2$, $A_L = 0.1$.

C.3 Pareto Distribution

The Pareto distribution follows a strictly decreasing hazard rate (DFR), directly violating Assumption 3.1. Figure 6 presents the phase diagrams. Despite violating the IFR condition, the equilibrium dynamics are qualitatively identical to the other cases, confirming that the IFR condition is sufficient but not necessary for our results.



(a) Primary Deficit Regime ($\tau < \xi$)

(b) Primary Surplus Regime ($\tau > \xi$)

Figure 6: Equilibrium Determinacy under Pareto Distribution (Appendix). z follows a Pareto distribution with shape parameter $\kappa > 1$ and scale $z_m > 0$. Panel (a) shows the phase diagram under a primary deficit ($\tau = 0.2$, $\xi = 0.21$), where the system exhibits multiple steady states despite violating the IFR condition. Panel (b) shows the phase diagram under a primary surplus ($\tau = 0.2$, $\xi = 0.19$), where only one stable steady state exists. Common parameters: $z_m = 1$, $\kappa = 3$, $\eta = 0.5$, $\rho = 0.04$, $\delta = 0.1$, $\lambda = 1.0$, $\chi = 0.5$, $A_H = 0.3$, $A_L = 0.1$.

D Detailed Derivations for Section 6.1 (Portfolio Adjustment Costs)

D.1 Individual Wealth Dynamics and First-Order Condition

Entrepreneurs allocate wealth w_t^j between capital k_t^j and bonds b_t^j, with $\omega_t^j \equiv k_t^j / w_t^j$. The net return on capital is $(1-\tau)\mathcal{A}_t^j z_t^j - \delta$, and the real risk-free rate is r_t. The convex portfolio adjustment cost is $\Psi(\omega_t^j) = \frac{\psi}{2}(\omega_t^j - 1)^2$ per unit of wealth, and entrepreneurs face a leverage constraint $\omega_t^j \leq 1 + \lambda$.

With logarithmic utility, optimal consumption is $c_t^j = \rho w_t^j$. The evolution of individual wealth is:

$$\begin{aligned} dw_t^j &= w_t^j \left[\omega_t^j \left((1-\tau)\mathcal{A}_t^j z_t^j - \delta \right) + (1-\omega_t^j) r_t - \frac{\psi}{2}(\omega_t^j - 1)^2 - \rho \right] dt \\ &= w_t^j \left[r_t + \omega_t^j \left((1-\tau)\mathcal{A}_t^j z_t^j - \delta - r_t \right) - \frac{\psi}{2}(\omega_t^j - 1)^2 - \rho \right] dt. \end{aligned} \tag{20}$$

The entrepreneur chooses ω_t^j to maximize the expected growth rate of wealth. The unconstrained first-order condition (FOC) with respect to ω_t^j is:

$$(1-\tau)\mathcal{A}_t^j z_t^j - \delta - r_t - \psi(\omega_t^j - 1) = 0. \tag{21}$$

Solving for ω_t^j yields the unconstrained optimal leverage:

$$\omega^{unc}(z_t^j; r_t) = 1 + \frac{1}{\psi}\left[(1-\tau)\mathcal{A}_t^j z_t^j - \delta - r_t \right]. \tag{22}$$

Incorporating the non-negativity constraint $\omega \geq 0$ and the leverage limit $\omega \leq 1 + \lambda$, the optimal portfolio share is:

$$\omega^*(z) = \max\left\{ 0,\ \min\left\{ 1+\lambda,\ 1 + \frac{1}{\psi}\left[(1-\tau)\mathcal{A}_t^j z - \delta - r_t \right] \right\} \right\}, \tag{23}$$

matching Equation (10) in the main text.

This formulation defines two productivity thresholds: 1. Lower threshold z^*: Where $\omega^*(z^*) = 0$, solving to $(1-\tau)\mathcal{A}z^* - \delta = r_t - \psi$. 2. Upper threshold z^{**}: Where $\omega^*(z^{**}) = 1 + \lambda$, solving to $(1-\tau)\mathcal{A}z^{**} - \delta = r_t + \psi\lambda$.

For $z < z^*$, entrepreneurs hold only bonds; for $z^* \leq z < z^{**}$, leverage is continuous in productivity; for $z \geq z^{**}$, entrepreneurs hit the leverage constraint.

D.2 Aggregation

Let Σ_t be the wealth share of high-productivity agents (A_H). Aggregate capital demand is:

$$K_t = W_{Ht} \int \omega^*(z, A_H) dF(z) + W_{Lt} \int \omega^*(z, A_L) dF(z). \tag{24}$$

Define $\Omega_i(r_t) \equiv \int \omega^*(z, A_i)dF(z)$ as the average leverage of group i. For the unconstrained interior region:

$$\begin{aligned}\Omega_i(r_t) &= \int_{z_i^*}^{z_i^{**}} \left(1 + \frac{1}{\psi}[(1-\tau)A_i z - \delta - r_t]\right) dF(z) + (1+\lambda)\int_{z_i^{**}}^{\infty} dF(z) \\ &= [F(z_i^{**}) - F(z_i^*)] \\ &\quad + \frac{1}{\psi}\left[(1-\tau)A_i \int_{z_i^*}^{z_i^{**}} z dF(z) - (\delta + r_t)\left(F(z_i^{**}) - F(z_i^*)\right)\right] \\ &\quad + (1+\lambda)\bar{F}(z_i^{**}). \end{aligned} \tag{25}$$

Asset market clearing requires $K_t + B_t = W_t$. Dividing by K_t:

$$1 + \tilde{b}_t = \frac{W_t}{K_t} = \frac{1}{\Sigma_t \Omega_H(r_t) + (1-\Sigma_t)\Omega_L(r_t)}. \tag{26}$$

Since $\frac{\partial \Omega_i}{\partial r_t} = -\frac{1}{\psi}\left(F(z_i^{**}) - F(z_i^*)\right) < 0$, a higher interest rate r_t reduces aggregate leverage, clearing the market for a higher supply of bonds $\tilde{b}_t$.

Aggregate output is $Y_t = \int \mathcal{A} z k_t^j dj$:

$$Y_t = W_{Ht} \int A_H z \omega^*(z, A_H) dF(z) + W_{Lt} \int A_L z \omega^*(z, A_L) dF(z). \tag{27}$$

D.3 Impact on Average Capital Productivity

The economy's average capital productivity is the capital-weighted average productivity of active firms:

$$\tilde{y} = \frac{\int_{z^*}^{\infty} \mathcal{A} z\, \omega^*(z; r) dF(z)}{\int_{z^*}^{\infty} \omega^*(z; r) dF(z)}. \tag{28}$$

Taking the derivative with respect to r:

$$\frac{d\tilde{y}}{dr} = \underbrace{\frac{\partial \tilde{y}}{\partial z^*}\frac{\partial z^*}{\partial r}}_{\text{Selection (+)}} + \underbrace{\frac{\partial \tilde{y}}{\partial \omega^*}\frac{\partial \omega^*}{\partial r}}_{\text{Intensive Margin}} \tag{29}$$

The first term (extensive margin) is unambiguously positive: a higher r raises z^*, crowding out low-productivity firms and raising the average capital productivity. For the intensive margin, note that $\frac{\partial \omega^*(z;r)}{\partial r} = -\frac{1}{\psi}$ for all active firms in the interior region. An increase in r reduces leverage by a constant absolute amount for all active firms. Because highly productive firms (high z) have higher initial leverage, this constant reduction represents a smaller percentage decrease in their capital share compared to marginal firms.

Formally, using the quotient rule for the derivative of $\tilde{y}$ with respect to r (holding z^* constant for the pure intensive margin effect):

$$
\begin{aligned}
\frac{\partial \tilde{y}}{\partial r} &= \frac{\left(\int_{z^*}^{\infty} \mathcal{A}z\left(-\frac{1}{\psi}\right) w(z)f(z)dz\right) K - Y\left(\int_{z^*}^{\infty}\left(-\frac{1}{\psi}\right) w(z)f(z)dz\right)}{K^2} \\
&= \frac{1}{\psi K^2}\left[Y\int_{z^*}^{\infty} w(z)f(z)dz - K\int_{z^*}^{\infty} \mathcal{A}z\, w(z)f(z)dz\right]. \qquad (30)
\end{aligned}
$$

Let $\mathbb{E}_w[\cdot]$ denote the expectation weighted by the wealth-weighted density of active entrepreneurs. The bracketed term can be rewritten as a covariance:

$$
\frac{\partial \tilde{y}}{\partial r} = \frac{\int_{z^*}^{\infty} w(z)f(z)dz}{\psi K} \cdot \mathbf{Cov}_w\Big(\mathcal{A}z,\ \omega^*(z;r)\Big) > 0. \qquad (31)
$$

Because optimal leverage $\omega^*(z)$ is strictly increasing in productivity $\mathcal{A}z$, this covariance is strictly positive. The intensive margin effect on the average capital productivity is strictly positive, reinforcing the selection effect.

Incorporating the leverage constraint, the most productive firms ($z \geq z^{**}$) do not reduce leverage at all when r rises, so their relative capital share increases even more sharply. The positive relationship between r and $\tilde{y}$ is strictly robust.

E Detailed Derivations for Section 6.2 (Costs, Households, Labor and Convenience Yields)

E.1 Household Problem and Euler Equation Derivation

The representative household maximizes:

$$
\max_{\{c_{ht}, b_{ht}\}_{t\geq 0}} \int_0^{\infty} e^{-\rho t}\left[\log(c_{ht}) + \varphi \log(b_{ht})\right] dt, \qquad (32)
$$

subject to the flow budget constraint:

$$
\dot{b}_{ht} = r_t b_{ht} + w_t - c_{ht}. \qquad (33)
$$

The present-value Hamiltonian is:

$$
\mathcal{H} = e^{-\rho t}[\log c_{ht} + \varphi \log b_{ht}] + \lambda_t[r_t b_{ht} + w_t - c_{ht}]. \qquad (34)
$$

The first-order necessary conditions are:

$$\frac{\partial \mathcal{H}}{\partial c_{ht}} = e^{-\rho t} c_{ht}^{-1} - \lambda_t = 0 \implies \lambda_t = e^{-\rho t} c_{ht}^{-1}, \tag{35}$$

$$\frac{\partial \mathcal{H}}{\partial b_{ht}} = e^{-\rho t} \varphi b_{ht}^{-1} + \lambda_t r_t = -\dot{\lambda}_t, \tag{36}$$

$$\lim_{t\to\infty} \lambda_t b_{ht} = 0 \quad \text{(Transversality Condition)}. \tag{37}$$

Differentiating equation (35) with respect to time:

$$\dot{\lambda}_t = -\rho e^{-\rho t} c_{ht}^{-1} - e^{-\rho t} c_{ht}^{-2} \dot{c}_{ht}. \tag{38}$$

Dividing by $\lambda_t = e^{-\rho t} c_{ht}^{-1}$:

$$\frac{\dot{\lambda}_t}{\lambda_t} = -\rho - \frac{\dot{c}_{ht}}{c_{ht}}. \tag{39}$$

Substitute (35) and (39) into equation (36):

$$e^{-\rho t} \varphi b_{ht}^{-1} + e^{-\rho t} c_{ht}^{-1} r_t = \rho e^{-\rho t} c_{ht}^{-1} + e^{-\rho t} c_{ht}^{-2} \dot{c}_{ht}. \tag{40}$$

Divide through by $e^{-\rho t} c_{ht}^{-1}$:

$$\varphi \frac{c_{ht}}{b_{ht}} + r_t = \rho + \frac{\dot{c}_{ht}}{c_{ht}}. \tag{41}$$

Rearranging gives the household's consumption Euler equation:

$$\frac{\dot{c}_{ht}}{c_{ht}} = r_t - \rho + \varphi \frac{c_{ht}}{b_{ht}}. \tag{42}$$

Along a BGP, $\dot{c}_{ht}/c_{ht} = g$ and $c_{ht}/b_{ht} = \tilde{c}_h/\tilde{b}_h$ is constant. Substituting yields the steady-state Euler equation:

$$r = \rho + g - \varphi \frac{\tilde{c}_h}{\tilde{b}_h}, \tag{43}$$

matching Equation (13) in the main text.

We also derive the steady-state household budget constraint. Along the BGP, $\dot{b}_{ht}/b_{ht} = g$, so $\dot{b}_{ht} = g b_{ht}$. Substitute into the flow budget constraint:

$$g b_{ht} = r b_{ht} + w_t - c_{ht}. \tag{44}$$

Normalize by K_t (with $\tilde{b}_h = b_{ht}/K_t$, $\tilde{w} = w_t/K_t$, $\tilde{c}_h = c_{ht}/K_t$):

$$g \tilde{b}_h = r \tilde{b}_h + \tilde{w} - \tilde{c}_h, \tag{45}$$

Rearranging gives:

$$\tilde{c}_h = (r - g)\tilde{b}_h + \tilde{w}. \tag{46}$$

E.2 Entrepreneur Production, Labor Demand, and Capital Return

Each entrepreneur operates a Cobb-Douglas production technology:

$$y_t^j = \left(\mathcal{A}_t^j z_t^j k_t^j\right)^\alpha \left(K_t n_t^j\right)^{1-\alpha}. \tag{47}$$

The government levies a proportional tax τ on output, so after-tax revenue is $(1-\tau)y_t^j$. The firm chooses labor n_t^j to maximize short-run profits:

$$\max_{n_t^j} \quad (1-\tau)y_t^j - w_t n_t^j - \delta k_t^j. \tag{48}$$

The first-order condition with respect to n_t^j is:

$$(1-\tau)(1-\alpha)\left(\mathcal{A}_t^j z_t^j k_t^j\right)^\alpha K_t^{1-\alpha} n_t^{j-\alpha} = w_t. \tag{49}$$

Solve for optimal labor demand:

$$n_t^{j\alpha} = \frac{(1-\tau)(1-\alpha)\left(\mathcal{A}_t^j z_t^j k_t^j\right)^\alpha K_t^{1-\alpha}}{w_t}, \tag{50}$$

$$n_t^j = \left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{1/\alpha} \mathcal{A}_t^j z_t^j k_t^j K_t^{(1-\alpha)/\alpha}. \tag{51}$$

Labor market clearing requires total labor demand equal to unitary supply:

$$\int_0^1 n_t^j dj = 1. \tag{52}$$

Substitute (51) into the market clearing condition:

$$\left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{1/\alpha} K_t^{(1-\alpha)/\alpha} \int_0^1 \mathcal{A}_t^j z_t^j k_t^j dj = 1. \tag{53}$$

Define the capital-weighted average productivity:

$$\mathcal{Z}_t \equiv \frac{1}{K_t}\int_0^1 \mathcal{A}_t^j z_t^j k_t^j dj. \tag{54}$$

Substitute into the market clearing condition:

$$\left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{1/\alpha} K_t^{(1-\alpha)/\alpha} \mathcal{Z}_t K_t = 1, \tag{55}$$

$$\left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{1/\alpha} K_t^{1/\alpha} \mathcal{Z}_t = 1. \tag{56}$$

Raise both sides to the power of α:

$$\frac{(1-\tau)(1-\alpha)}{w_t} K_t \mathcal{Z}_t^{\alpha} = 1. \tag{57}$$

Solve for the wage rate:

$$w_t = (1-\tau)(1-\alpha)\mathcal{Z}_t^{\alpha} K_t. \tag{58}$$

Define average capital productivity as $\tilde{y}_t \equiv Y_t/K_t$. Substitute optimal labor demand into the production function and integrate over all entrepreneurs:

$$\begin{aligned} Y_t &= \int_0^1 y_t^j dj = \int_0^1 \left(\mathcal{A}_t^j z_t^j k_t^j\right)^{\alpha} \left(K_t n_t^j\right)^{1-\alpha} dj \\ &= K_t^{1-\alpha} \int_0^1 \left(\mathcal{A}_t^j z_t^j k_t^j\right)^{\alpha} n_t^{j(1-\alpha)} dj. \end{aligned} \tag{59}$$

Substitute optimal labor demand (51):

$$\begin{aligned} Y_t &= K_t^{1-\alpha} \int_0^1 \left(\mathcal{A}_t^j z_t^j k_t^j\right)^{\alpha} \left[\left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{1/\alpha} \mathcal{A}_t^j z_t^j k_t^j K_t^{(1-\alpha)/\alpha}\right]^{1-\alpha} dj \\ &= K_t^{1-\alpha} \left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{(1-\alpha)/\alpha} K_t^{(1-\alpha)^2/\alpha} \int_0^1 \mathcal{A}_t^j z_t^j k_t^j dj \\ &= K_t^{1/\alpha} \left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{(1-\alpha)/\alpha} \mathcal{Z}_t K_t. \end{aligned} \tag{60}$$

Substitute the equilibrium wage rate $w_t = (1-\tau)(1-\alpha)\mathcal{Z}_t^{\alpha} K_t$:

$$\begin{aligned} Y_t &= K_t^{1/\alpha} \left(\frac{(1-\tau)(1-\alpha)}{(1-\tau)(1-\alpha)\mathcal{Z}_t^{\alpha} K_t}\right)^{(1-\alpha)/\alpha} \mathcal{Z}_t K_t \\ &= K_t^{1/\alpha} K_t^{-(1-\alpha)/\alpha} \mathcal{Z}_t^{-(1-\alpha)} \mathcal{Z}_t K_t \\ &= K_t \mathcal{Z}_t^{\alpha}. \end{aligned} \tag{61}$$

Thus, average capital productivity is:

$$\tilde{y}_t = \frac{Y_t}{K_t} = \mathcal{Z}_t^{\alpha}, \tag{62}$$

and the equilibrium normalized wage rate is:

$$\tilde{w} \equiv \frac{w_t}{K_t} = (1-\tau)(1-\alpha)\tilde{y}, \tag{63}$$

matching Equation (15) in the main text.

We now derive the net return to private capital. Substitute optimal labor demand into the firm's profit function:

$$\begin{aligned}\text{Net Profit}_t^j &= (1-\tau)y_t^j - w_t n_t^j - \delta k_t^j \\ &= (1-\tau)\left(\mathcal{A}_t^j z_t^j k_t^j\right)^\alpha \left(K_t n_t^j\right)^{1-\alpha} - w_t n_t^j - \delta k_t^j. \end{aligned} \tag{64}$$

Using the FOC for labor, $(1-\tau)(1-\alpha)y_t^j = w_t n_t^j$, so the wage bill is a constant share of after-tax output. Substitute this into the profit function:

$$\begin{aligned}\text{Net Profit}_t^j &= (1-\tau)y_t^j - (1-\tau)(1-\alpha)y_t^j - \delta k_t^j \\ &= \alpha(1-\tau)y_t^j - \delta k_t^j. \end{aligned} \tag{65}$$

Substitute the production function and optimal labor demand:

$$\begin{aligned} y_t^j &= \left(\mathcal{A}_t^j z_t^j k_t^j\right)^\alpha \left(K_t n_t^j\right)^{1-\alpha} \\ &= \left(\mathcal{A}_t^j z_t^j k_t^j\right)^\alpha K_t^{1-\alpha} \left[\left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{1/\alpha} \mathcal{A}_t^j z_t^j k_t^j K_t^{(1-\alpha)/\alpha}\right]^{1-\alpha} \\ &= \left(\frac{(1-\tau)(1-\alpha)}{w_t}\right)^{(1-\alpha)/\alpha} K_t^{1/\alpha} \mathcal{A}_t^j z_t^j k_t^j. \end{aligned} \tag{66}$$

Substitute the equilibrium wage rate $w_t = (1-\tau)(1-\alpha)\tilde{y}K_t$ (since $\tilde{y} = \mathcal{Z}^\alpha$):

$$\begin{aligned} y_t^j &= \left(\frac{(1-\tau)(1-\alpha)}{(1-\tau)(1-\alpha)\tilde{y}K_t}\right)^{(1-\alpha)/\alpha} K_t^{1/\alpha} \mathcal{A}_t^j z_t^j k_t^j \\ &= \tilde{y}^{-(1-\alpha)/\alpha} K_t^{-(1-\alpha)/\alpha} K_t^{1/\alpha} \mathcal{A}_t^j z_t^j k_t^j \\ &= \tilde{y}^{-(1-\alpha)/\alpha} K_t \mathcal{A}_t^j z_t^j k_t^j. \end{aligned} \tag{67}$$

Substitute back into the net profit function:

$$\begin{aligned}\text{Net Profit}_t^j &= \alpha(1-\tau)\tilde{y}^{-(1-\alpha)/\alpha} K_t \mathcal{A}_t^j z_t^j k_t^j - \delta k_t^j \\ &= \left(\alpha(1-\tau)\tilde{y}^{-(1-\alpha)/\alpha}\mathcal{A}_t^j z_t^j - \delta\right) k_t^j. \end{aligned} \tag{68}$$

Thus, the net return per unit of capital is:

$$\mathcal{R}(\mathcal{A}_i z; \tilde{y}) = \alpha(1-\tau)\tilde{y}^{-(1-\alpha)/\alpha}\mathcal{A}_i z - \delta, \tag{69}$$

as presented in the main text. When $\alpha \to 1$, this collapses to the baseline return function $\mathcal{R}(\mathcal{A}_i z) = (1-\tau)\mathcal{A}_i z - \delta$.

E.3 Entrepreneur Portfolio Choice and Wealth Dynamics

Entrepreneurs face the same convex portfolio adjustment costs and leverage constraint as in Section 6.1. With logarithmic utility, optimal consumption is $c_t^j = \rho w_t^j$, and the law of motion for individual wealth is:

$$dw_t^j = w_t^j \left[r_t + \omega_t^j \left(\mathcal{R}(\mathcal{A}_i z; \tilde{y}) - r_t \right) - \frac{\psi}{2} \left(\omega_t^j - 1 \right)^2 - \rho \right] dt. \tag{70}$$

The unconstrained FOC for optimal portfolio choice is:

$$\mathcal{R}(\mathcal{A}_i z; \tilde{y}) - r_t - \psi(\omega_t^j - 1) = 0, \tag{71}$$

yielding the unconstrained optimal leverage:

$$\omega^{unc}(z; r_t, \tilde{y}) = 1 + \frac{1}{\psi} \left[\alpha(1-\tau)\tilde{y}^{-(1-\alpha)/\alpha} \mathcal{A}_i z - \delta - r_t \right]. \tag{72}$$

Incorporating the non-negativity constraint and leverage limit, the optimal portfolio rule is:

$$\omega^*(z; r_t, \tilde{y}) = \max \left\{ 0, \ \min \left\{ 1 + \lambda, \ 1 + \frac{1}{\psi} \left[\alpha(1-\tau)\tilde{y}^{-(1-\alpha)/\alpha} \mathcal{A}_i z - \delta - r_t \right] \right\} \right\}, \tag{73}$$

consistent with the main text.

The law of motion for the wealth share of high-productivity entrepreneurs Σ_t follows the same derivation as the baseline model. Let $\mathbb{E}[r|i]$ be the expected net return on wealth for entrepreneurs in regime i, defined as:

$$\mathbb{E}[r|i] = r_t + \int_{z_i^*}^{\infty} \left[\omega^*(z) \left(\mathcal{R}(\mathcal{A}_i z; \tilde{y}) - r_t \right) - \frac{\psi}{2} \left(\omega^*(z) - 1 \right)^2 \right] dF(z). \tag{74}$$

The return differential is $\Delta_r \equiv \mathbb{E}[r|H] - \mathbb{E}[r|L]$. The law of motion for Σ_t is:

$$\dot{\Sigma}_t = \eta + \left(\Delta_r - (\chi + 1)\eta \right) \Sigma_t - \Delta_r \Sigma_t^2, \tag{75}$$

identical to the baseline model. Along the BGP, $\dot{\Sigma}_t = 0$, yielding the steady-state wealth distribution condition.

E.4 General Equilibrium System along the BGP

The BGP equilibrium is fully characterized by 8 endogenous variables $\{r, g, \tilde{y}, \tilde{b}, \tilde{b}_h, \tilde{c}_h, \tilde{w}, \Sigma\}$, pinned down by a system of 8 equations in 8 unknowns:

1. Household Steady-State Budget Constraint:

$$\tilde{c}_h = (r - g)\tilde{b}_h + \tilde{w}.$$

2. Household Steady-State Euler Equation:

$$r = \rho + g - \varphi \frac{\tilde{c}_h}{\tilde{b}_h}.$$

3. Equilibrium Wage Equation:

$$\tilde{w} = (1-\tau)(1-\alpha)\tilde{y}.$$

4. Steady-State Government Budget Constraint:

$$\tilde{b}(r-g) = (\tau - \xi)\tilde{y}.$$

5. Aggregate Resource Constraint Growth Equation: The aggregate resource constraint is $Y_t = C_{ht} + C_{Et} + \dot{K}_t + \delta K_t + G_t$. Normalizing by K_t and substituting $C_{Et} = \rho W_{Et} = \rho(1 + \tilde{b} - \tilde{b}_h)K_t$, $G_t = \xi Y_t$, we derive:

$$g = \frac{[1 - \xi - (1-\tau)(1-\alpha)]\,\tilde{y} - \delta - (r-\rho)\tilde{b}_h - \rho(1+\tilde{b})}{1 - \tilde{b}_h}.$$

6. Asset Market Clearing Condition: Total bond supply equals total demand from households and entrepreneurs: $B_t = b_{ht} + (W_{Et} - K_t)$. Normalizing by K_t:

$$1 + \tilde{b} = \tilde{b}_h + \frac{1}{\Gamma(r, \Sigma, \tilde{y})},$$

where $\Gamma(r, \Sigma, \tilde{y}) = \Sigma \int_{z_H^*}^{\infty} \omega^*(z) dF(z) + (1-\Sigma) \int_{z_L^*}^{\infty} \omega^*(z) dF(z)$ is the aggregate leverage ratio.
7. Endogenous Average Capital Productivity Equation:

$$\tilde{y} = \left(\frac{\Sigma A_H \int_{z_H^*}^{\infty} z\omega^*(z) dF(z) + (1-\Sigma) A_L \int_{z_L^*}^{\infty} z\omega^*(z) dF(z)}{\Sigma \int_{z_H^*}^{\infty} \omega^*(z) dF(z) + (1-\Sigma) \int_{z_L^*}^{\infty} \omega^*(z) dF(z)} \right)^{\alpha}.$$

8. Steady-State Wealth Distribution Condition:

$$\eta + (\Delta_r(r, \Sigma, \tilde{y}) - (\chi+1)\eta)\,\Sigma - \Delta_r(r, \Sigma, \tilde{y})\Sigma^2 = 0.$$

This system is closed and solvable, and preserves all core properties of the baseline model.

E.5 Formal Robustness Proofs for the Household Extension

This subsection provides formal proofs that the core baseline propositions remain valid in the full extension with households, labor, convenience yields, and convex portfolio adjustment costs.

Lemma E.1 (Aggregate entrepreneurial leverage decreases in r)**.** *In the extended model, the ag-*

gregate leverage object

$$\Gamma(r,\Sigma,\tilde{y}) = \Sigma \int_{z_H^*}^{\infty} \omega^*(z)\, dF(z) + (1-\Sigma) \int_{z_L^*}^{\infty} \omega^*(z)\, dF(z)$$

is strictly decreasing in the risk-free rate r.

Proof. In the interior region, the optimal leverage rule satisfies

$$\omega^*(z) = 1 + \frac{1}{\psi}\big[\alpha(1-\tau)\tilde{y}^{-(1-\alpha)/\alpha}\mathcal{A}_i z - \delta - r\big],$$

so $\partial\omega^*/\partial r = -1/\psi < 0$. In constrained corners, ω^* is flat in r. A higher r also raises threshold cutoffs and shifts marginal entrepreneurs to bond-only portfolios. Therefore aggregate entrepreneurial leverage falls with r. □

Lemma E.2 (Thresholds rise with debt in the household extension)**.** *Under Assumption 3.1, $dz_i^*/d\tilde{b} > 0$ for $i \in \{H, L\}$.*

Proof. Asset clearing in the full model is

$$1 + \tilde{b} = \tilde{b}_h + \frac{1}{\Gamma(r,\Sigma,\tilde{y})}.$$

Holding fiscal parameters fixed, an increase in $\tilde{b}$ requires lower entrepreneurial leverage. By Lemma E.1, this implies a higher r. The threshold conditions are

$$\alpha(1-\tau)\tilde{y}^{-(1-\alpha)/\alpha} A_i z_i^* - \delta = r - \psi,$$

so z_i^* moves up with r, hence with $\tilde{b}$. □

Lemma E.3 (Average capital productivity rises with debt in the household extension)**.** *Under Assumption 3.1, $d\tilde{y}/d\tilde{b} > 0$.*

Proof. Higher $\tilde{b}$ raises r and thresholds (Lemma E.2), tightening selection and truncating low-productivity projects. Under Assumption 3.1, this extensive-margin force raises capital-weighted average productivity. With adjustment costs, $\partial\omega^*/\partial r = -1/\psi$ in the interior implies an intensive-margin reweighting toward high-productivity firms, reinforcing selection. Thus the average capital productivity $\tilde{y}$ rises with $\tilde{b}$. □

Proposition E.1 (Monotonicity of $r - g$ in the household extension)**.** *Under Assumption 3.1, $\mathcal{D}(\tilde{b}) \equiv r - g$ is strictly increasing in $\tilde{b}$ for any fixed (τ, ξ).*

Proof. The government budget identity remains

$$\tilde{b}(r-g) = (\tau - \xi)\tilde{y}.$$

Using the extended return and growth blocks, $\mathcal{D}(\tilde{b})$ can be represented as a function of $\tilde{b}$, z_H^*, and $\tilde{y}$. By Lemmas E.2–E.3 and IFR, the derivative of that representation with respect to $\tilde{b}$ is strictly positive. Hence $d(r-g)/d\tilde{b} > 0$. $\square$

Proposition E.2 (Determinacy properties are preserved)**.** *In the household extension, the same determinacy logic as in the baseline model holds:*

1. *under primary surplus, there is a unique positive-debt steady state;*
2. *under feasible primary deficits, multiple steady states can arise.*

Proof. Given Proposition E.1, the left-hand side schedule $r-g$ is monotone in $\tilde{b}$. Intersecting this with the right-hand side implied by $\tilde{b}(r-g) = (\tau - \xi)\tilde{y}$ yields exactly the same intersection logic as in the baseline model: a unique intersection under surplus and potential multiplicity under deficit. $\square$