

# Sets of Models and Prices of Uncertainty\*

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December 28, 2015

## Abstract

A decision maker constructs a convex set of nonnegative martingales to use as likelihood ratios that represent parametric alternatives to a baseline model and also non-parametric models statistically close to both the baseline model and the parametric alternatives. Max-min expected utility over that set gives rise to equilibrium prices of model uncertainty expressed as worst-case distortions to drifts in a representative investor's baseline model. We offer quantitative illustrations for baseline models of consumption dynamics that display long-run risk. We describe a set of parametric alternatives that generates countercyclical prices of uncertainty.

**Keywords**— Risk, uncertainty, asset prices, Chernoff entropy, robustness, shock price elasticities, exponential quadratic stochastic discount factor

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\*We thank Julian Kozlowski, Lloyd Han, and Balint Szöke for thoughtful criticisms of earlier drafts. We thank Scott Lee, Botao Wu, and especially Lloyd Han and Paul Ho for carrying out the computations. Lloyd Han and Paul Ho played a central role in the construction of Appendices B, C and D. Hansen acknowledges support from the National Science Foundation, under grant 0951576 “DMUU: Center for Robust Decision Making on Climate and Energy Policy” and from the Policy Research Program of the MacArthur Foundation under the project “The Price of Policy Uncertainty.”

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In what circumstances is a minimax solution reasonable? I suggest that it is reasonable if and only if the least favorable initial distribution is reasonable according to your body of beliefs. Good (1952)

## 1 Introduction

Acknowledging that a model is an approximation concedes that one of a myriad of statistically similar alternative models might be correct. This paper proposes a new way to imagine how a decision maker forms that set of alternative models and then provides an application to equilibrium asset pricing.<sup>1</sup> We extend work by Hansen and Sargent (2001) and Hansen et al. (2006) that described a decision maker who expresses distrust of a single baseline probability model having a finite number of parameters by surrounding it with an infinite dimensional family of difficult-to-learn-about alternative models. The decision maker represents these alternative models by multiplying baseline probabilities with likelihood ratios whose entropies relative to the baseline model are less than a bound that makes alternative models stay statistically close to the baseline model. He wants to evaluate outcomes under these alternative models.<sup>2</sup>

This paper differs from Hansen et al. (2006) by refining how a decision maker forms a set of models surrounding a baseline model. A new object appears here: a quadratic function of a Markov state that describes alternative parametric models that we use to “tilt” discrepancy measures so that statistical neighborhoods include these alternative parametric models. The decision maker wants valuations that are robust to these models in addition to unspecified models expressed as before by multiplying the baseline model by likelihood ratios. The quadratic function can be constructed to include alternatives with either fixed or time varying parameters, and also less structured models inside a convex set  $\bar{\mathcal{Z}}$  of martingales that we use to pose a robust decision problem. We offer a quantitative example that illustrates how the set  $\bar{\mathcal{Z}}$  more concisely expresses concerns about particular parametric alternatives than does a set  $\mathcal{Z}^*$  used by Hansen et al. (2006).

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<sup>1</sup>Tractable ways to specify priors and compute posteriors facilitated a revolution in applied Bayesian statistics. We require an analogous practical science if the max-min expected utility decision theory elegantly axiomatized by Gilboa and Schmeidler (1989), Maccheroni et al. (2006a,b), and Strzalecki (2011) is to enlist a community of applied researchers. Viewing a set of models as a decision maker’s way of coping with approximation issues is a perspective that complements theoretical work about axioms.

<sup>2</sup>Applications of what Hansen and Sargent (2001) and Maccheroni et al. (2006a,b) call multiplier preferences to macroeconomic policy design and dynamic incentive problems include Karantounias (2013) and Bhandari (2014).

We apply our approach to an investor who represents “the market” and whose specification uncertainty affects prices of exposures to underlying economic shocks. We describe how our tilted discrepancy method for constructing the set of probability models  $\bar{\mathcal{Z}}$  affects uncertainty components of these shock exposures. Our continuous-time specification simplifies asset pricing. A key object is an endogenously determined vector of worst-case drift distortions to a baseline model.<sup>3</sup> The negative of the drift distortion vector equals the vector of market prices of model uncertainty that compensate investors for facing ambiguity about probabilities that describe random fluctuations.

A new mechanism amplifies and makes uncertainty prices fluctuate. We introduce no new risks associated with stochastic volatility.<sup>4</sup> Instead, we amplify the prices of exposures to the “original” shocks. Fluctuations in those prices reflect investors’ struggles to confront doubts about the baseline model. We study how uncertainty prices vary across investment horizons.

Section 2 specifies an investor’s baseline probability model and martingale perturbations to it. Section 3 describes statistical measures of discrepancies between martingales and uses one such measure to construct and characterize a convex set of probability measures. This set contains neighborhoods around both a baseline probability model and members of a family of parametric alternatives to the baseline model. We express the set of probabilities in terms of a set  $\bar{\mathcal{Z}}$  of martingales. This martingale representation proves to be a tractable way for us to formulate robust decision problems in sections 5 and 8.

Section 4 uses Chernoff entropy to construct a set of martingales  $\check{\mathcal{Z}}$  associated with discriminating between competing specifications of probabilities. Section 5 formulates a robust planning problem that generates a worst-case model that we use in conjunction with the Chernoff entropy set  $\check{\mathcal{Z}}$  to calibrate key parameters of the convex set  $\bar{\mathcal{Z}}$ . By extending estimates from Hansen and Sargent (2010) and Hansen et al. (2008), respectively, section 6 calculates key objects for two quantitative versions of a baseline model together with convex sets of alternative models that concern a robust investor and a robust planner. Section 7 uses one of our quantitative models to compare the convex set  $\bar{\mathcal{Z}}$  with two other sets featured in Anderson et al. (2003) and Hansen and Sargent (2010), the set  $\check{\mathcal{Z}}$  based on Chernoff entropy, and a set  $\mathcal{Z}^*$  based on relative entropy. Section 8 constructs a

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<sup>3</sup>That object also played a central role in the analysis of Hansen and Sargent (2010).

<sup>4</sup>By way of contrast, models in which a representative investor’s consumption process has innovations with stochastic volatility introduce new risk exposures in the form of the shocks to volatilities. Their presence induces time variation in equilibrium compensations for exposures to shocks that include both the stochastic volatility shocks as well as the “original” shocks whose volatilities now move.

recursive representation of a competitive equilibrium of an economy with a representative investor.<sup>5</sup> Then it links the worst-case model that emerges from a robust planning problem to equilibrium compensations that the representative investor earns for bearing model uncertainty. Section 9 describes a term structure of market prices of uncertainty. Our equilibrium features an exponential quadratic stochastic discount factor whose mathematical form closely resembles one that Ang and Piazzesi (2003) used in a no-arbitrage statistical model of asset prices and macroeconomic variables. Section 10 explains why our approach differs in conceptually and practically important ways from one advocated by Epstein and Schneider (2003). Section 11 offers concluding remarks. Six technical appendices include formulas that we used to create quantitative examples.

## 2 Models and perturbations

This section describes nonnegative martingales that perturb a probability model. Section 3 then describes how we use a family of parametric alternative to the baseline model to form a convex set of martingales that in later sections we use to pose robust decision problems.

### 2.1 Mathematical framework

For concreteness, we use a specific benchmark model here and then in section 3 a specific family of parametric alternatives. A consumer cares about a stochastic process  $C \doteq \{C_t : t \geq 0\}$  that he describes by the baseline model<sup>6</sup>

$$\begin{aligned} d \log C_t &= (.01) \left( \hat{\mu} + \hat{\beta} X_t \right) dt + (.01) \alpha \cdot dW_t \\ dX_t &= \hat{\phi} dt - \hat{\kappa} X_t dt + \sigma \cdot dW_t, \end{aligned} \tag{1}$$

where  $W$  is a multivariate Brownian motion,  $X$  is a scalar process initialized at a random variable  $X_0$ , and  $\hat{\mu} + \hat{\beta} X_t + \frac{.01}{2} |\alpha|^2$  is the time  $t$  growth rate of  $C$  expressed as a percent. The sextet  $(\hat{\mu}, \hat{\phi}, \hat{\beta}, \hat{\kappa}, \alpha, \sigma)$  characterizes the baseline model.<sup>7</sup>

Because he does not trust the baseline model, the consumer also cares about  $C$  under probability models obtained by multiplying probabilities associated with the baseline model

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<sup>5</sup>The representative investor stands in for “the market”.

<sup>6</sup>We let  $X$  denote a stochastic process,  $X_t$  the process at time  $t$ , and  $x$  a realized value of the process.

<sup>7</sup>In earlier papers, we sometimes referred to what we now call the baseline model as the decision maker’s approximating model or benchmark model.

(1) by likelihood ratios. Following Hansen et al. (2006), we represent a likelihood ratio by a positive martingale  $Z^H$  with respect to the baseline model that satisfies<sup>8</sup>

$$dZ_t^H = Z_t^H H_t \cdot dW_t \quad (2)$$

or

$$d \log Z_t^H = H_t \cdot dW_t - \frac{1}{2} |H_t|^2 dt, \quad (3)$$

where  $H$  is progressively measurable with respect to the filtration  $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$  associated with the Brownian motion  $W$ . We allow  $\int_0^t |H_u|^2 du$  to be infinite with positive probability and adopt the convention that  $Z_t^H$  is zero when this happens. In the event that

$$\int_0^t |H_u|^2 du < \infty \quad (4)$$

with probability one, the stochastic integral  $\int_0^t H_u \cdot dW_u$  is an appropriate probability limit. Imposing the initial condition  $Z_0^H = 1$ , we express the solution of a stochastic differential equation (2) as the stochastic exponential:

$$Z_t^H = \exp \left( \int_0^t H_u \cdot dW_u - \frac{1}{2} \int_0^t |H_u|^2 du \right). \quad (5)$$

The stochastic exponential is a local martingale, but not necessarily a martingale.<sup>9</sup>

**Definition 2.1.**  $\mathcal{Z}$  denotes the set of all martingales  $Z^H$  constructed as a stochastic exponential via representation (5) with some progressively measurable  $H$  relative to  $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$  and satisfying (4).

Starting from the probability distribution associated with the baseline model (1),  $H$  represents another probability distribution conditioned on  $\mathcal{F}_0$ . To construct this alternative probability measure, take any  $\mathcal{F}_t$ -measurable random variable  $Y_t$  and multiply it by  $Z_t^H$  before computing expectations conditioned on  $X_0$ . Associated with  $H$  are probabilities defined by

$$E^H [B_t | \mathcal{F}_0] = E [Z_t^H B_t | \mathcal{F}_0]$$

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<sup>8</sup>James (1992), Chen and Epstein (2002), and Hansen et al. (2006) used this representation.

<sup>9</sup>While there are sufficient conditions for the stochastic exponential to be a martingale such as Kazamaki's or Novikov's, they are not convenient here. Instead we will verify that an extremum does indeed result in a martingale.

for any  $t \geq 0$  and any bounded  $\mathcal{F}_t$ -measurable random variable  $B_t$ . Thus, the positive random variable  $Z_t^H$  acts as a Radon-Nikodym derivative for the date  $t$  conditional expectation operator  $E^H[\cdot | X_0]$ . The martingale property of the process  $Z^H$  ensures that conditional expectations operators for different dates satisfy a Law of Iterated Expectations. Moreover, under the  $Z^H$  probability measure,  $\int_0^t |H_u|^2 du$  is finite with probability one for each  $T$ .

Although  $W$  is a standard Brownian motion under the baseline model, under the alternative  $H$  model it has increments

$$dW_t = H_t dt + dW_t^H, \tag{6}$$

where  $W^H$  is a now standard Brownian motion. Moreover, under the  $Z^H$  probability measure,  $\int_0^t |H_u|^2 du$  is finite with probability one for each  $t$ . While (3) expresses the evolution of  $\log Z^H$  in terms of increment  $dW$ , the evolution in terms of  $dW^H$  is:

$$d \log Z_t^H = H_t \cdot dW_t^H + \frac{1}{2} |H_t|^2 dt. \tag{7}$$

In light of (7), we can write model (1) as:

$$\begin{aligned} d \log C_t &= (.01) \left( \hat{\mu} + \hat{\beta} X_t \right) dt + (.01) \alpha \cdot H_t dt + (.01) \alpha \cdot dW_t^H \\ dX_t &= \hat{\phi} dt - \hat{\kappa} X_t dt + \sigma \cdot H_t dt + \sigma \cdot dW_t^H, \end{aligned}$$

which implies that  $C$  has a (local) growth rate  $\hat{\mu} + \hat{\beta} X_t + \alpha \cdot H_t + \frac{.01}{2} |\alpha|^2$  under the  $H$  model.<sup>10</sup>

### 3 Quantifying probability distortions

We retreat cautiously from rational expectations by assuming that because agents bear model uncertainty they want to evaluate outcomes under alternative models that are difficult to distinguish on the basis of a finite history of data.<sup>11</sup> We formulate families of models by using measures of statistical discrepancy around a baseline model. By characterizing alternatives to a baseline model in terms of likelihood ratio processes sufficiently close to unity, we include a vast set of models with nonlinearities, time-varying parameters, history

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<sup>10</sup>The growth rate includes a multiplication by 100 that offsets one of the .01's.

<sup>11</sup>Rational expectations assumes that something (an infinite history of data?) has eliminated decision makers' model uncertainty.

dependence, and other difficult-to-learn features.

### 3.1 Measures of discrepancy

We construct discounted versions of statistical discrepancy measures. Consider a martingale  $Z^H$  in  $\mathcal{Z}$  and a twice differentiable convex function  $f(z)$  defined for  $z \geq 0$  and satisfying  $f(1) = 0$  and  $f''(1) = 1$ . By Jensen's Inequality

$$E [f (Z_t^H) | \mathcal{F}_0] \geq 0, \tag{8}$$

which holds with equality when  $Z_t^H$  is one with probability one conditioned on  $\mathcal{F}_0$ . Examples of  $f$  include a family suggested by Cressie and Read (1984):

$$f(z) = \frac{1}{r(1+r)} (z^{r+1} - 1) \tag{9}$$

for alternative choices of  $r$ .

Except when  $r = -\frac{1}{2}$ , these discrepancy measures are not metrics. Nevertheless, they can be used to define neighborhoods of a given model. The class of discrepancies defined by (9) contains interesting special cases. When  $r = -1$ , the discrepancy defined by (9) is the negative of the expected log-likelihood. When  $r = -\frac{1}{2}$ , we get the Hellinger discrepancy. The discrepancy  $E [f (Z_t^H)]$  for the  $r = 0$  case,

$$f(z) = z \log z,$$

is called relative entropy and is of especial interest to us. This discrepancy is an expected log likelihood ratio with respect to the  $z$ -perturbed model.

Using the Cressie and Read specification (9), the process  $\{f(Z_t^H)\}$  defined in (5) and (9) evolves as an Ito process with drift (also called a local mean)

$$\mu_t = (Z_t^H)^{1+r} \frac{1}{2} |H_t|^2.$$

Write the conditional mean of  $f(Z_t^H)$  in terms of the history of local means

$$E [f(Z_t^H) | X_0] = E \left( \int_0^t \mu_u du | X_0 \right).$$

For the time being, we will simply impose this equality and perform some calculations.<sup>12</sup> Later we will refer to a more rigorous justification in the context of an important special case in which  $r = 0$ .

To construct a discrepancy measure for decision problems, we discount  $f$  and integrate across time:

$$\begin{aligned}
\Delta(Z^H|x) &= \delta \int_0^\infty \exp(-\delta t) E \left[ f(Z_t^H) \mid X_0 = x \right] dt \\
&= \int_0^\infty \exp(-\delta t) E(\mu_t \mid X_0 = x) dt \\
&= \frac{1}{2} \int_0^\infty \exp(-\delta t) E \left[ (Z_t^H)^{1+r} |H_t|^2 \mid X_0 = x \right] dt \\
&\geq 0,
\end{aligned} \tag{10}$$

where the second equality follows from integration by parts. Formula (10) quantifies how a martingale  $Z^H$  distorts baseline model probabilities. We call the  $r = 0$  formulation discounted relative entropy after Hansen and Sargent (2001). Hansen et al. (2006) provide a formal justification for (10) when  $r = 0$ .<sup>13</sup> The  $r = 0$  case is the most tractable one to use in decision problems. Later we will compare sets of models associated with several measures of statistical discrepancy.

### 3.2 Tilting the discrepancy measure

The statistical discrepancy measure defined in (10) allows a decision maker to include a large class of statistically similar models, but it is not the most convenient way to include particular parametric models. To do that it is better to tilt the discrepancy measure in the following way. Let

$$\hat{f}(z, \zeta) = f(z) - \zeta z. \tag{11}$$

Evidently,  $\hat{f}$  is strictly convex in  $z$ . Construct a process

$$\Xi_t = \frac{1}{2} \int_0^t \xi(X_u) du, \tag{12}$$

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<sup>12</sup>There exist a variety of sufficient conditions that justify this equality.

<sup>13</sup>See Claims 6.1 and 6.2.

where

$$\xi(x) \doteq \xi_0 + 2\xi_1 x + \xi_2 x^2 \geq 0. \quad (13)$$

In section 3.3, we exhibit  $\xi(x)$  functions that represent specific alternative models.

As an initial step to form a tractable set of models for discounted dynamic control problems, we set  $r = 0$  and  $\delta$  to the same rate that discounts instantaneous utilities. This yields:

$$\widehat{\Delta}(Z^H|x) = \frac{1}{2} \int_0^\infty \exp(-\delta t) E \left[ (Z_t^H) [ |H_t|^2 - \xi(X_t) ] \middle| X_0 = x \right] dt.$$

The (random) function  $\widehat{\Delta}$  is convex in the martingale  $Z^H$ , implying that

$$\widehat{\mathcal{Z}}(x) = \left\{ Z^H \in \mathcal{Z} : \widehat{\Delta}(Z^H|x) \leq 0 \right\} \quad (14)$$

is a convex set of martingales that is not empty because it contains  $Z^0 = 1$ . We are interested in the non degenerate case in which  $\xi(x)$  satisfies

**Condition 3.1.**

$$\frac{1}{2} \int_0^\infty \exp(-\delta t) E \left[ \xi(X_t) \middle| X_0 = x \right] dt > 0.$$

One way to satisfy condition 3.1 is to set  $\xi(x) > 0$  for all  $x$ . There are other ways too. In some of our applications, we set  $\xi(0) = 0$  but still satisfy condition 3.1.

So far, we have used a convex function to build a convex set of martingales, but have not constructed an associated tilted discrepancy. We could construct a discrepancy by minimizing  $\widehat{\Delta}(Z^H|x)$  by choice of  $H$ . Call the minimized objective  $\Delta^*(x)$ . Since  $H = 0$  is a feasible choice,  $\Delta^*(x)$  satisfies

$$\Delta^*(x) \leq -\frac{1}{2} \int_0^\infty \exp(-\delta t) E \left[ \xi(X_t) \middle| X_0 = x \right] dt.$$

The resulting discrepancy is :

$$\widetilde{\Delta}(Z^H|x) \doteq \widehat{\Delta}(Z^H|x) - \Delta^*(x).$$

Tilting means that this discrepancy is zero at the minimizing choice of  $H$  that attains  $\Delta^*(x)$ , not at  $H = 0$ . The convex set is the object that matters in our analysis, not the discrepancy or the threshold used to construct the set. That means that to carry out the analysis that follows, it is not necessary for us to minimize  $\widehat{\Delta}(Z^H|x)$ .

### 3.3 Alternative parametric models

We can include specific alternative models within the set  $\widehat{\mathcal{Z}}(x)$  defined in (14) by choosing  $\xi \geq 0$  appropriately. Notice that  $\widehat{\Delta}(1|x) = 0$ . If

$$|H_t|^2 \leq \xi(X_t), \quad (15)$$

then

$$\widehat{\Delta}(Z^H|x) \leq 0, \quad (16)$$

so that the corresponding  $Z^H \in \widehat{\mathcal{Z}}(x)$ .

We now construct  $\xi$  by specifying parametric alternatives to our baseline model. Consider alternatives to baseline model (1) of the following form:

$$\begin{aligned} d \log C_t &= .01 (\mu + \beta X_t) dt + .01 \alpha \cdot dW_t^H \\ dX_t &= \phi dt - \kappa X_t dt + \sigma \cdot dW_t^H, \end{aligned} \quad (17)$$

where  $W^H$  is a Brownian motion and (6) continues to describe the relationship between the processes  $W$  and  $W^H$ . Here  $(\hat{\mu}, \hat{\phi}, \hat{\beta}, \hat{\kappa})$  are parameters of the baseline model (1),  $(\mu, \phi, \beta, \kappa)$  are parameters of model (17), and  $(\alpha, \sigma)$  are parameters common to both models. Not all specifications included in the set of models that concern the decision maker will be of this form. However, a worst-case model that plays an important role in shaping uncertainty prices will be.

Suppose that we want drift distortions  $H$  for  $W$  to represent models in the parametric class defined by (17). We can express model (17) in terms of our section 2.1 structure by setting

$$H_t = \eta(X_t) \equiv \eta_0 + \eta_1 X_t$$

and using (1), (6), and (17) to deduce the following restrictions on  $\eta_0$  and  $\eta_1$  as functions of  $(\mu, \phi, \beta, \kappa)$ :

$$\begin{aligned} \begin{bmatrix} \alpha' \\ \sigma' \end{bmatrix} \eta_0 &= \begin{bmatrix} \mu - \hat{\mu} \\ \phi - \hat{\phi} \end{bmatrix} \\ \begin{bmatrix} \alpha' \\ \sigma' \end{bmatrix} \eta_1 &= \begin{bmatrix} \beta - \hat{\beta} \\ \hat{\kappa} - \kappa \end{bmatrix}. \end{aligned} \quad (18)$$

By imposing restrictions (18), we can find pairs  $(\eta_0, \eta_1)$  that represent members of a class

of models having parametric form (17).

We further restrict a family of parametric models by using the nonnegative quadratic function  $\xi(x)$  defined in (13) to express a collection of alternatives to a baseline model. For example, to induce  $\xi$  to include a prespecified  $\tilde{\kappa}$ , form

$$\xi(x) \doteq \xi_0 + 2\xi_1 x + \xi_2 x^2 = \frac{1}{|\sigma|^2} (\tilde{\kappa} - \hat{\kappa})^2 x^2.$$

With this choice of  $\xi$ ,  $\kappa = \tilde{\kappa}$  and  $\kappa = 2\hat{\kappa} - \tilde{\kappa}$  become alternative parameter configurations defining models included in a neighborhood of the baseline model. More generally, we can set  $(\tilde{\mu}, \tilde{\phi}, \tilde{\beta}, \tilde{\kappa})$  and then compute  $\tilde{\eta}_0$  and  $\tilde{\eta}_1$  by solving a counterpart to (18). Then

$$\xi(x) = |\tilde{\eta}_0 + \tilde{\eta}_1 x|^2.$$

**Definition 3.2.**  $\mathcal{Z}^o$  is a set of martingales  $Z^H$  constructed by (i) selecting a pair  $(\eta_0, \eta_1)$  that satisfies (18) for some  $(\mu, \phi, \beta, \kappa)$ , (ii) pinning down an associated  $H_t = \eta_0 + \eta_1 X_t$ , (iii) constructing an implied martingale  $Z^H$  via (2), and (iv) restricting  $|H_t|^2 \leq \xi(X_t)$ .

By in effect setting  $\xi$  to a constant, Hansen and Sargent (2001) and Hansen et al. (2006) included only parametric alternatives that alter  $\hat{\mu}$  and  $\hat{\phi}$ .

Next we shall construct a larger convex set of martingales that contains  $\mathcal{Z}^o$  as well as models that depart from the parametric structure (17). Applying Bayes' rule to mixture models provides one motivation for being interested in a larger set.

### 3.4 Convexity and Bayes rule

Convexity of a set of martingales allows us to use Bayes rule to update probabilities and motivates our interest in weighted averages of martingales. We now apply Bayes' rule to two models characterized by  $H^1$  and  $H^2$  for which  $Z^{H^1}$  and  $Z^{H^2}$  are in  $\hat{\mathcal{Z}}(x)$ ; for example,  $H^1$  and  $H^2$  might both be affine functions of  $X$  and correspond to two time invariant parameter models. Let  $\pi_0^1$  be a prior probability on model  $H^1$  and  $\pi_0^2 = 1 - \pi_0^1$  be a prior on model  $H^2$ . A martingale

$$M = \pi_0^1 M^{H^1} + \pi_0^2 M^{H^2}$$

corresponds to a mixture of the  $H^1$  and  $H^2$  models. Like  $M^{H^1}$  and  $M^{H^2}$ , the mathematical expectation of  $M$  conditioned on date zero information equals unity. The law of motion

for  $M$  is

$$\begin{aligned} dM_t &= \pi_0^1 dM_t^{H_1} + \pi_0^2 dM_t^{H_2} \\ &= \pi_0^1 M_t^{H_1} H_t^1 \cdot dW_t + \pi_0^2 M_t^{H_2} H_t^2 \cdot dW_t \\ &= M_t (\pi_t^1 H_t^1 + \pi_t^2 H_t^2) \cdot dW_t \end{aligned}$$

where  $\pi_t^j$  is the date  $t$  posterior

$$\pi_t^j = \frac{\pi_0^j M_t^{H_j}}{M_t}.$$

The drift distortion

$$H_t = \pi_t^1 H_t^1 + \pi_t^2 H_t^2$$

associated with the mixture model is typically not an affine function of  $X$  even when both  $H_1$  and  $H_2$  are.

To construct a larger convex set of martingales that contains  $\mathcal{Z}^o$  as well as models that depart from the parametric structure (17) we define:

**Definition 3.3.**

$$\tilde{\mathcal{Z}} = \{Z^H \in \mathcal{Z} : |H_t|^2 \leq \xi(X_t)\} \tag{19}$$

for all  $t$  with probability one.

While martingales  $Z^H$  in  $\mathcal{Z}^o$  represent models with time-invariant parameters, martingales  $Z^H$  in  $\tilde{\mathcal{Z}}$  represent models whose parameters vary over time. We want also to include models that statistically are arbitrarily close to models implied by martingales in  $\tilde{\mathcal{Z}}$ , many of which don't satisfy an instant-by-instant constraint like the inequality on the right side of (3.3). For this reason, we construct a larger family of martingales by replacing the instant-by-instant inequality in the definition (19) of  $\tilde{\mathcal{Z}}$  with restrictions cast in terms of intertemporal probability-weighted averages of  $|H_t|^2$  that allow the intertemporal trade-offs associated with likelihood-ratio statistical model discrimination criteria.<sup>14</sup>

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<sup>14</sup>The ambiguity averse decision maker of Chen and Epstein (2002) considers a set of models characterized by martingales that are generated by  $h$  processes that satisfy instant-by-instant constraints on  $h$  like (19). Anderson et al. (1998) also explored consequences of this type of constraint without the state dependence in  $\xi$ .

### 3.5 Relative entropy neighborhoods

Building on Petersen et al. (2000), we constructed the set  $\widehat{\mathcal{Z}}(x)$  to include relative entropy neighborhoods of alternative parametric models as well as of a baseline model. This differentiates our approach from work by James (1992), Hansen and Sargent (2001), and Hansen et al. (2006), who constructed a set of models as a relative entropy neighborhood of a baseline model only.

To construct relative entropy with respect to the probability model affiliated with a martingale  $Z^{\widehat{H}}$ , we use a likelihood ratio  $\log Z_t^H - \log Z_t^{\widehat{H}}$  with respect to the  $Z_t^{\widehat{H}}$  model rather than a likelihood ratio  $\log Z_t^H$  with respect to the original baseline model to arrive at

$$\begin{aligned} \Delta \left( Z^H; Z^{\widehat{H}} | x \right) &= \delta \int_0^\infty \exp(-\delta t) E \left[ Z_t^H \left( \log Z_t^H - \log Z_t^{\widehat{H}} \right) \middle| X_0 = x \right] dt \\ &= \frac{1}{2} \int_0^\infty \exp(-\delta t) E \left[ Z_t^H | H_t - \widehat{H}_t |^2 \middle| X_0 = x \right] dt. \end{aligned}$$

We have recycled the notation  $\Delta$  to include two martingales as arguments. Implicitly the second argument  $Z^{\widehat{H}}$  was set to one in our previous analysis.

Given this measure, we form relative entropy neighborhoods about a martingale  $Z^H$ :

$$\{ Z^H : \Delta \left( Z^H; Z^{\widehat{H}} | x \right) < \epsilon \}$$

for alternative  $\epsilon > 0$ . Small neighborhoods contain martingales that imply probability measures that are statistically close to the measure implied by  $Z^{\widehat{H}}$ . The set  $\widehat{\mathcal{Z}}(x)$  includes small neighborhoods of the martingales in  $\mathcal{Z}^\circ$  constructed from parametric alternatives.

**Proposition 3.4.** *Suppose that*

$$\widehat{H}_t = \widehat{\eta}_0 + \widehat{\eta}_1 X_t.$$

where

$$\widehat{\Delta} \left( Z^{\widehat{H}} | x \right) < 0.$$

*There exists an  $\epsilon > 0$  such that the set  $\widehat{\mathcal{Z}}(x)$  defined in (14) contains the relative entropy neighborhood  $\{ Z^H \in \mathcal{Z} : \Delta(Z^H; Z^{\widehat{H}} | x) < \epsilon \}$ .*

We prove this result in Appendix A. Our inclusion of relative entropy neighborhoods of parameter models shapes our Section 10 analysis of a proposal by Epstein and Schneider

(2003) to expand sets of models enough to make them satisfy a property that they call rectangularity.

### 3.6 Averaging over an initial state distribution

So far we have conditioned on the initial state  $x$ , as was often done in previous research. In this paper, we will have cause to average of the this state using a probability distribution  $Q$  over initial states:

$$\bar{\Delta}(Z^H) \equiv \int \hat{\Delta}(Z^H|x)Q(dx). \tag{20}$$

The probability distribution  $Q$  can be a point mass at some initial state or it can be a stationary probability distribution of  $X$  under the baseline model. Later we suggest yet other ways to choose  $Q$ .

We construct a convex set  $\bar{\mathcal{Z}}$  of martingales  $Z^H$ :

**Definition 3.5.**

$$\bar{\mathcal{Z}} = \{Z^H \in \mathcal{Z} : \bar{\Delta}(Z^H) \leq 0\}. \tag{21}$$

Choosing  $\xi(x)$  appropriately makes  $\bar{\mathcal{Z}}$  include martingales that are associated with parametric probability models like those studied in this section as well as many other specifications that are not included in a parametric family and that may not be Markovian.

In section 8, we pose a robust decision problem in which  $\bar{\mathcal{Z}}$  serves as a set of positive martingales multiplying baseline model probabilities. Evidently, both the baseline model (1) and the alternative models captured by the quadratic function  $\xi(x)$  play important roles in constructing the set  $\bar{\mathcal{Z}}$ . The quadratic function  $\xi(x)$  also shapes another convex set  $\hat{\mathcal{Z}}$  that we use to calibrate plausible sets of models in section 4.

The following set inclusions summarize our constructions so far:

set	$\mathcal{Z}^o$	$\subset$	$\tilde{\mathcal{Z}}$	$\subset$	$\bar{\mathcal{Z}}$
<b>contains</b>	parametric alternatives		time-varying parameters		relative-entropy neighborhoods

## 4 Chernoff entropy

The set  $\bar{\mathcal{Z}}$  defined in (21) is tractable to use in decision problems, but an alternative set  $\tilde{\mathcal{Z}}$  emerges from studying how, by disguising probability distortions of a baseline model, Brownian motions make it challenging to distinguish models statistically. To construct

$\check{Z}$  we use Chernoff (1952) entropy, which differs from discounted relative entropy. While Chernoff entropy's connection to a statistical decision problem makes it interesting, it is less tractable than relative entropy for formulating robust decision problems.

In the spirit of Anderson et al. (2003), we use Chernoff (1952) entropy to measure a distortion  $Z$  to a baseline model. Think of a pairwise model selection problem that statistically compares the baseline model (1) with a model generated by the martingale  $Z^H$ . The logarithm of the martingale evolves according to

$$d \log Z_t^H = -\frac{1}{2}|H_t|^2 dt + H_t \cdot dW_t.$$

Consider a statistical model selection rule based on a data history of length  $t$  that takes the form  $\log Z_t^H \geq \log \tau$ , where  $Z_t^H$  is the likelihood ratio associated with the alternative model for a sample size  $t$ . Consider a model selection rule that incorrectly chooses the alternative model when the baseline model governs the data. We bound the probability of this outcome by using an argument from large deviations theory that starts from

$$\mathbf{1}_{\{\log Z_t^H \geq \tau\}} = \mathbf{1}_{\{-s\tau + s \log Z_t^H \geq 0\}} = \mathbf{1}_{\{\exp(-s\tau)(Z_t^H)^s \geq 1\}} \leq \exp(-s\tau)(Z_t^H)^s.$$

This inequality holds for  $0 \leq s \leq 1$ . The expectation of the term on the left side equals the probability of mistakenly selecting the alternative model when the data are a sample of size  $t$  generated by the baseline model. We bound this mistake probability for large  $t$  by following Donsker and Varadhan (1976) and Newman and Stuck (1979) and studying

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E [\exp(-s\tau) (Z_t^H)^s | X_0 = x] = \limsup_{t \rightarrow \infty} \frac{1}{t} \log E [(Z_t^H)^s | X_0 = x]$$

for alternative choices of  $r$ . The threshold  $\tau$  does not affect this limit. Furthermore, the limit is often independent of the initial state  $X_0 = x$ . To get the best bound, we compute

$$\inf_{0 \leq s \leq 1} \limsup_{t \rightarrow \infty} \frac{1}{t} \log E [(Z_t^H)^s | X_0 = x],$$

a limit that is typically negative because mistake probabilities decay with sample size. Chernoff entropy is then

$$\chi(Z^H, x) = - \inf_{0 \leq s \leq 1} \limsup_{t \rightarrow \infty} \frac{1}{t} \log E [(Z_t^H)^s | X_0 = x]. \quad (22)$$

Appendix B describes how to compute Chernoff entropy for competing parametric models. These calculations produce no dependence on the initial state  $x$ . This is to be expected for models with sufficient stationarity because discounting plays no role in constructing Chernoff entropy.

Setting  $\chi(Z^H, x) = 0$  would mean including only alternative models that cannot be distinguished on the basis of histories of infinite length. In effect, that is what is done in papers on self-confirming equilibria that extend the rational expectations equilibrium concept to allow probability models to be wrong off equilibrium paths, i.e., for events that do not occur infinitely often.<sup>15</sup> Because we want to include alternative parametric probability models associated with the martingales in  $\mathcal{Z}^+$ , we entertain positive values of  $\chi(Z^H, x)$ . Our decision theory differs from that typically used for self confirming equilibria because our decision makers formally acknowledge model uncertainty and adjust their decisions accordingly.

To help interpret  $\chi(Z^H, x)$ , consider the following argument. If the decay rate of mistake probabilities were constant, then mistake probabilities for two sample sizes  $T_i, i = 1, 2$ , would be

$$\text{mistake probability}_i = \frac{1}{2} \exp(-T_i \bar{\rho})$$

for  $\bar{\rho} = \chi(Z^H, x)$ . We define a ‘half-life’ as an increase in sample size  $T_2 - T_1 > 0$  that multiplies the mistake probability by a factor of one half:

$$\frac{1}{2} = \frac{\text{mistake probability}_2}{\text{mistake probability}_1} = \frac{\exp(-T_2 \bar{\rho})}{\exp(-T_1 \bar{\rho})}.$$

So the half-life is approximately

$$T_2 - T_1 = \frac{\log 2}{\bar{\rho}}. \tag{23}$$

The preceding back-of-the-envelope calculation justifies the detection error bound computed by Anderson et al. (2003). The bound on the decay rate should be interpreted cautiously because it is constant but the actual decay rate is not. Furthermore, the pairwise comparison oversimplifies the challenge truly facing a robust decision maker, which is statistically to discriminate among *multiple* models.

We can make a symmetrical calculation that reverses the roles of the two models and instead conditions on the perturbed model implied by martingale  $Z^H$ . It is straightforward to show that the limiting rate remains the same. Thus, when we select a model by com-

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<sup>15</sup>See Sargent (1999) and Fudenberg and Levine (2009).

paring a log likelihood ratio to a constant threshold, the two types of mistakes share the same asymptotic decay rate.

Chernoff (1952) entropy is related to the Cressie and Read (1984) discrepancy measures discussed in section 3. Notice that

$$\frac{1}{(s-1)s} (E [(Z_t^H)^s | X_0 = x] - 1)$$

is the special case  $r = 1 + s$  of the Cressie and Read family of discrepancies. When Chernoff entropy is strictly positive,  $E [(Z_t^H)^s | X_0 = x]$  tends to zero for some  $0 < s < 1$  as  $t$  tends to infinity. Thus, the corresponding Cressie and Read discrepancy converges to a positive number  $\frac{1}{s(1-s)}$  at an exponential rate as  $t$  gets large. The Chernoff entropy measure looks across these rates to obtain the best bound on mistake probabilities.

Our second convex set is a ball formed using Chernoff entropy (22).

**Definition 4.1.**

$$\check{\mathcal{Z}} = \{Z^H \in \mathcal{Z} : \chi(Z^H; x) \leq \check{\rho}\}. \tag{24}$$

We attain a specified half-life by adjusting the radius  $\check{\rho}$  of the ball.

## 5 Robust planning problem

To illustrate our formulation of sets of models, we deliberately consider a simple setup. We posit an exogenous consumption process and deduce the implied shadow prices of risk and uncertainty. Richer models would include production, capital accumulation, and distinct classes of decision makers with differential access to financial markets. Before adding such complications, we want to understand uncertainty in our simple environment. To construct uncertainty prices, we compute worst-case probabilities associated with martingales in the set  $\bar{\mathcal{Z}}$ . To do this, we solve a continuous-time optimization problem that leads to a Hamilton-Jacobi-Bellman equation.

In subsections 5.1 and 5.2, we formulate a robust planning problem for an economy with a representative consumer having an instantaneous utility function that is logarithmic in consumption. Associated with the worst-case probability from the robust planning problem is a greatest lower bound on expected discounted utility over the family  $\bar{\mathcal{Z}}$  of alternative probability distributions. In subsection 5.3, we represent the worst-case probability with a drift distortion to the multivariate Brownian motion in the baseline model (1). We use

that drift distortion to guide the calibration of parameters that determine the size of the set  $\bar{\mathcal{Z}}$ . In section 8, we show how that same worst-case drift distortion appears in a recursive representation of competitive equilibrium prices for an economy with a representative investor that participates in decentralized security markets. We deduce uncertainty prices and connect them to the worst-case drift distortion from our robust planning problem.

## 5.1 Parametric alternatives

Consider the baseline consumption process for  $C$  specified in equation (1). Guess a value function  $v(x, \theta, \gamma) + \log c$  where  $c$  is a realized value of the consumption process. We use the family of martingales in the set  $\bar{\mathcal{Z}}$  to represent alternative probabilities. We define the function  $\xi$  to be

$$\xi(x, \gamma) = \begin{cases} \gamma \hat{\xi}(x) & \text{or} \\ \gamma + \hat{\xi}(x), \end{cases}$$

where  $\hat{\xi}(x)$  is a pre-specified quadratic function and  $\gamma$  is temporarily an arbitrary parameter. Whether we choose an additive or a multiplicative scaling will depend on the application at hand. Eventually, we will calibrate  $\gamma$  to produce a model detection probability half-life defined in terms of Chernoff entropy. By bringing in other sources of information to help with calibration, we could allow both additive and multiplicative adjustments.

Let  $H$  be a (progressively measurable) control process that influences the evolution of  $Z^H$ :

$$\begin{aligned} dX_t &= \hat{\phi} dt - \hat{\kappa} X_t dt + \sigma \cdot H_t dt + \sigma \cdot dW_t \\ dZ_t^H &= Z_t^H H_t \cdot dW_t. \end{aligned}$$

The planner chooses  $H$  to minimize

$$\int_0^\infty \exp(-\delta t) E \left[ (Z_t^H) \log C_t \middle| X_0 = x \right] Q(dx)$$

subject to

$$\bar{\Delta}(Z^H) \leq 0. \tag{25}$$

Let  $\theta$  be a multiplier on constraint (25). In subsection 5.2, for a given  $(\theta, \gamma)$  we construct a recursive representation of a worst-case drift distortion. The distribution  $Q$  of the initial state plays no role in this initial calculation.

## 5.2 Recursive representation of worst-case drift distortion

We posit that the value function that solves the preceding control problem takes the form  $zv(x, \theta, \gamma)$ . Given  $(\gamma, \theta)$ , we compute  $H$  by solving Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \min_h -\delta v(x, \theta, \gamma) + (.01)(\hat{\mu} + \hat{\beta}x) + v'(x, \theta)(\hat{\phi} - \hat{\kappa}x) + \frac{1}{2}|\sigma|^2 v''(x, \theta, \gamma) + (.01)\alpha \cdot h + v'(x, \theta, \gamma)\sigma \cdot h + \frac{\theta}{2}|h|^2 - \frac{\theta}{2}\xi(x, \gamma). \quad (26)$$

Alternatively, we can view this HJB equation as associated with a *weak* solution to an optimization problem in which a drift distortion to a Brownian motion is the control process that holds under the probability distribution implied by  $Z^H$ . In this case, we would push the martingale into the background and instead make distributional statements about the implied Brownian motion.

The solution  $v$  of HJB equation (26) is derived in Appendix C and is quadratic in  $x$ :

$$v(x, \theta, \gamma) = -\frac{1}{2} [v_2(\theta, \gamma)x^2 + 2v_1(\theta, \gamma)x + v_0(\theta, \gamma)],$$

which implies that the minimizing  $h$  is affine in  $x$ :

$$h^* = \eta^*(x, \theta, \gamma) = -\frac{1}{\theta} [.01\alpha - \sigma v_2(\theta, \gamma)x - \sigma v_1(\theta, \gamma)]. \quad (27)$$

## 5.3 Determining $\theta$ and $\gamma$

To set  $\theta$ , we must decide how to model the initial state. Papers by Petersen et al. (2000) and Hansen et al. (2006) effectively conditioned on an initial state by setting  $Q$  to be a mass point at a single value of  $x$ . Here we suggest an alternative approach.

Under the baseline model,  $X$  has a stationary distribution  $\hat{Q}$  having density  $\hat{q}$  with mean  $\hat{\phi}/\hat{\kappa}$  and variance  $|\sigma|^2/(2\hat{\kappa})$ . Let  $g$  be a density that we use to alter  $\hat{Q}$ . Formally, we take a Borel measurable function  $g > 0$  satisfying  $\int g(x)\hat{Q}(dx) = 1$ . Let  $\mathcal{G}$  denote the collection of all such functions  $g(x)$ .

Consider any  $g \in \mathcal{G}$ . Introduce a nonnegative parameter  $\varsigma$  that we temporarily take as fixed, given  $(\theta, \gamma)$ . To make a conservative adjustment of the probability measure over the

initial state, compute  $g$  by solving

$$\min_{g \in \mathcal{G}} \int v(x, \theta, \gamma) g(x) \hat{Q}(dx) + \theta \left[ \int \log g(x) g(x) \hat{Q}(dx) - \varsigma \right]. \quad (28)$$

The minimizing  $g$  is an exponentially tilted density

$$\tilde{g}(x, \theta, \gamma) \propto \exp \left[ -\frac{1}{\theta} v(x, \theta, \gamma) \right] \quad (29)$$

that is evidently normal with precision

$$\omega(\theta, \gamma) = \frac{2\hat{\kappa}}{|\sigma|^2} - \frac{1}{\theta} v_2(\theta, \gamma)$$

and mean

$$\nu(\theta, \gamma) = \frac{1}{\omega} \left[ \frac{2\hat{\phi}}{|\sigma|^2} + \frac{1}{\theta} v_1(\theta, \gamma) \right].$$

To compute  $\theta$ , we substitute  $\tilde{g}$  into (28) to obtain the maximand in the following problem:

$$\max_{\theta > 0} -\theta \log \left[ \int \exp \left( -\frac{1}{\theta} v(x, \theta, \gamma) \right) \hat{Q}(dx) \right] - \theta \varsigma. \quad (30)$$

This maximization problem is concave in  $\theta$  given  $\varsigma$ . We use this problem to impose the restriction

$$\int \hat{\Delta}(Z^H|x) g(x) \hat{Q}(dx) + \left[ \int \log g(x) g(x) \hat{Q}(dx) - \varsigma \right] \leq 0.$$

Since the second term was not used in our construction of the set  $\bar{\mathcal{Z}}$ , instead of specifying  $\varsigma$  *a priori*, we choose it to satisfy

$$\int \tilde{g}[x, \theta, \gamma] \log \tilde{g}(x, \theta, \gamma) \hat{Q}(dx) = \varsigma. \quad (31)$$

Taken together, (30) and (31) determine  $\theta$  and  $\varsigma$  for a given value of  $\gamma$ .<sup>16</sup> This procedure gives us a value of  $\theta^*(\gamma)$  as a function of  $\gamma$ .

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<sup>16</sup>An alternative approach would have been to impose separately the constraint  $\int \log g(x) g(x) \hat{Q}(dx) - \varsigma \leq 0$  for an exogenously specified  $\varsigma$ . Here we avoid an additional “free parameter.”

## 5.4 Key steps in calibrating $\bar{\mathcal{Z}}$

By refining a suggestion of Anderson et al. (2003), we use a half-life for a model detection statistic to determine  $\gamma$ . Compute  $h^*[x, \theta^*(\gamma), \gamma]$  and evaluate the associated Chernoff entropy. Then adjust  $\gamma$  to match a target half-life as given by (23).<sup>17</sup> Call the resulting  $\gamma$ ,  $\gamma^*$  and let

$$g^*(x) = \tilde{g}[x, \theta^*(\gamma^*), \gamma^*].$$

## 6 Quantitative examples

We present two quantitative examples in which an  $X_t$  process contributes a predictable component of consumption growth.<sup>18</sup> We explore examples in which evidence for long-term predictability of consumption growth is intrinsically fragile, making it plausible that investors would want to explore alternative models. Chernoff entropy guides the magnitudes of the doubts that we impute to investors.

Our first example is patterned after Bansal and Yaron (2004), but like Hansen and Sargent (2010), we use data only on aggregate consumption. We use updated data to estimate parameters of the baseline model (1). We assume that  $X_t$  is hidden to us as econometricians, but that it is observed by both the planner and the representative investor.

Our second example blends Bansal and Yaron (2004) with Hansen et al. (2008). We estimate a VAR that builds in cointegrating relationships. Along with consumption, our VAR includes personal dividend income, a variable that we regard as a broad measure of business income. We use the VAR estimates to construct another quantitative version of our baseline model (1). We build this simplified model to approximate responses to permanent shocks. In contrast to Bansal and Yaron (2004), we introduce no stochastic volatility in order to focus on fluctuations in the endogenously determined uncertainty prices induced by the representative investor's model uncertainty.

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<sup>17</sup>We expect but haven't proved that the half-life is monotone in  $\gamma$ .

<sup>18</sup>While we appreciate the value of a more comprehensive empirical investigation with multiple macroeconomic time series, here our aim is to illustrate a mechanism within the context of relatively simple time series models of predictable consumption growth.

## 6.1 A long-term risk model

For our first set of calculations, we use our baseline model (1) evaluated at the following updates of maximum likelihood estimates computed by Hansen and Sargent (2010):<sup>19</sup>

$$\begin{aligned}
 \hat{\mu} &= .499 & \hat{\beta} &= 1 \\
 \hat{\phi} &= 0 & \hat{\kappa} &= .169 \\
 \alpha &= \begin{bmatrix} .424 \\ 0 \end{bmatrix} \\
 \sigma &= \begin{bmatrix} 0 \\ .195 \end{bmatrix}
 \end{aligned} \tag{32}$$

We study Chernoff entropy balls associated with half-lives of 60 quarters, 80 quarters, and 120 quarters. For comparison, we include a model with no concern for robustness, which is equivalent to a half-life equal to infinity. We let  $\xi(x, \gamma) = \gamma x^2$  in contrast to Hansen and Sargent (2001) or Hansen et al. (2006), who in effect set  $\xi(x, \gamma) = \gamma$ . When  $\xi$  does not depend on  $x$ , the worst-case drift distortions and the implied uncertainty prices are both constant.

Table 1 reports worst-case models that emerge from the robust planner's HJB equation (26). The worst-case models reduce  $\mu$  and impart a negative mean  $\frac{\phi}{\kappa}$  to  $X$ . We report the composite outcome for the growth rate of consumption  $\mu + \frac{\beta\phi}{\kappa}$ ;  $\beta$  turns out to be unity in all cases. The quantity  $\mu + \frac{\beta\phi}{\kappa} - \hat{\mu}$  traces out the implied long-term consequences of an adverse shift in the drift vector of the Brownian motion at a given date. Because an adverse shift can occur at *any* moment, the implied worst-case parameters reflect what would happen if an adverse drift shift occurred at *every* future moment.

As we reduce the half-life, the worst-case model makes the constant parameter adjustment smaller. When  $\xi(x, \gamma) = \gamma x^2$ , the worst-case model also increases the persistence of the growth-rate process. Although the minimizing agent could choose to increase persistence further, he instead chooses to allocate some of the entropy distortion to the constant terms.

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<sup>19</sup>The estimates are for per capita using consumption of nondurables and services for aggregate U.S. quarterly data over the period 1948 Q2 to 2015 Q2.

Table 1: Worst-case parameter values for  $\xi(x, \gamma) = \gamma x^2$  associated with HJB equation (26). The last column reports  $\underline{\kappa}$ , which is the smallest allowable value of  $\kappa$ .

Half-Life	$\mu$	$\phi$	$\beta$	$\kappa$	$\underline{\kappa}$	$\mu - \hat{\mu} + \frac{\beta\phi}{\kappa}$
$\infty$	.499	0	1	.169	.169	0
120	.477	-.0347	1	.134	.066	-.281
80	.475	-.0407	1	.122	.052	-.357
60	.475	-.0452	1	.112	.042	-.428

Figure 1 displays interdecile ranges of the distribution for consumption growth over alternative horizons. We depict deciles for both the baseline model and the worst-case model associated with a half-life of 80 according to formulas in Appendix E.2. The regions between the deciles describe *risk* in the consumption distribution. Variations across the baseline and worst-case models reflect *uncertainty* driven by skepticism about the baseline model. The upper decile of the worst-case model overlaps the lower decile of the baseline model.

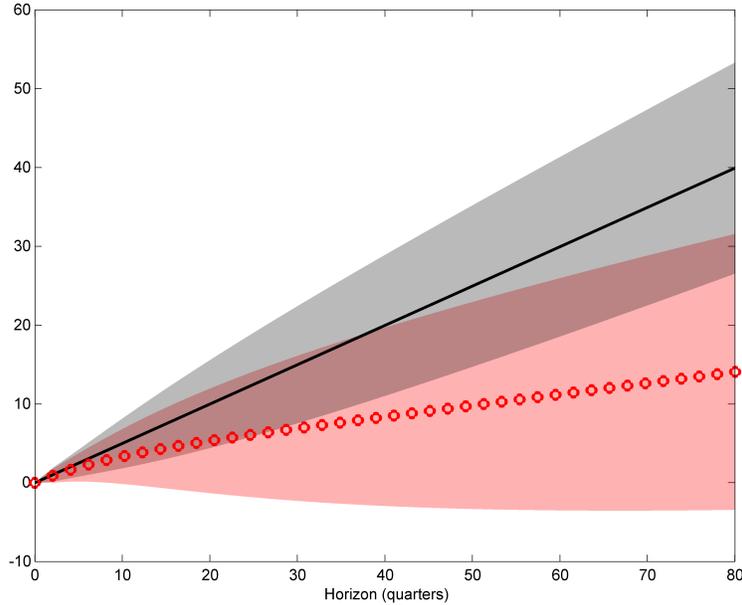


Figure 1: Expected values and interdecile ranges of  $\log C_t$  scaled by 100 for the baseline model and for a half-life of 90 when  $\xi(x, \gamma) = \gamma x^2$ . The shaded black and red areas show the .1 and .9 interdecile ranges under the baseline model and the worst-case model for a half-life of 80. The black line is the mean growth for the baseline model, and the red circle line is the mean growth for the worst-case model.

For the VAR construction, we follow Hansen et al. (2008) and use additional macroeconomic time series to infer information about long-term consumption growth. We report a calibration of our baseline model (1) constructed from a trivariate VAR for log consumption, log business income, and log personal dividend income. Business income is measured as proprietor’s income plus corporate profits per capita. Dividends are personal dividend income per capita. We fit a vector autoregression (VAR) to the consumption growth rate, the difference between logs of business income and consumption, and the difference between logs of personal dividend income and consumption.<sup>20</sup> By including proprietors’ income in addition to corporate profits, we use a broader measure of business income than Hansen et al. (2008) who used only corporate profits. Moreover, Hansen et al. (2008) did not

<sup>20</sup>The time series are quarterly data from 1948 Q1 to 2015 Q1. where our consumption measure is nondurables plus services consumption per capita. The business income data are from NIPA Table 1.12 and the dividend income from NIPA Table 7.10.

include personal dividends in their VAR analysis.

The VAR estimates imply:

$$\begin{aligned}
 \hat{\mu} &= .386 & \hat{\beta} &= 1 \\
 \hat{\phi} &= 0 & \hat{\kappa} &= .019
 \end{aligned}$$

$$\alpha = \begin{bmatrix} .488 \\ 0 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} .013 \\ .028 \end{bmatrix}$$
(33)

There are two important differences between this specification and the preceding one that uses consumption alone. First, now there is cross correlation between shocks. Second, since the baseline  $\kappa$  is much smaller, there is more persistence, leaving less room for a worst-case model to move away from the baseline before encountering a so-called breakdown probability model at which the objective is minus infinity. Even prior to attaining the breakdown point, we find that  $\underline{\kappa}$  becomes negative. To avoid these outcomes, we proceed as follows. First we compute

$$\xi_2 = \min_r \eta_1 \cdot \eta_1$$

where the minimization is subject to

$$\begin{bmatrix} \alpha' \\ \sigma' \end{bmatrix} \eta_1 = \begin{bmatrix} r \\ \hat{\kappa} - \underline{\kappa} \end{bmatrix}.$$

In this way we allow the continuous-time autoregressive coefficient to be as low as  $\underline{\kappa}$  without taking a stand on the magnitude of  $\beta$ . We then choose half-lives by picking  $\gamma$  so that

$$\xi(x) = \gamma + (\hat{\kappa} - \underline{\kappa})^2 x^2.$$

This construction makes  $\underline{\kappa}$  a feasible choice.

In table 2, we fix  $\underline{\kappa} = .005$  then set  $\gamma$  according to three half-lives. The minimizing decision maker sets  $\hat{\kappa} > \kappa > \underline{\kappa}$  and thereby increases the persistence of state variable relative to the baseline model, but less than he could. The minimization also appends adverse constant shifts to the Brownian increments. These adverse shifts in the Brownian

Table 2: Worst-case parameter values for  $\xi(x) = \gamma + (\hat{\kappa} - \underline{\kappa})^2 x^2$  associated with HJB equation (26)

Half-Life	$\mu$	$\phi$	$\beta$	$\kappa$	$\underline{\kappa}$	$\mu - \hat{\mu} + \frac{\beta\phi}{\kappa}$
$\infty$	.386	0	1	.019	.019	0
120	.334	-.0055	1.038	.013	.005	-.482
80	.322	-.0068	1.038	.013	.005	-.593
60	.312	-.0078	1.038	.013	.005	-.686

Table 3: Worst-case parameter values for  $\xi(x) = \gamma + (\hat{\kappa} - \underline{\kappa})^2 x^2$  associated with HJB equation (26).

Half-Life	$\mu$	$\phi$	$\beta$	$\kappa$	$\underline{\kappa}$	$\mu - \hat{\mu} + \frac{\beta\phi}{\kappa}$
$\infty$	.386	0	1	.019	.019	0
80	.316	-.0072	1.024	.015	.0075	-.550
80	.322	-.0068	1.038	.013	.0050	-.593
80	.332	-.0061	1.058	.010	.0025	-.682

increments have long-lasting effects on the growth rate of consumption determined by the worst-case persistence parameter  $\kappa$ . An associated increase in the slope coefficient  $\beta$  in the consumption evolution equation magnifies these effects. Reducing the half-life of the probability decay enhances all of these effects. Since  $\kappa$  is smaller under the worst-case model than under the approximating model, the long-term consequence of a mean distortion to the shock vector is further amplified. Unlike the approximating model, the worst-case model implies that  $\mu + \frac{\beta\phi}{\kappa}$  is negative, a consequence of allowing the mean of the shock vector to be misspecified and permitting this misspecification to recur in all future time periods.

In table 3 we explore the consequences of changing  $\underline{\kappa}$ . We fix and the half-life to be eighty quarters and consider three  $\underline{\kappa}$ 's: .0075, .0050, and .0025. The worst-case  $\kappa$  becomes smaller as we decrease  $\underline{\kappa}$ , but it exceeds  $\underline{\kappa}$ . The slope  $\beta$  also increases. To maintain the same half-life, the implied worst-case values for  $\mu$  and  $\phi$  become larger.

Figure 2 is a counterpart to figure 1 for our second model model calibration. Once again, especially near term, the grey and red shaded areas depicting interquartile ranges for the baseline and worst-case model share much coverage.

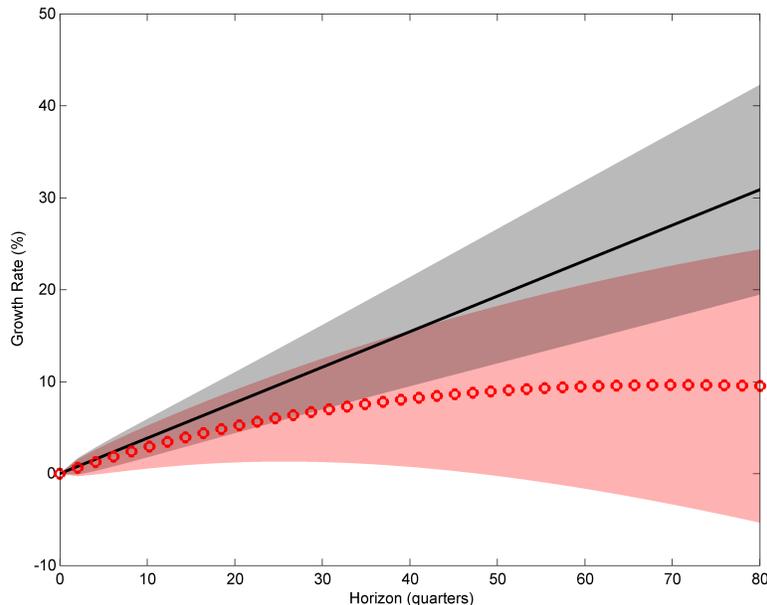


Figure 2: Expected values and interdecile ranges of  $\log C_t$  scaled by 100 for the baseline model and for a half-life of 80 and  $\underline{\kappa} = .005$  when  $\tilde{\xi} = \gamma + (\hat{\kappa} - \underline{\kappa})^2 x^2$ . The shaded black and red areas show the .1 and .9 interdecile ranges under the baseline model and the worst-case model for a half-life of 80. The black line is the mean growth for the baseline model, and the red circle line is the mean growth for the worst-case model.

## 7 Comparing sets of models

The set  $\overline{\mathcal{Z}}$  of martingales from definition 3.5 describes our robust representative investor's concerns about misspecification of his baseline model (1). In this section, we compare  $\overline{\mathcal{Z}}$  to an associated Chernoff entropy ball  $\check{\mathcal{Z}}$  and also to another set  $\mathcal{Z}^*$ , the smallest entropy ball of the type used by Hansen and Sargent (2001) and Hansen et al. (2006) that contains  $\overline{\mathcal{Z}}$ . We illustrate these comparisons using parameter values that emerged from our first method for calibrating the baseline model (1). A comparison of these sets provides a practical answer to the question: instead of using the set  $\overline{\mathcal{Z}}$ , why not simply expand the entropy ball  $\mathcal{Z}^*$  used by Hansen and Sargent (2001) and Hansen et al. (2006) enough to include the alternative parametric models that concern our decision maker? The answer is that doing that would lead to an entropy ball that is too big when measured by Chernoff-

entropy. We also show that for our example a Chernoff entropy ball closely approximates the set  $\bar{\mathcal{Z}}$  that we use to pose our robust planning problem.

## 7.1 A set $\mathcal{Z}^*$ that contains $\bar{\mathcal{Z}}$

In contrast to what we do in this paper, Anderson et al. (2003) and Hansen and Sargent (2010) formulated robust control problems in terms of discounted entropy balls surrounding baseline model like (1). To help understand the differences between those robust control problems and the ones posed in this paper, we construct a new set  $\mathcal{Z}^*$  defined as the smallest entropy ball that contains  $\bar{\mathcal{Z}}$ . An entropy ball surrounding a baseline model is a family of  $Z^H$ s that satisfy<sup>21</sup>

$$\mathcal{Z}^* \doteq \left\{ Z^H : \int \left[ \int_0^\infty \exp(-\delta t) E(Z_t^H \log Z_t^H | X_0 = x) \right] g^*(x) \hat{Q}(dx) \leq \frac{1}{2\delta} \xi^* \right\} \quad (34)$$

for the smallest constant  $\xi^* > 0$  such that

$$\bar{\mathcal{Z}} \subset \mathcal{Z}^*.$$

By constructing an entropy ball  $\mathcal{Z}^*$  that contains  $\bar{\mathcal{Z}}$ , we discover how large relative entropy can be for martingales in the set  $\bar{\mathcal{Z}}$ . By comparing  $\bar{\mathcal{Z}}$  and  $\mathcal{Z}^*$  with a Chernoff entropy ball that we use to calibrate  $\bar{\mathcal{Z}}$ , as we shall do in figure 4 below, we can appreciate some of the consequences of formulating robust control problems as we do in this paper rather than as we did in Hansen et al. (2006) and Hansen and Sargent (2001). We describe how to compute  $\xi^*$  in Appendix D using recursive methods.

## 7.2 Alternative sets

We compare intersections of  $\mathcal{Z}^o$  defined in (3.2) with each of the three sets  $\mathcal{Z}^*$ ,  $\bar{\mathcal{Z}}$ , and  $\check{\mathcal{Z}}$ . While it is tractable to use the sets  $\mathcal{Z}^*$  and  $\bar{\mathcal{Z}}$  to formulate robust decision problems, these sets are not directly linked to statistical discrimination problems. The set  $\check{\mathcal{Z}}$  is closely linked to “statistical discrimination,” but for forming robust decision problems it is not as tractable as the other two. It would be comforting if  $\mathcal{Z}^*$  were to approximate  $\hat{\mathcal{Z}}$  closely, at least in regions near the worst-case model that emerges from the robust planner’s HJB equation (26).

---

<sup>21</sup>We use the term ball loosely because typically a ball in mathematics is defined using a metric.

Using parameter values associated with the calibration strategy described in subsection 6.1, we report the projection  $(\check{\mathcal{Z}} \cap \mathcal{Z}^o)$  of the Chernoff ball on  $\mathcal{Z}^o$  for three half-lives in figure 3. We represent these projections using the three parameters  $(\mu, \kappa, \phi)$  that characterize  $\mathcal{Z}^o$ . For comparison, we also report  $(\bar{\mathcal{Z}} \cap \mathcal{Z}^o)$ . The sets are distinct, but the big differences occur for larger values of  $\kappa$  at which the Chernoff ball contains points not included in  $\mathcal{Z}^*$ . But large values of  $\kappa$  are not ones that the robust planner most fears. The two sets  $(\bar{\mathcal{Z}} \cap \mathcal{Z}^o)$  and  $(\check{\mathcal{Z}} \cap \mathcal{Z}^o)$  are closer for longer specifications of the Chernoff entropy half-life.

In figure 4, we compare the entropy ball  $(\mathcal{Z}^* \cap \mathcal{Z}^o)$  projection to both  $(\bar{\mathcal{Z}} \cap \mathcal{Z}^o)$  and  $(\check{\mathcal{Z}} \cap \mathcal{Z}^o)$  when  $\hat{\xi}(x) = x^2$ . The graphs on the left side show how large an entropy ball  $\mathcal{Z}^*$  would have to be to contain the set  $\bar{\mathcal{Z}}$  used in the robust planning problem affiliated with HJB equation (26). The figures on the right side compare the Chernoff balls  $\check{\mathcal{Z}}$  to the smallest entropy balls  $\mathcal{Z}^*$  that contain  $\bar{\mathcal{Z}}$ . These figures convey the message that the worst-case model associated with the entropy ball  $\mathcal{Z}^*$  that contains  $\bar{\mathcal{Z}}$  differs from the worst-case model affiliated with  $\bar{\mathcal{Z}}$  and that it also lies outside the Chernoff ball  $\check{\mathcal{Z}}$  that we use to calibrate  $\bar{\mathcal{Z}}$ . The half lives associated with the worst-case models for the sets  $\mathcal{Z}^*$  in figure 4 are 71, 43, and 30, roughly half of values for their  $\bar{\mathcal{Z}}$  counterparts. Figure 4 and these reduced half lives thus confirm the claim made in section 1 that the set  $\bar{\mathcal{Z}}$  more concisely expresses concerns about particular parametric alternatives than would the bigger set  $\mathcal{Z}^*$  used in the robust control problem posed by Hansen et al. (2006).

## 8 Robust portfolio choice and pricing

In this section, we describe equilibrium prices that make a representative consumer willing to bear risks presented by the environment described by baseline model (1) in light of his concerns about model misspecification as expressed with the set  $\mathcal{Z}^*$ . We construct equilibrium prices by appropriately extracting shadow prices from the robust planning problem of subsection 5.1. We decompose equilibrium risk prices into distinct compensations for bearing risk and for bearing model uncertainty. We also describe an equilibrium term structure of compensations for bearing model uncertainty. We begin by posing the representative investor's portfolio choice problem.

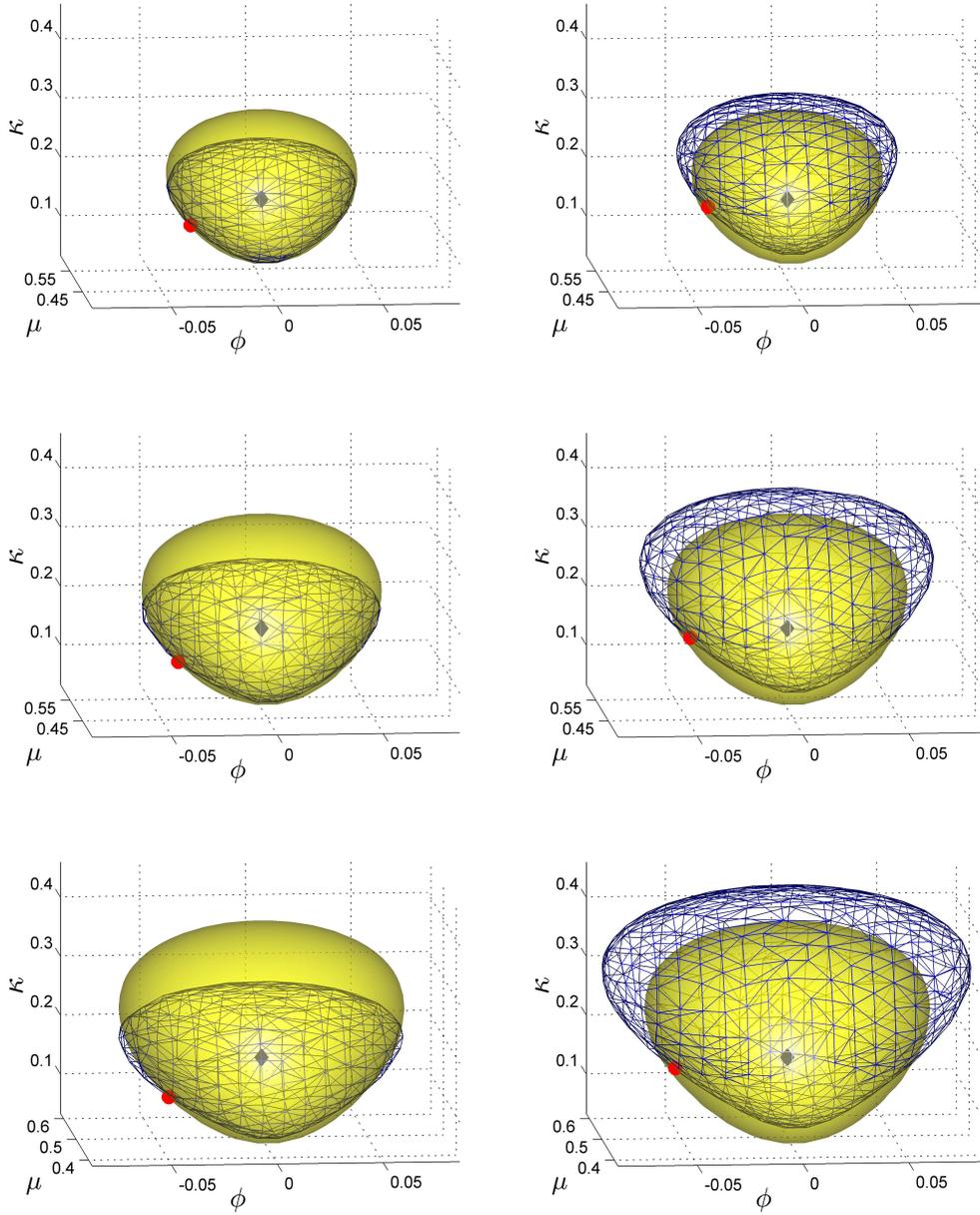


Figure 3: Projections of  $\bar{\mathcal{Z}}$  and the Chernoff entropy ball  $\check{\mathcal{Z}}$  onto three-parameter axis. From top to bottom: target half-lives 120, 80, and 60, respectively. Left column:  $\hat{\xi}(x) = x^2$ . Right column:  $\hat{\xi}(x) = 1$ .  $(\bar{\mathcal{Z}} \cap \mathcal{Z}^o)$  shown in blue mesh.  $(\check{\mathcal{Z}} \cap \mathcal{Z}^o)$  shown in yellow. The red points solve the robust planner's problems.

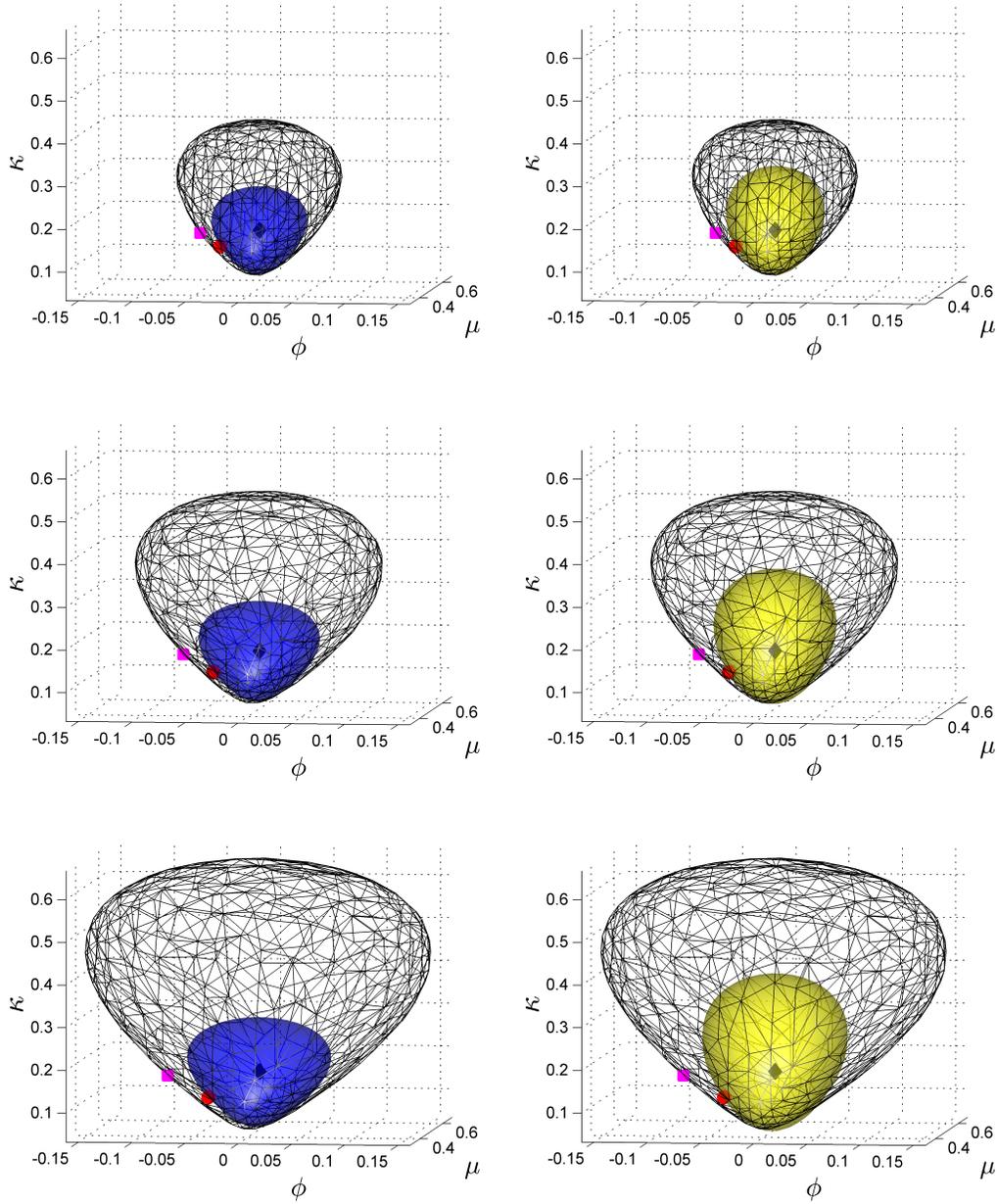


Figure 4: Sets  $Z^*$ ,  $\bar{Z}$ , and  $\hat{Z}$  when  $\hat{\xi} = x^2$ . From top to bottom: half-lives 120, 80, and 60, respectively.  $(Z^* \cap Z^o)$  shown in blue,  $(\bar{Z} \cap Z^o)$  shown in black mesh.  $(\hat{Z} \cap Z^o)$  shown in yellow. The black diamond shows the baseline model. The red points solve the robust planner's problems. The magenta squares solve the robust planner's problems under  $\bar{Z}$ .

## 8.1 Robust investor portfolio problem

A representative investor faces a continuous-time Merton portfolio problem in which individual wealth  $K$  evolves as

$$dK_t = -C_t dt + (.01)K_t \iota(X_t) dt + K_t A_t \cdot dW_t + (.01)K_t \pi(X_t) \cdot A_t dt, \quad (35)$$

where  $A_t = a$  is a vector of chosen risk exposures,  $\iota(x)$  is the instantaneous risk free rate expressed as a percent, and  $\pi(x)$  is the vector of risk prices evaluated at state  $X_t = x$ . Initial wealth is  $K_0$ . The investor has discounted logarithmic preferences and distrusts his probability model.

Key inputs to a representative investor's robust portfolio problem are the baseline model (1), the wealth evolution equation (35), the vector of risk prices  $\pi(x)$ , and the quadratic function  $\xi$  in (13) that defines the alternative explicit models that concern the representative investor. As in the robust planners problem analyzed in section 5.1, let  $\theta$  be a penalty parameter or a Lagrange multiplier on the constraint (25). For the recursive competitive equilibrium, we take  $(\theta, \gamma)$  as given. We calibrated these parameters as described in section 5.

Under a guess that the value function takes the form  $\tilde{v}(x, \theta, \gamma) + \log k + \log \delta$ , the HJB equation for the robust portfolio allocation problem is

$$\begin{aligned} 0 = \max_{a,c} \min_h & -\delta \tilde{v}(x, \theta, \gamma) - \delta \log k - \delta \log \delta + \delta \log c - \frac{c}{k} + (.01)\iota(x) \\ & + (.01)\pi(x) \cdot a + a \cdot h - \frac{|a|^2}{2} + \left( \hat{\phi} - \hat{\kappa}x \right) \tilde{v}'(x, \theta, \gamma) + h \cdot \sigma \tilde{v}'(x, \theta, \gamma) \\ & + \frac{|\sigma|^2}{2} \tilde{v}''(x, \theta, \gamma) + \theta \left[ \frac{|h|^2}{2} - \frac{1}{2} \tilde{\xi}(x, \gamma) \right]. \end{aligned} \quad (36)$$

First-order conditions for consumption are

$$\frac{\delta}{c^*} = \frac{1}{k},$$

which implies that  $c^* = \delta k$ , an implication that flows partly from the representative investor's unitary elasticity of intertemporal substitution. First-order conditions for  $a$  and  $h$  are

$$(.01)\pi(x) + h^* - a^* = 0 \quad (37a)$$

$$a + \theta h^* + \tilde{v}'(x, \theta, \gamma)\sigma = 0. \quad (37b)$$

We can appeal to arguments like those in Hansen and Sargent (2008, ch. 7) to justify stacking first-order conditions as a way to collect equilibrium conditions for the two-person zero-sum game.<sup>22</sup>

## 8.2 Competitive equilibrium prices

We show here that the drift distortion  $h^*$  that emerges from the robust planner's problem of subsection 5.1 determines prices that a competitive equilibrium awards to people who bear model uncertainty. To compute a vector  $\pi(x)$  of competitive equilibrium risk prices, we find a robust planner's marginal valuations of exposures to the  $W$  shocks. We decompose that price vector into separate compensations for bearing risk and for accepting model uncertainty.

Noting from the robust planning problem that the shock exposure vectors for  $\log K$  and  $\log C$  must coincide implies

$$a^* = (.01)\alpha.$$

From (37b),

$$h^* = \eta^*(x, \theta, \gamma)$$

where  $\eta^*$  is the worst-case drift from the robust planning problem provided that we show that  $\bar{v} = v$ , where  $v$  is the value function for the robust planning problem. Thus, from (37a),  $\pi = \pi^*$ , where

$$\pi^*(x) = \alpha - 100\eta^*(x, \theta, \gamma) = \alpha - 100h^*. \quad (38)$$

Similarly, in the problem for a representative investor within a competitive equilibrium, the drifts for  $\log K$  and  $\log C$  must coincide:

$$-\delta + (.01)\iota(x) + (.01)[(.01)\alpha - \eta^*(x, \theta, \gamma)] \cdot \alpha - \frac{.0001}{2}\alpha \cdot \alpha = (.01)(\hat{\mu} + \hat{\beta}x),$$

---

<sup>22</sup>If we were to use a timing protocol that allows the maximizing player to take account of the impact of its decisions on the minimizing agent, we would obtain the same equilibrium decision rules described in the text.

so that  $\iota = \iota^*$ , where

$$\iota^*(x) = 100\delta + (\hat{\mu} + \hat{\beta}x) + \alpha \cdot \eta^*(x, \theta, \gamma) - \frac{.01}{2}\alpha \cdot \alpha. \quad (39)$$

We use these formulas for equilibrium prices to construct a solution to the HJB equation of a representative investor in a competitive equilibrium by letting  $\bar{v} = v$ .

### 8.3 Reinterpreting worst-case portfolio problem

Hansen et al. (2006) described an ordinary (i.e., non-robust) portfolio selection problem for a representative investor who has a completely trusted model for the exogenous state dynamics and whose decision rule attains the value function  $\psi$  associated with the HJB equation (36) for a robust representative investor portfolio problem. This investor's completely trusted model of course differs from the baseline model (1) and is associated with what Hansen et al. (2006) called an *ex post* problem because it comes from exchanging orders of maximization and minimization in the two-person zero-sum game that gives rise to the robust portfolio choice rule. This *ex post* portfolio problem is a special case of a Merton problem with a state evolution that is distorted relative to the baseline model. The distorted evolution imputes to the process  $W$  a drift  $\eta^*(X)$  so that

$$\begin{aligned} dW_t &= \eta^*(X_t)dt + dW_t^* \\ &= (\eta_0^* + \eta_1^* X_t) dt + dW_t^*, \end{aligned}$$

where  $W^*$  is a multivariate standard Brownian motion under the  $h^*$  probability distribution. Thus,

$$\begin{aligned} d \log C_t &= .01(\hat{\mu} + \hat{\psi} X_t)dt + (.01)\alpha \cdot \eta^*(X_t)dt + (.01)\alpha \cdot dW_t^* \\ dX_t &= \hat{\phi}dt - \hat{\kappa}X_t dt + \sigma \cdot \eta^*(X_t)dt + \sigma \cdot dW_t^*. \end{aligned}$$

A value function  $\tilde{v} + \log k$  satisfies the HJB equation

$$\begin{aligned} 0 = \max_{a,c} & -\delta \tilde{v}(x) - \delta \log k + \delta \log c - \frac{c}{k} + (.01)\iota(x) + (.01)\pi(x) \cdot a + a \cdot (\eta_0 + \eta_1^* x) - \frac{|a|^2}{2} \\ & [-\hat{\kappa}x + \eta_0^* + \eta_1^* x] \tilde{v}'(x) + \frac{|\sigma|^2}{2} \tilde{v}''(x). \end{aligned}$$

First-order conditions are

$$\begin{aligned} \frac{\delta}{c^*} - \frac{1}{k} &= 0 \\ (.01)\pi(x) + \eta_0^* + \eta_1^*x - a^* &= 0, \end{aligned}$$

which lead to decision rules  $c^* = \delta k$  and

$$a^* = (.01)\pi(x) + \eta_0^* + \eta_1^*x.$$

Because the respective exposures and drifts for  $\log K$  and  $\log C$  should coincide in equilibrium, it follows that

$$-\delta + (.01)\iota(x) + .01\alpha \cdot h^*(x) + .0001\alpha \cdot \alpha - \frac{.0001}{2}\pi \cdot \alpha = (.01) \left[ \hat{\mu} + \hat{\beta}x + \alpha \cdot h^*(x) \right].$$

Thus, the ordinary decision rules that solve the *ex post* portfolio problem imply the same equilibrium prices as the robust portfolio problem, so that  $\pi = \pi^*$  and  $\iota = \iota^*$ , as given by (38) and (39), respectively.

## 9 Term structure of uncertainty prices

In a continuous-time formulation, a pertinent notion of a shock that occurs during a “next instant” is an incremental change that carries with it incremental effects on all future outcomes, which for us include cash flows, stochastic discounts, and asset prices. This section applies a type of nonlinear impulse response function defined by Borovička et al. (2011) and Borovička et al. (2014) to describe how competitive equilibrium uncertainty prices vary over investment horizons. Borovička et al.’s (2014) counterpart of an impulse response function is a state- and horizon-dependent elasticity of an outcome at date  $t + j$  to a shock at date  $t$ , a collection of objects they call “shock elasticities.” The joint nonlinear dynamics of stochastic discount factor processes, payout processes, and asset prices make these elasticities depend both on the horizon and the Markov state, which in our application is a growth state. In this section, we compute these elasticities and use them to construct what we regard as a dynamic value decomposition. We present quantitative examples in subsections 9.3 and 9.4.

## 9.1 Local uncertainty prices

The equilibrium stochastic discount factor process for our robust representative investor economy is

$$d \log S_t = -\delta dt - .01 \left( \hat{\mu} + \hat{\beta} X_t \right) dt - .01\alpha \cdot dW_t + H_t^* \cdot dW_t - \frac{1}{2} |H_t^*|^2 dt. \quad (40)$$

The log stochastic discount factor has a linear local mean and a quadratic local variance. Such exponential-quadratic specification have been used extensively in empirical asset pricing applications. A leading example is the work of Ang and Piazzesi (2003) who estimated a term structure model with an exponential quadratic stochastic discount factor process driven by macroeconomic state variables.

Components of the vector  $\pi^*(X_t)$  given by (38) equal minus the local exposures to the Brownian shocks. These are usually interpreted as local “risk prices,” but we shall reinterpret them. Motivated by the decomposition

$$\begin{array}{lcl} \text{minus stochastic discount factor exposure} & = & .01\alpha \quad \quad \quad -H_t^*, \\ & & \text{risk price} \quad \quad \quad \text{uncertainty price} \end{array}$$

we prefer to think of  $.01\alpha$  as risk prices induced by the curvature of log utility and  $-H_t^*$  as “uncertainty” prices induced by a representative investor’s doubts about the baseline model. Here  $H_t^* = \eta_0^* + \eta_1^* X_t$ , as described in equation (27). When  $\eta_1^* = 0$ ,  $H_t^*$  is constant; but when  $\eta_1^*$  differs from zero, the uncertainty prices  $-H_t^* = -h^*(X_t)$  are time varying and depend linearly on the growth state  $X_t$ . When  $h^*$  depends positively on  $x$ , uncertainty prices are higher in bad times than in good times. Countercyclical uncertainty prices emerge endogenously from a baseline model that excludes stochastic volatility in the underlying consumption risk as an exogenous input. Such fluctuations emerge endogenously from a baseline model that excludes stochastic volatility in the underlying consumption risk as an exogenous input. Stochastic volatility models introduce new risks to be priced while also inducing fluctuations in the prices of the “original” risks. The mechanism in this paper simultaneously enhances and induces fluctuations in the uncertainty prices, but it introduces no new sources of risk. Instead, the mechanism features investors’ responses to their uncertainty about those risks.

## 9.2 Uncertainty prices over alternative investment horizons

In the context of our quantitative models, we now report the price-shock elasticities that Borovička et al. (2014) showed are horizon-dependent uncertainty prices of risk exposures. Shock price elasticities describe the dependence of logarithms of expected returns on an investment horizon. The logarithm of the expected return from a consumption payoff at date  $t$  consists of two terms:

$$\log E \left( \frac{C_t}{C_0} \middle| X_0 = x \right) - \log E \left[ S_t \left( \frac{C_t}{C_0} \right) \middle| X_0 = x \right]. \quad (41)$$

The first term is the expected payoff and the second is the cost of purchasing that payoff. Malliavin derivatives tell how a shock in the next instance affects consumption and stochastic discount factor processes. There is one Malliavin derivative for each Brownian increment. Following Borovička et al. (2014), we call the Malliavin derivatives of expected returns across horizons with respect to a shock in the next instant the *price elasticity* function, and the Malliavin derivatives of expected payoffs across horizons with respect to a shock in the next instant a payoff *exposure* elasticity. For our model, we give quasi-analytical formulas for these two elasticities in Appendix E.1.

## 9.3 Shock price elasticities for quantitative example 1

Red lines and areas in figure 5 displays shock elasticities evaluated at the median and the two deciles of the stationary distribution for  $X$  when  $\hat{\xi}(x) = x^2$ ; blue circles show shock elasticities when  $\hat{\xi} = 1$ . As for the red lines associated with  $\hat{\xi} = x^2$ , interdecile ranges between .1 and .9 are shaded. The uncertainty price elasticities for the second shock but not the first depend on the initial state  $x$ . The price elasticity trajectories are larger for second shock, the one to the consumption growth rate, than for the first shock, which directly hits the level of log consumption. Both price elasticity trajectories are nearly constant across horizons. Increasing the concern for robustness, i.e., lowering the associated Chernoff half-lives, makes the elasticities larger and increases their variation across horizons for the growth rate shock.

For comparison, figure 5 also includes the Hansen and Sargent (2001)–Hansen et al. (2006) formulation of concerns about robustness that sets  $\hat{\xi} = 1$ . Here even the price elasticities for the growth shock are constant and are larger than the corresponding median elasticities when  $\hat{\xi} = x^2$ . Evidently, an interesting trade-off between variability and levels

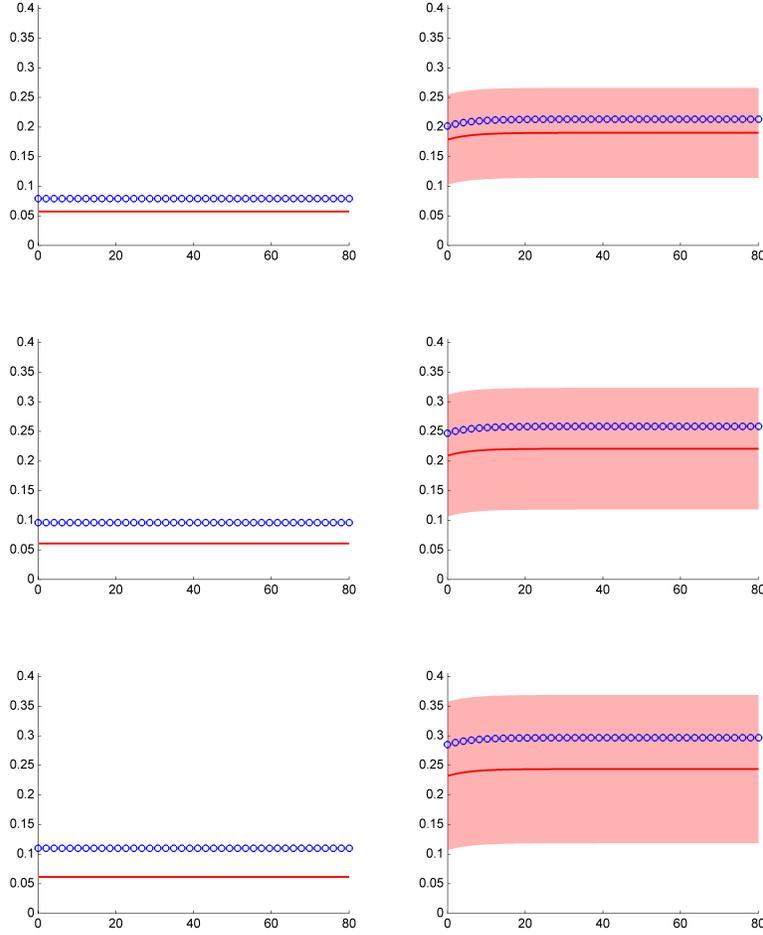


Figure 5: Shock-price elasticity to a shock to  $X$  for three target half-lives when and  $\xi(x, \gamma) = \gamma x^2$  using the first set of parameter estimates. From top to bottom: half-lives 120, 80, and 60, respectively. The left column gives the elasticities for the first shock (which hits the growth rate of consumption) and the right column for the second shock (which hits the level of log consumption). The shaded regions show the .1 to .9 interdecile ranges of the shock-price elasticities under the stationary distribution for  $X$ . The blue circles are the elasticities when  $\xi(x, \gamma) = \gamma$ .

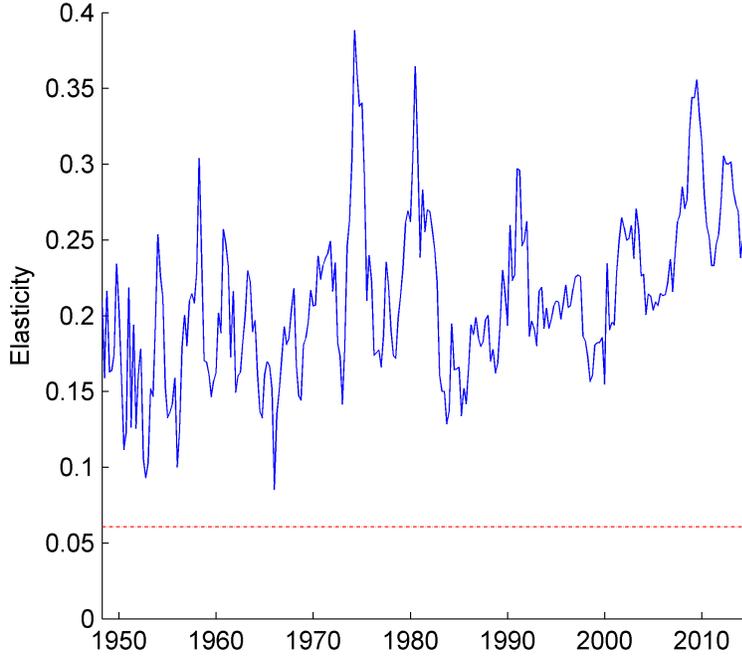


Figure 6: Components of uncertainty prices  $-H_t^* = -\eta_1^* - \eta_2^* \tilde{X}_t$ ;  $\tilde{X}_t$  is filtered estimate of  $X_t$ . The uncertainty prices are the contributions to the instantaneous elasticities. The red dashed line is the drift distortion for the first shock and the blue solid line is for second one. Results are based on the first set of parameter estimates and a half-life of 80 quarters.

of uncertainty prices is traced out by altering how much state dependence we include in  $\hat{\xi}$ . The  $\hat{\xi} = x^2$  specification activates the new mechanism for uncertainty price fluctuations proposed in this paper, while the  $\hat{\xi} = 1$  specification shuts it down.

Figure 6 depicts our estimates as econometricians of the local uncertainty prices associated with the two shocks. These estimates were obtained by first using the Kalman filter to estimate  $\tilde{X}_t = E[X_t | \log C_t - \log C_{t-1}, \dots, \log C_1 - \log C_0]$  under the baseline model evaluated at the parameters associated with our first quantitative example.<sup>23</sup> We then constructed a bivariate time series of uncertainty prices as  $\{-h_t^* : t = 1, 2, \dots, N\}$  where  $H_t^* = \eta_1^* + \eta_2^* \tilde{X}_t$ . The uncertainty price for the first shock, which affects consumption directly, is time invariant; but secular fluctuations are evident in the second shock, which

<sup>23</sup>We initialized the Kalman filter using the mean and variance of  $X$  in the stationary distribution implied by the baseline model.

hits the consumption growth rate. The uncertainty shock affiliated with the second shock makes uncertainty prices increase during times of diminished consumption growth.

We have structured our quantitative examples to investigate a particular mechanism for generating fluctuations in uncertainty prices from statistically plausible amounts of uncertainty. We inferred the baseline parameters for these examples solely from time series of macroeconomic quantities, thus completely ignoring asset prices during calibration. As a consequence, we do not expect to track closely the high frequency movements in financial markets. By limiting our empirical inputs, we consciously set aside concerns that Hansen (2007) and Chen et al. (2015) expressed about using asset market data to calibrate macro-finance models that assign a special role to investors' beliefs about the future asset prices.<sup>24</sup> An alternative approach would have been to calibrate partly to capture variations over time in cross-sectional distributions of asset returns, an approach that would impose cross-equation restrictions between time series of macroeconomic quantities and a vector of asset returns. We recognize that exploring cross-equations restrictions promises a fuller analysis of connections between financial markets and aggregate quantities. We would want to add features to our model including exogenous stochastic contributions to the volatilities of shocks to macroeconomic quantities and financial markets.

## 9.4 Shock price elasticities for quantitative example 2

In figure 7 we display the shock elasticities evaluated at the median and the two quartiles of the stationary distribution for  $X$ . We shade in interquartile ranges. Now uncertainty price elasticities for both shocks depend on the initial state  $x$ . Recall that for this parameterization the first shock has a direct impact on consumption and on the consumption growth rate. For the second quantitative example, there is state dependence in elasticities for the first shock and these elasticities are notably larger than those for the second shock. Allowing a greater range of  $\kappa$  with lower values of  $\underline{\kappa}$  enhances the state dependence in the elasticities.

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<sup>24</sup>Hansen (2007) and Chen et al. (2015) describe situations in which it is the behavior of rates of return on assets that, through the cross-equation restrictions, lead an econometrician to make confident inferences about the behavior of macroeconomic quantities like consumption that are much more confident than can be made from the quantity data alone. That opens questions about how the investors who are supposedly putting those cross-equation restrictions into returns came to know those quantity processes before *they* observed returns.

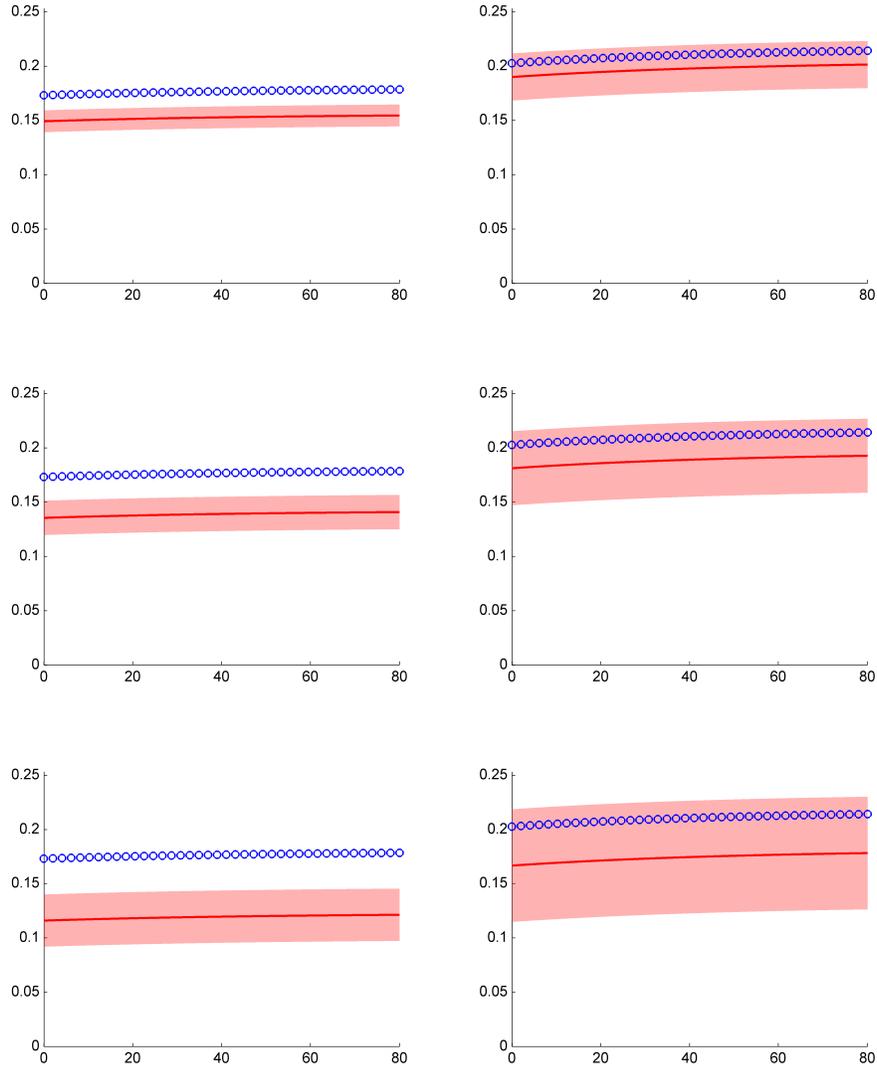


Figure 7: Shock-price elasticity to a shock to  $X$  for the target half-life 80 and the three target  $\underline{\kappa}$ 's when  $\tilde{\xi}(x, \gamma) = \gamma + (\underline{\kappa} - \hat{\kappa})^2 x^2$  using the second set of parameter estimates. From top to bottom,  $\underline{\kappa} = .0075, .005, .0025$  respectively. The first column gives the elasticities for the first shock and similarly for the second column. The shaded regions show the .1 to .9 interdecile ranges of the shock-price elasticities under the stationary distribution for  $X$ . The blue circles are the elasticities when  $\xi(x, \gamma) = \gamma$ .

Figure 8 depicts secular movements in estimated local uncertainty prices for our second calibrated example. Again, these are constructed as  $H_t^* = \eta_1^* + \eta_2^* \check{X}_t$  where  $\check{X}_t$  is the filtered estimate of  $X_t$ . Since the shocks are correlated under the baseline model for this calibration, both estimated local uncertainty prices now fluctuate over time. The fluctuations are proportional because they are constructed as an affine function of the estimates of the single state variable  $X_t$ .

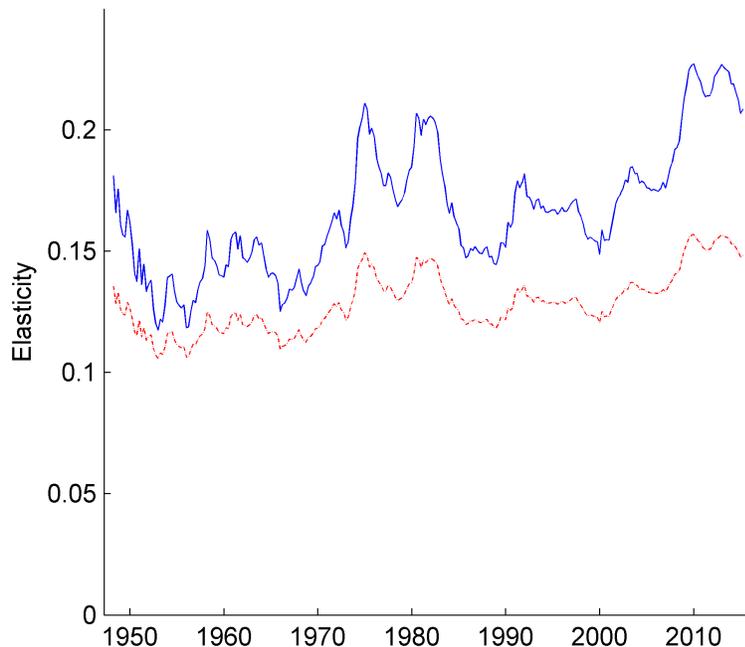


Figure 8: Estimates of local uncertainty prices under the second calibration. Uncertainty prices are the contributions to the instantaneous elasticities. The red dashed line is for the first shock and the blue solid line is for second one. Results are based on the second set of parameter estimates a half-life 80 quarters, and  $\underline{\kappa} = .005$ .

## 10 Rectangularity

Chen and Epstein (2002) advocated what they called rectangular sets of probabilities in the context of continuous-time specifications with ambiguity aversion. Epstein and Schneider (2003) formulated decision theoretic axioms that justified rectangular sets of models.

Proposition 3.4 shows that the set  $\overline{\mathcal{Z}}$  that we have used to pose robust decision problems contains relative entropy neighborhoods centered on alternative parametric models described with a function  $\xi(x)$ . If we had instead imposed only the restrictions

$$H_t \cdot H_t \leq \xi(X_t) \tag{42}$$

on drift distortions  $H_t$ , then our set of alternative probabilities would have been rectangular. In this section, we revisit the point made by Hansen et al. (2006) that any set like ours that includes relative entropy neighborhoods violates Epstein and Schneider’s rectangularity property.

We did not follow Epstein and Schneider’s recommendation to expand our set to become rectangular because we share a point of view expressed by Good (1952) in the quotation with which we began this paper. Good’s perspective guides how we construct convex sets of models.<sup>25</sup> As in Anderson et al. (2003), in section 4 we use statistical model selection criteria to evaluate the plausibility of least favorable models. Epstein and Schneider (2003) criticize robust control theory for violating a notion of dynamic consistency and propose a remedy that expands the set of models beyond those typically employed in robust control problems. They propose adding enough models to attain a rectangular set of models (or priors over models). They construct this larger set of models to represent preferences that satisfy axioms that they like. They state that

there is an important conceptual distinction between the set of probability laws that the decision maker views as possible, such as *Prob*, and the set of priors *P* that is part of the representation of preference.

We now show that Epstein and Schneider’s set expansion proposal results in an implementation of max-min utility theory that is not subjectively reasonable even when we apply it to an original set of models that is reasonable in Good’s sense.

Epstein and Schneider’s procedure for achieving rectangularity focuses attention on restrictions that a set like  $\overline{\mathcal{Z}}$  imposes on implied local state transitions. In our setting, these transitions are completely specified by local drift distortions. Rectangularity conditions (42) impose restrictions on drift distortions that are disconnected over time as in Chen and Epstein (2002) and in section 4.2 of Anderson et al. (1998). The set  $\overline{\mathcal{Z}}$  necessarily contains martingales of the following type. For a fixed date  $\tau$ , consider a random vector  $\overline{H}_\tau$  that is

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<sup>25</sup>See Berger (1994) and Chamberlain (2000) for related discussions.

observable at that date and that satisfies

$$E(|\bar{H}_\tau|^2 | X_0 = x) < \infty. \quad (43)$$

Form a stochastic process

$$H_t^u = \begin{cases} 0 & 0 \leq t < \tau \\ \bar{H}_\tau & \tau \leq t < \tau + u \\ 0 & t \geq \tau + u. \end{cases} \quad (44)$$

The martingale  $Z^{H^u}$  associated with  $H^u$  equals one both before time  $\tau$  and after time  $\tau + u$ .

Compute relative entropy:

$$\begin{aligned} \Delta(Z^{H^u} | x) &= \left(\frac{1}{2}\right) \int_\tau^{\tau+u} \exp(-\delta t) E \left[ Z_t^{H^u} |\bar{H}_\tau|^2 dt \middle| X_0 = x \right] dt \\ &= \left[ \frac{1 - \exp(-\delta u)}{2\delta} \right] \exp(-\delta \tau) E(|\bar{H}_\tau|^2 | X_0 = x). \end{aligned}$$

Evidently, relative entropy  $\Delta(Z^{H^u} | x)$  can be made arbitrarily small by shrinking  $u$  to zero. This means that any rectangular set that contains  $\bar{\mathcal{Z}}$  must allow for a drift distortion  $\bar{h}_\tau$  at date  $\tau$ . We summarize this argument in the following proposition:

**Proposition 10.1.** *Any rectangular set of probabilities that contains the probabilities induced by martingales in a conditional relative entropy neighborhood*

$$\{Z^H \in \mathcal{Z} : \Delta(Z^{H^u} | x) < \epsilon\}$$

*for some  $\epsilon > 0$  must also contain the probabilities induced by martingales in  $\bar{\mathcal{Z}}$ .*

This proposition targets neighborhoods of baseline model, but it is straightforward to extend it to relative entropy neighborhoods of other centering points.

Confining ourselves to any such rectangular set of martingales allows us too much freedom in setting both the date  $\tau$  and the random vector  $\bar{h}_\tau$ . Restriction (43) implies instant-by-instant drift distortions that are just too flexible: they need satisfy only measurability and second-moment restrictions. In particular, all martingales in the set  $\mathcal{Z}$  identified in definition 2.1 are included in the rectangular set. That set is too big to use as part of a meaningful robust decision problem.

Despite the fact that we decline to use the rectangular embedding that Epstein and Schneider (2003) advocate in order to satisfy their notion of dynamic consistency, it re-

mains true that the dynamic decision problems that we formulate give rise to optimization problems associated with two-person zero-sum games solvable via dynamic programming.

## 11 Concluding remarks

This paper formulates and applies a tractable model of the effects on equilibrium prices of exposures to macroeconomic uncertainties. Our analysis uses models' consequences for discounted expected utilities to quantify traders' concerns about model misspecification. We characterize the effects of concerns about misspecification of a baseline stochastic process for individual consumption as shadow prices for a planner's problem that supports competitive equilibrium prices.

To illustrate our approach, we have focused on the growth rate uncertainty featured in the "long-run risk" literature initiated by Bansal and Yaron (2004). Other applications seem natural. For example, the tools developed here could shed light on a recent public debate between two groups of macroeconomists, one prophesizing secular stagnation because of technology growth slowdowns, the other dismissing those pessimistic forecasts. The tools that we describe can be used, first, to quantify how challenging it is to infer persistent changes in growth rates, and, second, to guide macroeconomic policy design in light of available empirical evidence.

Specifically, we have produced a quadratic model (40) of the log stochastic discount factor whose uncertainty prices reflect a robust planner's worst-case drift distortions  $H_t^*$ . We have argued that these drift distortions should be interpreted as prices of model uncertainty. The dependency of these uncertainty prices  $H_t^*$  on the growth state  $x$ , and thus whether they are pro cyclical or countercyclical, is shaped partly by a function  $\hat{\xi}(x)$  that describes parametric alternatives to a baseline model. In this way, the theory of countercyclical risk premia in this paper is all about how our robust investor responds to the presence of the alternative parametric models among a huge set of unspecified alternative models that also concern him. We have demonstrated that this is a simple way of generating countercyclical risk premia.

It is worthwhile comparing this paper's way of inducing countercyclical risk premia with three other macro/finance models that also get them. Campbell and Cochrane (1999) proceed in the standard rational expectations single-known-probability-model tradition and so exclude any fears of model misspecification from the mind of their representative investor. They construct a history-dependent utility function in which the history of consumption

expresses an externality. This history dependence makes the investor’s local risk aversion depend in a countercyclical way on the economy’s growth state. Ang and Piazzesi (2003) use an exponential quadratic stochastic discount factor in a no-arbitrage statistical model and explore links between the term structure of interest rates and other macroeconomic variables. Their approach allows movements in risk prices to be consistent with historical evidence without specifying an explicit general equilibrium model. A third approach introduces stochastic volatility into the macroeconomy by positing that the volatilities of shocks driving consumption growth are themselves stochastic processes. A stochastic volatility model induces time variation in risk prices via exogenous movements in the conditional volatilities impinging on macroeconomic variables.

In Hansen and Sargent (2010), countercyclical risk prices are driven by a representative investor’s robust model averaging. The investor carries along two difficult-to-distinguish models of consumption growth, one asserting i.i.d. log consumption growth, the other asserting that the growth in log consumption is a process with a slowly moving conditional mean.<sup>26</sup> The investor uses observations on consumption growth to update a Bayesian prior over these two models, starting from an initial prior probability of .5. The prior wanders over a post WWII sample period, but ends where it started. Each period, the Hansen and Sargent representative investor expresses his specification distrust by pessimistically exponentially twisting a posterior over the two baseline models. That leads the investor to interpret good news as temporary and bad news as persistent, causing him to put countercyclical uncertainty components into equilibrium “risk” prices.

In this paper, we propose a different way to make risk prices vary. We exclude learning and instead consider alternative models with parameters whose future variations cannot be inferred from historical data. These time-varying parameter models differ from the decision maker’s baseline model, a fixed parameter model whose parameters can be well estimated from historical data. The alternative models include ones that allow parameters persistently to deviate from those of the baseline model in statistically subtle and time-varying ways. In addition to this particular class of alternative models, the decision maker also includes other statistical specifications in the set of models that concern him. The

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<sup>26</sup>Bansal and Yaron (2004) and Hansen and Sargent (2010) both start from the observation that two such models are difficult to distinguish empirically, but they draw different conclusions from that observation. Bansal and Yaron use the observation to justify a representative consumer who with complete confidence embraces one of the models (the long-run risk model with persistent log consumption growth), while Hansen and Sargent use the observation to justify a representative consumer who initially puts prior probability .5 on both models and who continues to carry along both models when evaluating prospective outcomes.

robust planner's worst-case model responds to these forms of model ambiguity partly by having more persistence than the baseline models. Our approach acquires tractability because the worst-case model turns out to be a time-invariant model in which projections for long-term growth are more cautious and stochastic growth is more persistent than in the baseline model. Worst-case shock distributions shift adversely and imply additional persistence that gives rise to enduring effects on uncertainty prices. Adverse shifts in the shock distribution that drive up the absolute magnitudes of uncertainty prices were also present in some of our earlier work (for example, see Hansen et al. (1999) and Anderson et al. (2003)). In this paper, we induce state dependence in uncertainty prices in a different way, namely, by specifying a set of alternative models to capture concerns about the baseline model's specification of persistence in consumption growth.

Relative to rational expectations models, models of robustness and ambiguity aversion bring new parameters. In this paper, we extend work by Anderson et al. (2003) that calibrated those additional parameters by exploiting connections between models of statistical model discrimination and our way of formulating robustness. We build on mathematical formulations of Newman and Stuck (1979), Petersen et al. (2000), and Hansen et al. (2006). We pose an *ex ante* robustness problem that pins down a robustness penalty parameter  $\theta$  by linking it to an asymptotic measure of statistical discrimination between models. This asymptotic measure allows us to quantify a half-life for reducing the mistakes in selecting between competing models based on historical evidence. A large statistical discrimination rate implies a short half-life for reducing discrimination mistake probabilities. Anderson et al. (2003) and Hansen (2007) had studied the connection between conditional discrimination rates and uncertainty prices that clear security markets. By following Newman and Stuck (1979) and studying asymptotic discrimination rates, we link statistical discrimination half-lives to calibrated equilibrium uncertainty prices.

# Appendices

## A Relative entropy neighborhoods

In this appendix we establish Proposition 3.4. We start by constructing the following decomposition of  $\widehat{\Delta}$  to support our study of relative entropy neighborhoods:

$$\begin{aligned}\widehat{\Delta}(Z^H|x) &= \Delta(Z^H; Z^{\widehat{H}}|x) + \delta \int_0^\infty \exp(-\delta t) E\left(Z_t^H \log Z_t^{\widehat{H}} \middle| X_0 = x\right) dt \\ &\quad - \frac{1}{2} \int_0^\infty \exp(-\delta t) E\left[Z_t^H \xi(X_t) \middle| X_0 = x\right] dt\end{aligned}\tag{45}$$

A direct calculation gives us:

$$\begin{aligned}\delta \int_0^\infty \exp(-\delta t) E\left(Z_t^H \log Z_t^{\widehat{H}} \middle| X_0 = x\right) dt \\ = \int_0^\infty \exp(-\delta t) E\left[Z_t^H \left(H_t \cdot \widehat{H}_t - \frac{1}{2} \widehat{H}_t \cdot \widehat{H}_t\right) \middle| X_0 = x\right] dt.\end{aligned}$$

Substituting this result into the right-hand side of (45)

$$\begin{aligned}\widehat{\Delta}(Z^H|x) &= \Delta(Z^H; Z^{\widehat{H}}|x) + \int_0^\infty \exp(-\delta t) E\left[Z_t^H \left(H_t \cdot \widehat{H}_t\right) \middle| X_0 = x\right] dt \\ &\quad - \frac{1}{2} \int_0^\infty \exp(-\delta t) E\left[Z_t^H \left[\widehat{H}_t \cdot \widehat{H}_t + \xi(X_t)\right] \middle| X_0 = x\right] dt\end{aligned}\tag{46}$$

By restricting the size of the relative entropy neighborhoods of  $Z^{\widehat{H}}$ , we restrict the first term on the right-hand side to be small. We now consider the remaining terms. We consider a  $Z^{\widehat{H}}$  corresponding to one of our parametric alternatives:

$$\widehat{H}_t = \hat{\eta}(X_t) = \hat{\eta}_0 + \hat{\eta}_1 X_t.$$

In this case we study an optimization problem:

$$\max_H -\frac{1}{2} \int_0^\infty \exp(-\delta t) E\left(Z_t^H \left[\xi(X_t) + |\hat{\eta}(X_t)|^2 - 2H_t \cdot \hat{\eta}(X_t)\right] \middle| X_0 = x\right) dt.\tag{47}$$

subject to

$$\Delta(Z^H; Z^{\widehat{H}}|x) \leq \epsilon\tag{48}$$

We solve this problem in steps. We first solve a family of optimization problems with objectives indexed by a Lagrange multiplier  $\theta \geq 0$  on constraint (48), namely,

$$\max_H -\frac{1}{2} \int_0^\infty \exp(-\delta t) E \left( Z_t^H \left[ \xi(X_t) + |\hat{\eta}(X_t)|^2 - 2H_t \cdot \hat{\eta}(X_t) + \theta |H_t - \hat{\eta}(X_t)|^2 \right] \middle| X_0 = x \right) dt. \quad (49)$$

where  $H$  is a (progressively measurable) control process that influences the evolution of  $Z^H$ :

$$\begin{aligned} dX_t &= \hat{\phi} dt - \hat{\kappa} X_t dt + \sigma \cdot H_t dt + \sigma \cdot dW_t \\ dZ_t^H &= Z_t^H H_t \cdot dW_t. \end{aligned}$$

This problem is amenable to recursive methods for continuous-time optimization. We posit a value function of the form  $z\bar{v}(x)$ . The HJB equation for this problem is :

$$\begin{aligned} 0 = \max_h & -\delta \bar{v}(x, \theta) - \frac{1}{2} \tilde{\xi}(x) - \frac{1}{2} |\hat{\eta}(x)|^2 + \bar{v}'(x, \theta) \left[ \hat{\phi} - \hat{\kappa} x + \sigma \cdot h \right] + \frac{1}{2} |\sigma|^2 \bar{v}''(x, \theta) \\ & - \frac{\theta}{2} |h - \hat{\eta}(x)|^2 + h \cdot \hat{\eta}(x). \end{aligned} \quad (50)$$

where  $\hat{\eta}(x) = \hat{\eta}_0 + \hat{\eta}_1 x$ . Alternatively, we can view this HJB equation as producing a *weak* solution to an optimization problem in which a drift distortion to a Brownian motion is the control process that holds under the probability distribution implied by  $Z^H$ . In this case we suppress reference to the martingale and impose distributional properties on the implied Brownian motion.

Prior to solving the HJB equation, apply the change variables:  $d = h - \hat{\eta}$ . Then the the new HJB equation is:

$$\begin{aligned} 0 = \max_d & -\delta \tilde{v}(x, \theta) - \frac{1}{2} \xi(x) + \frac{1}{2} \hat{\eta}(x) \cdot \hat{\eta}(x) \\ & + \tilde{v}'(x, \theta) [\hat{\phi} - \hat{\kappa} x + \sigma \cdot \hat{\eta}(x) + \sigma \cdot d] + \frac{1}{2} |\sigma|^2 \tilde{v}''(x, \theta) \\ & - \frac{\theta}{2} d \cdot d + \hat{\eta}(x) \cdot d. \end{aligned}$$

We conjecture the following solution of this HJB equation:

$$\tilde{v}(x, \theta) = -\frac{1}{2} [\tilde{v}_2(\theta) x^2 + 2\tilde{v}_1(\theta) x + \tilde{v}_0(\theta)]$$

which implies a decision rule

$$d^*(x, \theta) = \frac{1}{\theta} [-\sigma \tilde{v}_2(\theta)x - \sigma \tilde{v}_1(\theta) + \hat{\eta}_1 x + \hat{\eta}_0].$$

Consider first terms in  $x^2$ :  $A[\tilde{v}_2(\theta)]^2 + B\tilde{v}_2(\theta) + C = 0$  where<sup>27</sup>

$$\begin{aligned} A &= -\frac{1}{2\theta}|\sigma|^2 + \frac{1}{\theta}|\sigma|^2 = \frac{1}{2\theta}|\sigma|^2 \\ B &= \hat{\kappa} + \frac{\delta}{2} - \sigma \cdot \hat{\eta}_1 - \frac{1}{\theta}\sigma \cdot \hat{\eta}_1 \\ C &= -\frac{1}{2}\xi_2 + \frac{1}{2}\hat{\eta}_1 \cdot \hat{\eta}_1 + \frac{1}{2\theta}\hat{\eta}_1 \cdot \hat{\eta}_1 \end{aligned}$$

We solve this quadratic equation for  $\tilde{v}_2(\theta)$

$$\tilde{v}_2(\theta) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

which is guaranteed to have a solution for  $\theta$  sufficiently large. Letting  $\theta$  tend to  $\infty$ , we find that

$$\tilde{v}_2(\infty) = \frac{\xi_2 - \hat{\eta}_1 \cdot \hat{\eta}_1}{2\hat{\kappa} + \delta - 2\sigma \cdot \hat{\eta}_1}.$$

Compute  $\tilde{v}_1$  by equating the coefficients on  $x$  in the HJB equation:

$$\tilde{v}_1(\theta) = \frac{\tilde{v}_2(\theta) \left[ \theta \hat{\phi} + (1 + \theta)(\sigma \cdot \hat{\eta}_0) \right] - (1 + \theta)(\hat{\eta}_0 \cdot \hat{\eta}_1) + \theta \xi_1}{\theta(\delta + \hat{\kappa}) + |\sigma|^2 \tilde{v}_2 - (1 + \theta)(\sigma \cdot \hat{\eta}_1)}$$

The limiting value is:

$$\tilde{v}_1(\infty) = \frac{\tilde{v}_2(\infty) \left[ \hat{\phi} + (\sigma \cdot \hat{\eta}_0) \right] - (\hat{\eta}_0 \cdot \hat{\eta}_1) + \xi_1}{(\delta + \hat{\kappa}) - (\sigma \cdot \hat{\eta}_1)}$$

Finally, compute  $\tilde{v}_0$  by solving by equating coefficients on the constant terms in the HJB equation:

$$\tilde{v}_0(\theta) = \frac{-|\sigma|^2 (\tilde{v}_1(\theta)^2 - \theta \tilde{v}_2(\theta)) - (1 + \theta)|\eta_0|^2 + 2\tilde{v}_1(\theta) \left[ \theta \hat{\phi} + (1 + \theta)(\sigma \cdot \hat{\eta}_0) \right] + \theta \xi_0}{\delta \theta}$$

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<sup>27</sup>We have recycled some notation including  $C$ . In this formula  $C$  denotes a real number and not the consumption process.

The limiting value is:

$$\tilde{v}_0(\infty) = \frac{|\sigma|^2 \tilde{v}_2(\theta) - |\eta_0|^2 + 2\tilde{v}_1(\infty) \left[ \hat{\phi} + \sigma \cdot \hat{\eta}_0 \right] + \xi_0}{\delta}$$

For each  $x$ , the function  $\tilde{v}(x, \cdot)$  of  $\theta$  is decreasing and strictly convex. Moreover, the limiting function of  $x$  as  $\theta$  tends to  $\infty$  is

$$\begin{aligned} \tilde{v}(x, \infty) &= \frac{1}{2} \int_0^\infty \exp(-\delta t) E \left( Z^{\hat{H}} [-\xi(X_t) + \hat{\eta}(X_t) \cdot \hat{\eta}(X_t)] \middle| X_0 = x \right) dt \\ &= \hat{\Delta} \left( Z^{\hat{H}} | x \right) \end{aligned}$$

which is the objective evaluated at the  $h = \hat{\eta}(x)$ .

The next step in solving optimization problem (47)-(48) is to set  $\theta$  to satisfy constraint (48) at equality. The function  $\tilde{v}$  is convex in  $\theta$  with a finite limit as  $\theta$  tends to  $\infty$ . We solve

$$\min_{\theta > 0} \tilde{v}(x, \theta) + \epsilon \theta. \quad (51)$$

Decreasing  $\epsilon$  increases the minimizing  $\theta$ . As  $\epsilon$  approaches zero, the minimizing  $\theta$  necessarily tends to  $\infty$ , driving the minimized value in (51) arbitrarily close to  $\tilde{v}(x, \infty)$ . The conclusion of Proposition 3.4 follows provided that  $\tilde{v}(x, \infty) < 0$ .

## B Operationalizing Chernoff entropy

In this appendix we show how to compute Chernoff entropies for parametric models of the form (17). Because the  $H$ 's associated with them take the form

$$H_t = \eta(X_t),$$

these alternative models are Markovian. This allows us to compute Chernoff entropy by using an eigenvalue approach of Donsker and Varadhan (1976) and Newman and Stuck (1979). We start by computing the drift of  $(Z_t^H)^s g(X_t)$  for  $0 \leq s \leq 1$  at  $t = 0$ :

$$\begin{aligned} [\mathbb{G}(s)g](x) &\doteq \frac{(-s + s^2)}{2} |\eta(x)|^2 g(x) + s g(x)' \sigma \cdot \eta(x) \\ &\quad - g'(x) \kappa x + \frac{g''(x)}{2} |\sigma|^2, \end{aligned}$$

where  $[\mathbb{G}(\mathbf{s})g](x)$  is the drift given that  $X_0 = x$ . Next we solve the eigenvalue problem

$$[\mathbb{G}(\mathbf{s})]e(x, \mathbf{s}) = -\rho(\mathbf{s})e(x, \mathbf{s}),$$

whose eigenfunction  $e(x, \mathbf{s})$  is the exponential of a quadratic function of  $x$ . We compute Chernoff entropy numerically by solving:

$$\chi(Z^H, x) = \max_{\mathbf{s} \in [0,1]} \rho(\mathbf{s}).$$

To deduce a corresponding equation for  $\log e$ , notice that

$$(\log e)'(x) = \frac{e'(x)}{e(x)}$$

and

$$(\log e)''(x) = \frac{e''(x)}{e(x)} - \left[ \frac{e'(x)}{e(x)} \right]^2.$$

For a positive  $g$

$$\begin{aligned} \frac{[\mathbb{G}(\mathbf{s})g](x)}{g(x)} &\doteq \frac{(-\mathbf{s} + \mathbf{s}^2)}{2} |\eta(x)|^2 + \mathbf{s}(\log g)'(x)\sigma \cdot \eta(x) - (\log g)'(x)\kappa x \\ &+ \frac{\log g''(x)}{2} |\sigma|^2 + \frac{[\log g'(x)]^2}{2} |\sigma|^2. \end{aligned} \quad (52)$$

Using formula (52), define

$$[\overline{\mathbb{G}}(\mathbf{s}) \log g](x) = \frac{[\mathbb{G}(\mathbf{s})g](x)}{g(x)}.$$

Then we can solve

$$[\overline{\mathbb{G}}(\mathbf{s}) \log e](x, \mathbf{s}) = -\rho(\mathbf{s})$$

for  $\log e$  to compute the positive eigenfunction  $e$ .

These calculations allow us numerically to compute the largest and smallest Chernoff entropies attained by members of the set  $\mathcal{Z}^o$ . In our analysis we consider alternative models of the form:

$$\begin{aligned} d \log C_t &= (.01) (\mu + \beta X_t) dt + (.01) \alpha \cdot d\widetilde{W}_t \\ dX_t &= \phi dt - \kappa X_t dt + \sigma \cdot d\widetilde{W}_t. \end{aligned}$$

1. Input values of  $\kappa$ ,  $\mu$ , and  $\phi$ . Construct the implied  $h(x) = \eta_1 x + \eta_0$  by solving;

$$\begin{bmatrix} \mu - \hat{\mu} \\ \phi - \hat{\phi} \end{bmatrix} = \begin{bmatrix} \alpha' \\ \sigma' \end{bmatrix} \eta_0$$

$$\begin{bmatrix} \beta - \hat{\beta} \\ \hat{\kappa} - \kappa \end{bmatrix} = \begin{bmatrix} \alpha' \\ \sigma' \end{bmatrix} \eta_1$$

for  $\eta_1$  and  $\eta_0$ .

2. For a given  $\mathbf{s}$ , construct  $\zeta_0$ ,  $\zeta_1$ ,  $\zeta_2$ ,  $\check{\kappa}$ ,  $\check{\beta}$ ,  $\check{\phi}$ , and  $\check{\mu}$  from:

$$(-\mathbf{s} + \mathbf{s}^2)|h(x)|^2 = (-\mathbf{s} + \mathbf{s}^2)|\eta_0 + \eta_1 x|^2 = -(\zeta_0 + 2\zeta_1 x + \zeta_2 x^2)$$

$$\check{\kappa} = (1 - \mathbf{s})\hat{\kappa} + \mathbf{s}\kappa$$

$$\check{\beta} = (1 - \mathbf{s})\hat{\beta} + \mathbf{s}\beta$$

$$\check{\phi} = (1 - \mathbf{s})\hat{\phi} + \mathbf{s}\phi$$

$$\check{\mu} = (1 - \mathbf{s})\hat{\mu} + \mathbf{s}\mu$$

3. Solve

$$-\rho(\mathbf{s}) = -\frac{1}{2}(\zeta_0 + 2\zeta_1 x + \zeta_2 x^2) + (\check{\phi} - x\check{\kappa})(\log e)'(x)$$

$$+ \frac{(\log e)''(x)}{2}|\sigma|^2 + \frac{[(\log e)'(x)]^2}{2}|\sigma|^2$$

where  $\log e(x, \mathbf{s}) = \lambda_1 x + \frac{1}{2}\lambda_2 x^2$ . Thus,

$$\lambda_2 = \frac{\check{\kappa} - \sqrt{(\check{\kappa})^2 + \zeta_2|\sigma|^2}}{|\sigma|^2}.$$

Given  $\lambda_2$ ,  $\lambda_1$  solves

$$-\zeta_1 - \check{\kappa}\lambda_1 + \lambda_1\lambda_2|\sigma|^2 + \check{\phi}\lambda_2 = 0$$

or

$$\lambda_1 = \frac{\zeta_1 - \check{\phi}\lambda_2}{\lambda_2|\sigma|^2 - \check{\kappa}} = -\frac{\zeta_1 - \check{\phi}\lambda_2}{\sqrt{(\check{\kappa})^2 + \zeta_2|\sigma|^2}}.$$

Finally,

$$\rho(\mathbf{s}) = \frac{1}{2}\zeta_0 - \frac{1}{2}|\sigma|^2\lambda_2 - \frac{1}{2}|\sigma|^2(\lambda_1)^2 - \check{\phi}\lambda_1.$$

4. Repeat for alternative  $\mathbf{s}$ 's and maximize  $\rho(\mathbf{s})$  as a function of  $\mathbf{s}$ .

## C Robust value function

In this appendix we provide formulas for the robust planners problem. Specifically, we solve for  $v$  given  $\theta$ .

Consider the HJB equation:

$$\begin{aligned} 0 = \min_h & -\delta v(x) + (.01)(\hat{\mu} + \hat{\beta}x) + v'(x)(\hat{\phi} - \hat{\kappa}x) + \frac{1}{2}|\sigma|^2 v''(x, \theta) \\ & + (.01)\alpha \cdot h + v'(x, \theta)\sigma \cdot h + \frac{\theta}{2}|h|^2 - \frac{\theta}{2}(\xi_2 x^2 + 2\xi_1 x + \xi_0) \end{aligned}$$

Recall that the value function is quadratic

$$v(x, \theta) = -\frac{1}{2}[v_2(\theta)x^2 + 2v_1(\theta)x + v_0(\theta)]$$

which implies

$$\eta^*(x, \theta) = -\frac{1}{\theta}[.01\alpha - \sigma v_2(\theta)x - \sigma v_1(\theta)].$$

We can solve for  $v_2$ ,  $v_1$ , and  $v_0$  by matching the coefficients for  $x^2$ ,  $x$  and the constant terms, respectively. Solving first for  $v_2$  collection the  $x^2$  terms. This results in the quadratic equation of the form  $A[v_2(\theta)]^2 + Bv_2(\theta) + C = 0$  where

$$\begin{aligned} A &= \frac{1}{2\theta}|\sigma|^2 - \frac{1}{\theta}|\sigma|^2 = -\frac{1}{2\theta}|\sigma|^2 \\ B &= \hat{\kappa} + \frac{\delta}{2} \\ C &= -\frac{\theta}{2}\xi_2. \end{aligned}$$

Then

$$v_2(\theta) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$= \theta \left[ \frac{\delta + 2\hat{\kappa} - \sqrt{(\delta + 2\hat{\kappa})^2 - 4|\sigma|^2 \xi_2}}{2|\sigma|^2} \right]$$

$$v_1(\theta) = 2 \left[ \frac{-.01\hat{\beta} - \frac{.01}{\theta} (\alpha \cdot \sigma) v_2(\theta) + \hat{\phi}v_2(\theta) + \theta\xi_1}{\delta + \sqrt{(\delta + 2\hat{\kappa})^2 - 4|\sigma|^2 \xi_2}} \right]$$

$$v_0(\theta) = \frac{1}{\delta} \left[ -.02\hat{\mu} + 2\hat{\phi}v_1(\theta) + |\sigma|^2 v_2(\theta) + \frac{1}{\theta} |.01\alpha - \sigma v_1(\theta)|^2 + \theta\xi_0 \right]$$

## D Construction of an entropy ball

In section 7 we construct an entropy ball  $\mathcal{Z}^*$  that contains  $\bar{\mathcal{Z}}$ . In forming the entropy ball, we consider  $Z^H$ 's that satisfy

$$\bar{\Delta}(Z^H) \leq 0.$$

At the boundary of this set

$$\begin{aligned} & \frac{1}{2} \int \left[ \int_0^\infty \exp(-\delta t) E \left( Z_t^H |H_t|^2 \middle| X_0 = x \right) dt \right] Q(dx) \\ &= \frac{1}{2} \int \left[ \int_0^\infty \exp(-\delta t) E \left( Z_t^H \xi(X_t) \middle| X_0 = x \right) dt \right] Q(dx). \end{aligned}$$

Given this equality, we lead to compute the maximal relative entropy over the the set  $\bar{\mathcal{Z}}$  by first solving:

$$\min_H -\frac{1}{2} \int_0^\infty \exp(-\delta t) E \left( Z_t^H [\xi(X_t) - \theta [|H_t|^2 - \xi(X_t)]] \middle| X_0 = x \right) dt. \quad (53)$$

where  $H$  is a (progressively measurable) control process that influences the evolution of  $Z^H$ :

$$\begin{aligned} dX_t &= \hat{\phi}dt - \hat{\kappa}X_t dt + \sigma \cdot H_t dt + \sigma \cdot dW_t \\ dZ_t^H &= Z_t^H H_t \cdot dW_t. \end{aligned}$$

We pose this as minimization problem to exploit is mathematical similarity to the robust planning problem.

Consider the HJB equation:

$$0 = \min_h -\delta v(x) + v'(x)(\hat{\phi} - \hat{\kappa}x) + \frac{1}{2}|\sigma|^2 v''(x, \theta) \\ + v'(x, \theta)\sigma \cdot h + \frac{\theta}{2}|h|^2 - \frac{\theta+1}{2}(\xi_2 x^2 + 2\xi_1 x + \xi_0)$$

Guess the value function is quadratic:

$$v(x, \theta) = -\frac{1}{2}[v_2(\theta)x^2 + 2v_1(\theta)x + v_0(\theta)]$$

which implies

$$h^* = \frac{1}{\theta}[\sigma v_2(\theta)x + \sigma v_1(\theta)].$$

We can solve for  $v_2$ ,  $v_1$ , and  $v_0$  by matching the coefficients for  $x^2$ ,  $x$  and the constant terms, respectively. Solving first for  $v_2$  collection the  $x^2$  terms. This results in the quadratic equation of the form  $A[v_2(\theta)]^2 + Bv_2(\theta) + C = 0$  where

$$A = \frac{1}{2\theta}|\sigma|^2 - \frac{1}{\theta}|\sigma|^2 = -\frac{1}{2\theta}|\sigma|^2 \\ B = \hat{\kappa} + \frac{\delta}{2} \\ C = -\frac{(\theta+1)}{2}\xi_2$$

Then

$$v_2(\theta) = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \\ = \theta \left[ \frac{\delta + 2\hat{\kappa} - \sqrt{(\delta + 2\hat{\kappa})^2 - 4\left(\frac{\theta+1}{\theta}\right)|\sigma|^2 \xi_2}}{2|\sigma|^2} \right]$$

We equate coefficients on terms involving  $x$  and find that

$$v_1(\theta) = 2 \left[ \frac{\hat{\phi}v_2(\theta) + (\theta+1)\xi_1}{\delta + \sqrt{(\delta + 2\hat{\kappa})^2 - 4|\sigma|^2 \xi_2 \frac{(\theta+1)}{\theta}}} \right].$$

Finally, by equating the constant contributions:

$$v_0(\theta) = \frac{1}{\delta} \left[ 2\hat{\phi}v_1(\theta) + (\theta + 1)\xi_0 + v_2(\theta)|\sigma|^2 + \frac{1}{\theta}v_1(\theta)^2|\sigma|^2 \right].$$

We solve for  $\theta$  by solving

$$\max_{\theta} \int v(x, \theta)Q(dx).$$

The negative of this solution scaled by  $2\delta$  is the constant  $\xi^*$  used in place of  $\xi$  when forming the relative entropy ball of interest.

## E Supplemental Appendix

### E.1 Computing shock elasticities

We compute shock price elasticities in four steps:

1. *Stochastic impulse response for*  $\log S$ . We solve the recursion:

$$\begin{aligned} d\log S_t^1 &= -.01\hat{\beta}X_t^1dt + H_t^1 \cdot dW_t - H_t^* \cdot H_t^1dt \\ dX_t^1 &= -\hat{\kappa}X_t^1dt \\ dX_t &= -\hat{\kappa}X_t + \sigma \cdot dW_t \\ H_t^* &= \eta_0^* + \eta_1^*X_t \\ H_t^1 &= \eta_1^*X_t^1 \end{aligned}$$

where  $X_0^1 = \sigma \cdot \mathbf{u}$  and  $\log S_0^1 = -.01\alpha \cdot \mathbf{u} + h^*(x) \cdot \mathbf{u}$ . The quadratic terms in the evolution equation of  $\log S$  make  $\log S_t^1$  stochastic.

2. *Deterministic impulse response for*  $\log C$ . We solve the recursion:

$$\begin{aligned} d\log C_t^1 &= .01\hat{\beta}X_t^1dt \\ dX_t^1 &= -\hat{\kappa}X_t^1dt, \end{aligned}$$

where  $\log C_0^1 = (.01)\alpha \cdot \mathbf{u}$  and  $X_0^1 = \sigma \cdot \mathbf{u}$ . Thus,

$$\log C_t^1 = \frac{.01\hat{\beta}}{\hat{\kappa}} [1 - \exp(-\hat{\kappa}t)] \sigma \cdot \mathbf{u} + (.01)\alpha \cdot \mathbf{u}$$

3. *Compute*

$$\frac{E(M_t \log S_t^1 | X_0 = x)}{E(M_t | X_0 = x)}$$

where  $M_t = S_t \left( \frac{C_t}{C_0} \right)$ . Note that

$$d\log M_t = -\delta dt + H_t^* \cdot dW_t - \frac{1}{2}|H_t^*|^2 dt.$$

Let  $dW_t$  have drift  $H_t$  and compute expectations conditioned on  $X_0 = x$  recursively:

$$\begin{aligned}
d \log S_t^1 &= -.01 \hat{\beta} X_t^1 dt + H_t^1 \cdot (H_t dt + d\widetilde{W}_t) - H_t^* \cdot H_t^1 dt \\
&= -.01 \hat{\beta} X_t^1 dt + H_t^1 \cdot d\widetilde{W}_t \\
dX_t^1 &= -\hat{\kappa} X_t^1 dt \\
H_t^* &= \eta_0^* + \eta_1^* X_t \\
H_t^1 &= \eta_1^* X_t^1,
\end{aligned}$$

where  $X_0^1 = \sigma \cdot \mathbf{u}$  and  $\log S_0^1 = -(.01)\alpha \cdot \mathbf{u} + h^*(x) \cdot \mathbf{u}$ . Thus,

$$\frac{E(M_t \log S_t^1 | X_0 = x)}{E(M_t | X_0 = x)} = -\frac{.01 \hat{\beta}}{\hat{\kappa}} [1 - \exp(-\hat{\kappa}t)] \sigma \cdot \mathbf{u} - .01 \alpha \cdot \mathbf{u} + h^*(x) \cdot \mathbf{u}.$$

#### 4. Construct elasticities:

(a) Shock-exposure elasticity for consumption:

$$\frac{E\left(\frac{C_t}{C_0} \log C_t^1 | X_0 = x\right)}{E\left(\frac{C_t}{C_0} | X_0 = x\right)} = \frac{.01 \hat{\beta}}{\hat{\kappa}} [1 - \exp(-\hat{\kappa}t)] \sigma \cdot \mathbf{u} + .01 \alpha \cdot \mathbf{u},$$

which is also the continuous time impulse response for  $\log C$ .

(b) Shock-price elasticity

$$\begin{aligned}
\frac{E\left(\frac{C_t}{C_0} \log C_t^1 | X_0 = x\right)}{E\left(\frac{C_t}{C_0} | X_0 = x\right)} - \frac{E[M_t (\log S_t^1 + \log C_t^1) | X_0 = x]}{E(M_t | X_0 = x)} &= -\frac{E(M_t \log S_t^1 | X_0 = x)}{E(M_t | X_0 = x)} \\
&= \frac{.01 \hat{\beta}}{\hat{\kappa}} [1 - \exp(-\hat{\kappa}t)] \sigma \cdot \mathbf{u} + .01 \alpha \cdot \mathbf{u} - h^*(x) \cdot \mathbf{u}.
\end{aligned}$$

## E.2 Consumption distributions

Starting at date zero, the expected growth rate at  $t$  is:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \exp\left(\begin{bmatrix} 0 & \beta \\ 0 & -\kappa \end{bmatrix} t\right) \begin{bmatrix} \mu \\ \phi \end{bmatrix} = \mu + \frac{\beta \phi}{\kappa} - \frac{\beta \phi}{\kappa} \exp(-\kappa t).$$

Integrating this growth rate over an interval  $[0, t]$  gives a worst-case trend for log consumption:

$$\left(\mu + \frac{\beta\phi}{\kappa}\right)t + \frac{\beta\phi}{\kappa^2} \exp(-\kappa t) - \frac{\beta\phi}{\kappa^2} \quad (54)$$

Notice that the initial growth trend growth rate is  $\mu$  and that the eventual growth rate is  $\mu + \frac{\phi}{\kappa}$ . In this calculation, we impose the distorted model starting at date zero and consider its implications going forward. The shift in the constant term for the evolution of  $X$  has no immediate impact on the growth of  $\log C$ . Its eventual impact is determined in part by the persistence parameter  $\kappa$ .

Next we consider the distributional impacts. The new information about  $\log C_t - \log C_0$  (scaled by 100) is:

$$\begin{aligned} \beta \int_0^t \int_0^u \exp[-\kappa(u-r)] \sigma \cdot dW_r du + \int_0^t \alpha \cdot dW_u \\ &= \beta \int_0^t \int_r^t \exp[-\kappa(u-r)] du \sigma \cdot dW_r + \int_0^t \alpha \cdot dW_u \\ &= \beta \frac{1}{\kappa} \int_0^t \exp(\kappa r) [\exp(-\kappa r) - \exp(-\kappa t)] \sigma \cdot dW_r \\ &\quad + \int_0^t \alpha \cdot dW_u \\ &= \frac{\beta}{\kappa} \int_0^t (1 - \exp[-\kappa(t-r)]) \sigma \cdot dW_r + \int_0^t \alpha \cdot dW_u \end{aligned}$$

The variance is .0001 times the following object

$$\begin{aligned} &\frac{\beta^2}{\kappa^2} \int_0^t [1 - 2 \exp[-\kappa(t-r)] + \exp[-2\kappa(t-r)]] |\sigma|^2 dr + |\alpha|^2 t \\ &\quad + 2 \frac{\alpha \cdot \beta}{\kappa} \int_0^t (1 - \exp[-\kappa(t-r)]) dr \\ &= \frac{\beta^2}{\kappa^2} |\sigma|^2 t + |\alpha|^2 t - \frac{2\beta^2}{\kappa^3} [1 - \exp(-t\kappa)] |\sigma|^2 + \frac{\beta^2}{2\kappa^3} [1 - \exp(-2\kappa t)] |\sigma|^2 \\ &\quad + 2 \frac{\beta(\alpha \cdot \sigma)}{\kappa^2} [\kappa t - 1 + \exp(-t\kappa)] \end{aligned}$$

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