Commodity and Token Monies

Thomas J. Sargent*

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Abstract

A government defines a dollar as a list of quantities of one or more precious metals. If issued in limited amounts, token money is a perfect substitute for precious metal money. Atemporal equilibrium conditions determine how quantities of precious metals and token monies affect an equilibrium price level. Within limits, a government can peg the relative price of two precious metals, confirming Fisher's (1911) response to a classic criticism of bimetallism. Monometallism dominates bimetallism according to a natural welfare criterion.

Keywords: Gold, silver, token money, bimetallism, quantity theory of money, gold exchange standard, (not a) fiscal theory of the price level

*New York University, E-mail: thomas.sargent@nyu.edu. I thank Francesco Lippi, Rodolfo Manuelli, Harald Uhlig, two referees, and editor Morten Raven for helpful comments. I thank Chenghan Hou and Ling Long for technical help.

1 Introduction

This paper analyzes (1) how supplies of two types of currency, one a precious metal, the other a token, affect the price level; and (2) how two sources of demand for a precious metal, one as a consumption good,[1] the other as money, affect the price of the precious metal relative to other consumption goods. These are two sides of the same coin. A commodity money system binds the price level to the price of a precious metal relative to other consumption goods.[2] A government defines a dollar as a number of ounces of a precious metal.[3] Government or private agents might also issue tokens that exchange one-for-one with money that is precious metal. Although they influence both the price level and the allocation of metals between monetary and non monetary uses, limited amounts of token monies are compatible with a commodity standard. But excess issues of token money can sever the link between the price level and the price of a precious metal relative to other consumption goods and thereby break a commodity standard. Nevertheless, David Ricardo (1816) urged issuing the maximum amount of token money consistent with leaving a commodity standard intact. We discuss Ricardo's proposal in section 4.3.

Our analysis is old fashioned and entirely static despite theories of the demand for currency naturally being intertemporal.[4] Our justification for conducting a static analysis is that instances of the atemporal inequalities that summarize the forces in our model prevail in dynamic settings too, as readers of Barro (1979), Velde and Weber (2000), Velde (2002) or appendix C of this paper, for example, will recognize. We start with a simple parametric theory of currency demand that we take for granted has emerged from a dynamic theory, then focus on atemporal forces shaping demands for distinct objects, some precious metals, others intrinsically worthless tokens, that comprise the currency stock.[5] The heart of the theory is that, under appropriate circumstances, these objects are perfect substitutes when used as money, but imperfect substitutes when not used as money.

We assume throughout that commodity monies are valued by weight and not by face

[1] Or maybe as an "industrial" good.

[2] Or maybe a bundle of consumption goods.

[3] Or in a multi-metallic standard, as given numbers of ounces of metal 1, metal 2, ... See section 5.

[4] Therefore, our analysis leaves unmentioned intertemporal forces emanating from a sequence of government budget constraints that are the focus of fiscal theories of inflation and the price level in the spirit of Sargent and Wallace (1981). In our framework, these forces would manifest themselves in the form of pressures to adjust outstanding amounts of token money m.

[5] Assuming outcomes of a more comprehensive but unarticulated theory is what Leontief (1937) called implicit theorizing. He used that term to characterize and criticize Keynes's *General Theory*. But appendix C sketches a dynamic model, the steady state of which matches our static analysis.

value.[6] Section 2 describes open and closed economy models of a monometallic standard. Sections 3 and 4 describe how the monometallic model expresses ideas and policy recommendations of Smith (1937), Ricardo (1816), Fisher (1911), Hawtrey (1919), and others. Section 5 describes a closed-economy model of a bimetallic standard. We use this model to discuss the "Crime of '73" and how the suboptimality of bimetallism that prevails in the model suggests that maybe those notorious events of 1873 should have been regarded not as a crime but as a good deed. Appendix B contains proofs.

2 Monometallism

2.1 Cast of characters and units

A representative consumer enjoys two consumption goods, one that we call "standard" consumption c, measured in units of that good, and silver s, measured in ounces. The standard consumption good has one use. Silver has two uses because an amount $\bar{s} - s \geqslant 0$ of silver is also used as money. There are competitive markets in silver and the standard consumption good. The consumer's preferences are ordered by $u(c, s) = Ac^{\alpha}s^{1-\alpha}$, $\alpha \in (0, 1), A > 0$. The ratio of the consumer's marginal utilities for s and c is

$$\frac{u_c}{u_s} = \frac{\alpha}{1-\alpha}\frac{s}{c}.$$

This ratio pins down the relative price of silver and the other consumption good. The government defines a dollar as consisting of $\frac{1}{e}$ ounces of silver. The exchange rate e has units of dollars per ounce of silver. The price of c relative to dollars is denoted p and is measured in dollars per unit of the consumption good; p is also called the nominal price level. The price of the standard consumption good relative to silver is $\frac{p}{e}$, whose units are ounces of silver per unit of consumption good. There is also a nonnegative amount of token money m, whose units are dollars.[7] When $0 < s < \bar{s}$, the total stock of currency is $M = e(\bar{s} - s) + m$.

[6] Sargent and Smith (1997) and Sargent and Velde (2002) describe equilibria in which commodity monies are valued by face value, i.e., "by tale", instead of by weight.

[7] Milton Friedman was fond of calling token money "fiduciary" money.

2.2 Structural equations

The model has two structural equations, one imposing equality between demand for and supply of real money balances, another imposing equality between demands for and supplies of silver and the other consumption good:

$$\frac{e(\bar{s}-s)+m}{p} = kc, \quad k > 0 \tag{1}$$

$$\frac{p}{e} = \frac{\alpha}{1-\alpha}\frac{s}{c} \tag{2}$$

Equation (1) is a "quantity theory of money" that equates the supply of real balances $\frac{M}{p}$ with the demand for real balances kc, where k is the "Cambridge k".[8] Equation (2) equates the price of silver relative to the consumption good on the left side to the ratio of marginal utilities $\frac{u_c}{u_s}$ on the right side.[9]

2.3 Closed and open economies

We study two versions of the model. In both we take the total supplies $\check{c}, \check{s}$ as exogenous.

A *small open economy* version restricts c and $\bar{s}$ by

$$c - \check{c} = -\frac{e}{p}(\bar{s} - \check{s}). \tag{3}$$

In this version $\check{c}, \check{s}$ are exogenous domestic endowments of the standard consumption good and silver. They can be traded internationally subject to the budget constraint (3).

A *closed economy* version imposes

$$c = \check{c}, \bar{s} = \check{s}. \tag{4}$$

In this version, $\check{c}, \check{s}$ are exogenous endowments of the standard consumption good and silver that must be allocated domestically.

[8]See Johnson (1962).

[9]Appendix C describes a dynamic version of the household's optimization problem that would emerge from a cash-in-advance constraint like those in Lucas (1982). The appendix model assumes that the standard consumption good is not durable and that silver is perfectly durable. A steady state of the associated dynamic model has versions of equations (1) and (2) that differ only in their constant terms. See equations (40) and (41).

2.4 Open economy monometallism

The structural equations for the open economy model are (1), (2), and (3). Endogenous variables are $s, \bar{s}, c$ and exogenous variables are $p, e, m, \check{s}, \check{c}$. Equilibrium c satisfies

$$c = \frac{\alpha}{1 + k\alpha}\left[\frac{e}{p}\check{s} + \check{c} + \frac{m}{p}\right] \tag{5}$$

and equation (2) implies that equilibrium s is proportional to c:

$$s = \frac{p}{e}\frac{1-\alpha}{\alpha}c. \tag{6}$$

To maintain convertibility of tokens for silver, m must respect

$$0 \leqslant m \leqslant k\alpha\left[\frac{e}{p}\check{s} + \check{c}\right]. \tag{7}$$

Equation (5) implies that if $m = 0$

$$c = \frac{\alpha}{1 + k\alpha}\left[\frac{e}{p}\check{s} + \check{c}\right],$$

and that if m is at the upper bound in inequality (7) associated with retaining convertibility, then

$$c = \alpha\left[\frac{e}{p}\check{s} + \check{c}\right].$$

2.5 Closed economy monometallism

In the closed economy, c and $\bar{s}$ can be taken as exogenous by virtue of (4). We delineate two regions of the parameter space.

In a first region, the quantity of token money m is small enough to allow the government to peg the exchange rate e by defining a dollar as being worth (i.e., exchanging for) a particular amount of silver. In this region, p, s are endogenous variables, while $e, m, \bar{s}$ $(= \check{s}), c$ $(= \check{c})$ are exogenous variables. Except at an upper bound on the stock of token money, some silver is used as money.

In a second region, the quantity of token money is so large that no silver is used as money. Here the endogenous variables are $s, \bar{s}, c$ while the exogenous variables are $p, e, m, \check{s}, \check{c}$. The quantity of token dollars is too large to allow the government to succeed in naming e as the quantity of silver that defines (i.e., that exchanges for) a dollar. No silver can be exchanged for token money at the government mandated price e. Instead, it trades at a price $\tilde{e}$ determined by "the market".

Proposition 2.1. *To be in the first regime in which silver coins and tokens coexist as money, the quantity of token money must respect the bounds:*

$$0 \leqslant m \leqslant k\frac{\alpha}{1-\alpha}\check{s}e. \tag{8}$$

Setting $m > k\frac{\alpha}{1-\alpha}\check{s}e$, *puts the economy in the second regime in which silver trades for token money at the exchange rate* $\tilde{e} = \frac{1-\alpha}{\alpha}\frac{\check{c}}{\check{s}}p > e$ *and the price level* p *satisfies* $\frac{m}{p} = k\check{c}$.

2.5.1 Equilibrium price level

Appendix A computes equilibrium p when m respects the bounds (8). In the interior of the set of token money m values that satisfy (8), the price level is an increasing function of m. Equation (2) captures the main force leading to this outcome. Since e is fixed, equation (2) describes an increase in the price level sufficient to bring about the reduced relative price of silver in terms of the standard consumption good that is required by the increase in the equilibrium $\frac{\alpha}{1-\alpha}\frac{s}{c}$ provoked by the increase in m.

Figure 1 offers a graphical version of the closed economy model. The downward sloping curve is a demand for nonmonetary uses of silver defined by equation (2). The upward sloping curves are supplies of nonmonetary silver associated with different levels of $\frac{m}{pc} \in [0, k]$ implied by (1). Equilibrium $\frac{e}{p}$ occurs at the intersection of demand and supply. Both supply and demand curves are rectangular hyperbolas, demand with respect to $s = 0$, supply with respect to $s - \bar{s}$.

3 Open economy doctrines and episodes

We can use the monometallic model to express several classic doctrines and to describe events that led monetary theorists to create them. We begin with doctrines about small open economies.

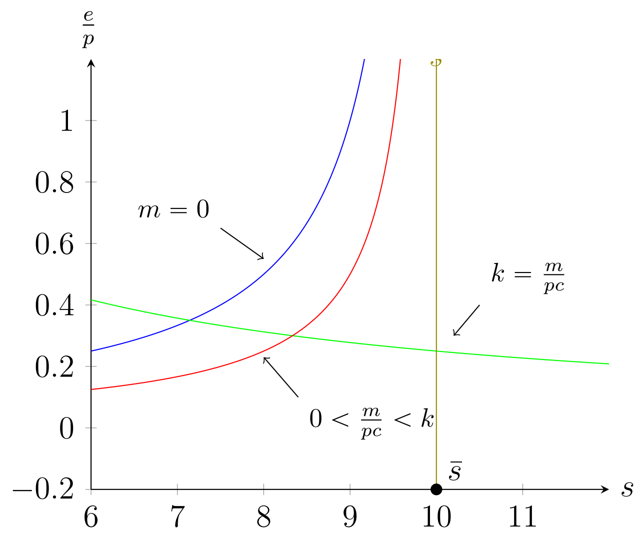

Figure 1: Downward sloping curve: $\frac{e}{p} = \frac{1-\alpha}{\alpha}\frac{\bar{c}}{s}$. Upward sloping curves: $\frac{e}{p} = \frac{\left(k - \frac{m}{p}\right)}{\bar{s}-s}$ for $m = 0$ and for $0 < \frac{m}{pc} < k$.

3.1 Adam Smith's consumption boon

Adam Smith (1937, pp. 304-305) asserted that a small open economy can foster a consumption boon by replacing commodity money with intrinsically value-less tokens. To express Adam Smith's reasoning, we state

Problem 3.1.

$$\max_{m \in M} u(c, s)$$

where the maximization is subject to equations (5) *and* (6) *and* M *is the set of* m*'s satisfying inequalities* (7).

Proposition 3.2. *The solution of problem 3.1 sets* m *at or above the upper bound in* (7).

If m is lower than its upper bound in (7), small increments in m increase both c and s but leave the price level unchanged. If m exceeds the upper bound in inequalities (7), the market price of silver is $\tilde{e} > e$ and the price level p denominated in dollars is proportional to m. In this case, units of account are dollars, which are not anchored to ounces of silver. However, it is possible that the *unit of account* can remain silver even when m exceeds the upper bound in inequalities (7). In this case, the price level measured in "silver dollars" equals the exogenous "world price" p denominated in "silver" (a.k.a. the "hard currency"), while "token" or paper money trades at a discount, exchanging for $\frac{1}{\tilde{e}}$ rather than $\frac{1}{e}$ vis a vis the "hard currency."

Token money was issued in amounts exceeding the upper bound in (7) during both the US War of Independence and the US Civil War. During the first of these wars, paper money traded at a discount relative to precious metal money, while during the second war, precious metal coins traded at a premium relative to the government issued paper money. During the War of Independence, the unit of account continued to be the Spanish silver dollar. After too many of them had been issued, paper dollars issued by the Continental Congress depreciated relative to the Spanish dollar; eventually $\tilde{e}$ rose by a factor of about 100. But commodity prices and bond contracts continued to be denominated in Spanish dollars.[10] By way of contrast, during the US Civil War, except on the West Coast, the unit of account was a token money called the greenback dollar issued by the Union government.[11] Goods prices and bond contracts were typically denominated in greenbacks. Gold dollars traded at a premium over greenbacks, rising to over two greenbacks per gold dollar in 1864, so that $\tilde{e}$ was more than twice e and prices denominated in greenbacks rose approximately proportionately with $\tilde{e}$.

Because our model is silent about what determines the unit of account, it cannot explain why the precious metal coin remained the unit of account during the War of Independence while the overissued token currency became the unit of account during the Civil War. An arrangement outside our model – the successful enforcement in Eastern and Midwestern states of the legal tender provision of the 1862 Civil War Act authorizing greenbacks – probably explains the difference.[12]

3.2 "Too much token money"

Accounts of several inflation episodes – e.g., the North American War of Independence, the North American Civil War – rest on calculating an upper bound on an amount of token currency consistent with sustaining convertibility to precious metal at the official exchange rate, like the one appearing in (7), as well as descriptions of inflations that occurred when a government issued a token currency that exceeded that bound. These

[10] For example, see the price series constructed by David and Solar (1977).

[11] Rothbard (2002, pp. 127-128) describes how California and Oregon businessmen and courts maintained the gold standard during the Civil War and accepted greenbacks only at discounts relative to gold dollars.

[12] See Dewey (1912, ch. XII). Adams (1887, p. 144) noted that "by declaring its promises to be a legal-tender in payment of of private debts, the government virtually threatens to withdraw full protection in matters of contract from every man who refuses to assist in floating its notes."

calculations typically consist of these parts: first, an estimate of the demand kc for real balances during a hard money period immediately preceding large issues of token currency; second, an estimate of additions to stocks of the token currency that would not imperil its convertibility at par into precious metal coins; and third, an estimate of the date at which issues of token currency passed the upper bound consistent with convertibility. For example, the stock of specie circulating in the 13 North American colonies at the beginning of the War of Independence was about 30 million Spanish dollars. In 1775, the Continental Congress began issuing papers dollars. Each Continental dollar promised one Spanish dollar. Continental dollars depreciated substantially relative to Spanish dollars only after 1776 when the Continental Congress had issued more than 30 million Continental dollars. Estimates of kc for 1860 immediately before the U.S. civil were about 200 million dollars. Gold and silver appreciated significantly relative to greenbacks only after the Union had issued more than 200 million dollars of greenbacks.[13]

4 Closed economy doctrines and episodes

The "small open economy" doctrines and instances described in section 3 take as given the relative price of silver and other consumption. We now turn to doctrines about a closed economy. These are about effects of alterations in the stock of tokens on the relative price of silver and other consumption goods.

4.1 Price level as relative price

When $m = 0$, equation (30) tells us that an increase in the supply of silver increases the price level, not through a "quantity theory" effect that makes p is proportional to m, but by decreasing the price of silver relative to other consumption goods. Various authors used that effect to explain rises in price levels after Spanish expropriations of silver and gold from native Americans were brought to Europe during the 16th and 17th centuries.[14] These authors explained price level changes in terms of "ordinary microeconomics", not a special "monetary economics".

[13] Sargent and Velde (2002, ch. 14) offer an account of an experiment in 17th century Spain in which moderate issues of copper token coins displaced silver but at first led to little inflation. But larger increases in tokens caused substantial inflation. Sargent and Velde (2002, chs. 13,15) describe similar episodes in other countries.

[14] See Parker (1974) and Sargent and Velde (2002, ch. 11).

When $m > 0$ respects the bounds (7), the relevant larger root of equation (24) of appendix A asserts the same relative price effect that we have just discussed. In addition, when m respects the bounds, equation (24) tells how increases in m affect the price level by changing the relative price of goods for silver, an effect that we dub a "Hawtrey effect", to which we now turn.

4.2 Hawtrey effect

An analysis of price level determination based on the larger root of (24) of appendix A expresses a line of causation of Hawtrey (1919) who described paths of nominal prices under England's convertible currency during the 1790s. After substantially increasing a paper currency called the assignat in 1791 and 1792, France went on a paper standard; silver and gold coins left France and pushed world prices of those precious metals downward relative to consumption goods. That led to inflows of precious metals and upward pressure on British prices. France returned to a specie standard in 1797 and precious metals flowed into France, causing upward pressure on prices of precious metals and downward pressure on the British price level. Hawtrey wrote about those 1790s events in 1919 because a number of countries that had been on the gold standard before World War I had issued large amounts of currency and abandoned convertibility to gold. Hawtrey meant to warn of the price level declines that would come if all leading countries returned to the pre WWI gold standard at the old par values. Hawtrey probably had in mind adverse effects on real activity coming through a Phillips curve that is missing from the models in this paper.[15]

To avoid the adverse effects on real activity like those associated with France's return to specie in 1797 and Britain's return to a specie standard in 1821 after the Napoleonic Wars that he feared would accompany Britain's return to gold after World War I, Hawtrey embraced a recommendation of David Ricardo (1816) that we discuss next.

[15] Friedman and Schwartz (1963, ch. 7, sec. 4) and Friedman (1992, ch. 7) present an analysis of a connection of President Roosevelt's silver purchase program to a 1930s fall in the price level in China that rests on this relative price effect. Rather than leading to a consumption boon in China, they assert that the fact that the price level is not "just another relative price" meant that the decline in Chinese prices caused adverse effects on real activity in China. Friedman and Schwartz are implicitly appealing to a Phillips curve. Friedman (1992, ch. 7) critically reviews papers that had challenged evidence of adverse effects on real activity in China coming from Roosevelt's silver purchase program.

4.3 Ricardo's proposal

We regard an influential proposal of David Ricardo as a refinement and extension of the closed economy proposition 3.2 that we attributed to Adam Smith. Proposition 2.1 and formula (33) in particular confirm Ricardo's recommendation:[16]

> The introduction of the precious metals for the purposes of money may with truth be considered as one of the most important steps towards the improvement of commerce, and the arts of civilised life; but it is no less true that, with the advancement of knowledge and science, we discover that it would be another improvement to banish them again from the employment to which, during a less enlightened period, they had been so advantageously applied. Ricardo (1816, p. 65)

To show how Ricardo's recommendation manifests itself in our closed economy model, we pose:

Problem 4.1.

$$\max_{m \in M} u(c, s)$$

where maximization is subject to equations (4) *and* (33) *and* M *is the set of* m*'s satisfying inequalities* (8).

Proposition 4.2. *The solution of problem 4.1 sets* m *at the upper bound, which in turn assures that* $s = \check{s}$ *and* $\bar{s} - s = 0$*, thereby banishing silver as money, as Ricardo recommended.*

Thus, given e, welfare as measured by $u(c, s)$ is maximized by setting the supply of token currency m at the upper bound associated with keeping token currency convertible into silver coins at exchange rate e.[17]

4.4 A "gold-exchange standard"

Ricardo's recommendation is the intellectual foundation of the normative case for the "gold exchange standard" advocated by Hawtrey (1919) and others. To fix ideas, we use:

[16] This recommendation was adopted in various versions by Alfred Marshall, Irving Fisher, John Maynard Keynes, Ralph Hawtrey, and Milton Friedman.

[17] The Ricardo proposal to set $\frac{m}{pc} = k$ at a fixed e to be discussed in section 4.3 drives the supply curve in figure 1 toward a half rectangle with value zero and slope 0 for $s < \bar{s}$ and slope $+\infty$ at $s = \bar{s}$.

Definition 4.3. *In a "classical silver" standard, all money is silver, so* $m = 0$*. In a pure "silver exchange" standard,* $s = \bar{s} = \check{s}$*, all money is token money, and token money* m *is at the upper bound of inequalities* (8)*.*

When monetary historians write about a classical gold standard before World War I, they have in mind that the metal s was "gold" rather than "silver" and that m was closer to the lower bound in (8) than to the upper bound. When they write about the gold exchange standard of the 1920s, they have in mind that m was closer to the upper bound.

5 Bimetallism

In this section, we describe a closed economy model of bimetallism. There are three distinct utility yielding goods, namely, silver s, gold g, and standard consumption c. There are potentially three types of currency: silver, gold, and token.[18] At an interior equilibrium, the three types of currency are perfect substitutes as money: token dollars are freely convertible into gold or silver dollars at exchange rates e_g and e_s named by the government. This aspect of the model captures provisions of the US Coinage Act of 1792 that defined the dollar as "371 4/16 grain (24.1 g) pure or 416 grain (27.0 g) standard silver" and the 10 dollar eagle as "247 4/8 grain (16.0 g) pure or 270 grain (17.5 g) standard gold".[19]

5.1 Background

A widely accepted criticism of bimetallism asserted: (1) that competitive markets determine a relative price of silver for gold that responds to fluctuations in supplies of and demands for gold and silver; (2) that if a government attempts to fix a time-invariant relative price $\frac{e_g}{e_s}$, that relative price will almost always either over- or under-value silver relative to gold relative to the price set by the market; and (3) that currency will be exclusively silver if it is overvalued by the mint, and exclusively gold if it is overvalued by the mint.[20]

Fisher (1911) used geometry to construct a counterexample. He argued that within well defined limits, by naming $\frac{e_g}{e_s}$, measured in ounces of silver per ounces of gold, the

[18] This is a poor man's version of the Velde and Weber (2000) model of bimetallism. We eliminate all of the interesting dynamics and retain only the essential static "currency plumbing" that is at the heart of their model as well of the "diaphragm model" of Fisher (1911, ch. VII) that convinced Friedman (1990a,b). The key inequalities that drive the Velde and Weber (2000) and Fisher (1911) models are atemporal.

[19] Friedman (1951) analyses good properties of commodity monies.

[20] A rallying cry of William Jennings Bryan and other advocates of bimetallism in the US during the 1890s was that $\frac{e_g}{e_s}$ should equal 16.

government can set the market price of silver relative to gold in the face of sufficiently moderate fluctuations in demands for and supplies of silver and gold. The shock absorber at an interior equilibrium of Fisher's model is that monetary uses of precious metals respond to fluctuations in demands for non monetary uses. Our model expresses economic forces underlying Fisher's idea and also circumstances that render the preceding criticism of bimetallism valid.[21]

5.2 The model

The utility function of a representative agent is $u(c, s, g) = Ac^{\alpha_1}s^{\alpha_2}g^{1-\alpha_1-\alpha_2}$, where α_1, α_2, and $1-\alpha_1-\alpha_2$ are all in $(0, 1)$. Let $\bar{p}$ be the relative price of silver and consumption goods, measured in ounces of silver per unit of consumption good, and let $\check{p}$ be the relative price of silver and gold, measured in units of ounces of silver per ounce of gold. The household's first-order necessary conditions for constrained maximization of $u(c, s, g)$ imply

$$\frac{u_c}{u_s} = \bar{p}$$

and

$$\frac{u_c}{u_g} = \frac{\bar{p}}{\check{p}}.$$

The units of $\frac{\bar{p}}{\check{p}}$ are ounces of gold per unit of consumption good. Equilibrium relative prices are

$$\begin{aligned} \frac{u_c}{u_g} &= \frac{\alpha_1}{1-\alpha_1-\alpha_2}\frac{g}{c} && \text{ounces of gold per unit consumption} \\ \frac{u_c}{u_s} &= \frac{\alpha_1}{\alpha_2}\frac{s}{c} && \text{ounces of silver per unit consumption} \\ \frac{u_g}{u_s} &= \frac{1-\alpha_1-\alpha_2}{\alpha_2}\frac{s}{g} && \text{ounces of silver per ounce of gold} \end{aligned} \tag{9}$$

The government names e_s and e_g, whose units are dollars per ounce of silver and dollars per ounce of gold, respectively. The amount of silver used as dollars is $\bar{s}-s$ and the amount of gold used as dollars is $\bar{g}-g$, where $\bar{s}, \bar{g}$ are the total amounts of silver and gold used either as dollars or for non-monetary pleasure (e.g., jewelry). Where m is the supply of

[21] Related representations of these same forces appear in Barro (1979) and Velde and Weber (2000).

paper dollars, the total supply of dollars is

$$M = e_s(\bar{s} - s) + e_g(\bar{g} - g) + m$$

and the quantity theory equation equating the supply of dollars to the demand is $\frac{M}{pc} = k$. In this closed economy model, $\bar{s}, \bar{g}$, and c are exogenous.[22] The structural equations are:[23]

$$\frac{p}{e_s} = \frac{\alpha_1}{\alpha_2}\frac{s}{c} \tag{10}$$

$$\frac{p}{e_g} = \frac{\alpha_1}{1 - \alpha_1 - \alpha_2}\frac{g}{c} \tag{11}$$

$$\frac{e_s(\bar{s} - s) + e_g(\bar{g} - g) + m}{p} = kc \tag{12}$$

We take as exogenous variables $e_s, e_g, m, \bar{s}, \bar{g}$. Then we regard (10), (11), and (12) as three equations to be solved for the three endogenous variables p, s, g.

5.2.1 A "natural" relative price

When $k = 0$, there is no monetary demand for silver or gold. In that case, the price of gold relative to silver is

$$\tilde{\phi} = \frac{1 - \alpha_1 - \alpha_2}{\alpha_2}\frac{\bar{s}}{\bar{g}}. \tag{13}$$

The criticism of bimetallism against which Irving Fisher contended was that if the government were to set $\frac{e_g}{e_s}$ at a value other than $\tilde{\phi}$, then either gold or silver coins would disappear. If $\frac{e_g}{e_s} > \tilde{\phi}$ there would be only gold coins and if $\frac{e_g}{e_s} < \tilde{\phi}$ there would be only silver coins. All coins would consist of the metal that is "overvalued at the mint", a version of "Gresham's law".[24] The heart of Fisher's response is that when $k > 0$, the presence of monetary demands for gold and silver opens room for the government to set $\frac{e_g}{e_s}$ not equal

[22]In a small open economy version, we would make endowments $\check{s}, \check{g}, \check{c}$ exogenous, take the price level p as exogenous, and add a trade balance equation

$$(c - \check{c}) + \frac{e_g}{p}(\bar{g} - \check{g}) + \frac{e_s}{p}(\bar{s} - \check{s}) = 0.$$

[23]Equation (12) builds in a remark of Friedman (1951) that the real value of the stock of currency is independent of the precious metal used as currency.

[24]For statements of Gresham's law(s) that distinguish "circulation by weight" from "circulation by tale", see Sargent and Smith (1997) and Sargent and Velde (2002).

to $\tilde{\phi}$.

5.3 Interior equilibria with $m = 0$

To begin, we follow Fisher (1911) and Velde and Weber (2000) by setting $m = 0$ so that money must be either gold or silver. An interior equilibrium respects the inequalities

$$0 \leqslant s \leqslant \bar{s} \tag{14}$$

$$0 \leqslant g \leqslant \bar{g} \tag{15}$$

At an interior equilibrium, algebra recovers the following formulas for s, g, p:[25]

$$s = \left(\frac{\alpha_2}{1 - \alpha_1 + \alpha_1 k}\right)(\bar{s} + \frac{e_g}{e_s}\bar{g}) \tag{16}$$

$$g = \frac{e_s}{e_g}\left(\frac{1 - \alpha_1 - \alpha_2}{1 - \alpha_1 + \alpha_1 k}\right)(\bar{s} + \frac{e_g}{e_s}\bar{g}) \tag{17}$$

$$p = \left(\frac{\alpha_1}{1 - \alpha_1 + \alpha_1 k}\right)\left(\frac{e_s\bar{s} + e_g\bar{g}}{c}\right) \tag{18}$$

Proposition 5.1. *When $m = 0$, inequalities (14) and (15) impose restrictions on the government's choice of $\frac{e_g}{e_s}$. We obtain an upper bound on $\frac{e_g}{e_s}$ by equating s to $\bar{s}$ and solving for $\frac{e_g}{e_s}$. We obtain a lower bound on $\frac{e_g}{e_s}$ by equating g to $\bar{g}$ and solving for $\frac{e_g}{e_s}$. The bounds are*

$$\frac{1 - \alpha_1 - \alpha_2}{\alpha_2 + \alpha_1 k}\frac{\bar{s}}{\bar{g}} \leqslant \frac{e_g}{e_s} \leqslant \frac{1 - \alpha_1 - \alpha_2 + \alpha_1 k}{\alpha_2}\frac{\bar{s}}{\bar{g}} \tag{19}$$

At the upper bound, $s = \bar{s}$ and at the lower bound $g = \bar{g}$. In the interior, $s < \bar{s}$ and $g < \bar{g}$, so that there are both gold and silver coins. As $k =\downarrow 0$, the bounds (19) collapse around $\frac{e_g}{e_s} = \tilde{\phi}$, the "natural" relative price.

5.4 $m > 0$ tightens the bounds on permissible $\frac{e_g}{e_s}$

We now consider how positive token money $m > 0$ affects an interior equilibrium. Let $\phi = \frac{e_g}{e_s}$. We begin by "back-solving" for an equilibrium in which $\frac{e_g}{e_s} = \tilde{\phi}$, where $\tilde{\phi}$ is the

[25]Evidently, $\frac{p}{e_s}$ pins down $\frac{s}{c}$ because $\frac{p}{e_s} = \frac{\alpha_1}{\alpha_2}\frac{s}{c}$ and $\frac{p}{e_g}$ pins down $\frac{g}{c}$ because $\frac{p}{e_g} = \frac{\alpha_1}{1-\alpha_1-\alpha_2}\frac{g}{c}$.

"natural" relative price of silver per unit of gold defined in equation (13). We seek an equilibrium in which $\phi = \tilde{\phi}$ and in which there are neither silver nor gold coins. The price of the standard consumption good relative to silver must be $\frac{\alpha_1}{\alpha_2}\frac{\bar{s}}{c}$ because no silver is used as money and the price of the standard consumption good relative to gold must be $\frac{\alpha_1}{1-\alpha_1-\alpha_2}\frac{\bar{g}}{c}$ because no gold is used as money. If e_s is to be an active exchange rate in the sense that e_s dollars actually exchange for an ounce of silver, then the price level has to satisfy $p = \hat{p} \equiv e_s \frac{\alpha_1}{\alpha_2}\frac{\bar{s}}{c}$. But for this to be the equilibrium price level, the stock of token money has to satisfy $m = \bar{m} \equiv k\hat{p}c$. If the money supply were less than $\bar{m}$, there would be a "shortage of money" and equation (12) would not hold at $s = \bar{s}, g = \bar{g}$. If more token money than $\bar{m}$ were issued, then the price level would have to rise above $\hat{p}$ and token currency would depreciate relative to the government set rate e_s.

This type of reasoning leads us to make the following

Proposition 5.2. *There is an upper bound $\bar{m}$ on the stock of paper money that is consistent with paper money exchanging for gold and silver at values e_g and e_s set by the government. At the upper bound, all money is token money and e_s and e_g must satisfy $\frac{e_g}{e_s} = \tilde{\phi}$, where $\tilde{\phi}$ is the "natural" relative price of gold and silver defined in* (13). *The following steps tell how to find the upper bound on m.*

1. *Set $e_s > 0$*
2. *Set $e_g = e_s \frac{1-\alpha_1-\alpha_2}{\alpha_2}\frac{\bar{s}}{\bar{g}}$. (This sets $\frac{e_g}{e_s}$ to its "natural" value $\tilde{\phi}$.)*
3. *Compute $\hat{p} = e_s \frac{\alpha_1}{\alpha_2}\frac{\bar{s}}{c} = e_g \frac{\alpha_1}{1-\alpha_1-\alpha_2}\frac{\bar{g}}{c}$. Here $\hat{p}$ is an equilibrium price level that prevails when there are neither gold nor silver coins, e_s dollars still buys one unit of silver, and there is just enough token money to satisfy the demand for money at consumption level c and nominal price level $\hat{p}$.*
4. *Compute the upper bound $\bar{m} = kc\hat{p}$; $\bar{m}$ is the amount of paper money that satisfies the demand for real balances at the equilibrium price $\hat{p}$.*

At these settings of e_s and e_g, $s = \bar{s}$, $g = \bar{g}$, $p = \hat{p}$ and $\frac{\hat{m}}{\hat{p}c} = k$. Further, when $0 < m < \bar{m}$, limits on $\frac{e_g}{e_s}$ become

$$\frac{1-\alpha_1-\alpha_2}{\alpha_2 + \alpha_1(k - \frac{m}{pc})}\frac{\bar{s}}{\bar{g}} \leqslant \frac{e_g}{e_s} \leqslant \frac{1-\alpha_1-\alpha_2+\alpha_1(k-\frac{m}{pc})}{\alpha_2}\frac{\bar{s}}{\bar{g}} \tag{20}$$

Within these bounds, there is room to set $\frac{e_g}{e_s} \neq \tilde{\phi}$. *At the upper bound,* $\bar{g} = g$, *at the lower bound,* $\bar{s} = s$, *and inside the bounds,* $\bar{s} - s > 0$ *and* $\bar{g} - g > 0$.

Appendix A describes how to compute the equilibrium value of p when $0 \leqslant m \leqslant \bar{m}$. It also provides formulas for equilibrium s and g.

5.4.1 Ricardo's recommendation eliminates Fisher's wedge

Inequalities (20) show that the lower and upper bounds on $\frac{e_g}{e_s}$ both converge to the natural relative price $\tilde{\phi}$ as $\frac{m}{pc} \uparrow k$. Under the David Ricardo (1816) proposal to banish precious metals from use as money by instead using tokens and thereby setting $\frac{m}{pc} = k$, the government loses the latitude to affect $\frac{e_g}{e_s}$ that it has in the Fisher (1911)-Barro (1979)-Velde and Weber (2000) model. Nevertheless, a policy that sets $\frac{m}{pc} = k$ produces an equilibrium allocation that maximizes $u(c, s, g)$, reaffirming Ricardo's recommendation.

5.5 Crime of '73

We can use the model to represent the alleged "Crime of '73" that occurred when Congress took the United States from a bimetallic to a monometallic standard (see Friedman (1990a,b)).[26] In this version of the model, the government declares that only gold, and not silver, can be used as the commodity money. Convertible token money is still allowed. The government defines a dollar as e_g ounces of gold. Structural equations determining the price level p now become

$$\begin{aligned} \frac{\alpha_1}{(1-\alpha_1-\alpha_2)} \frac{g}{c} &= \frac{p}{e_g} \\ \frac{e_g(\bar{g}-g)+m}{p} &= kc. \end{aligned} \tag{21}$$

For simplicity, we focus on the pure gold standard case in which $m = 0$ and compute the following equilibrium values for g, s, p:

$$\begin{aligned} g &= \left(\frac{1-\alpha_1-\alpha_2}{1-\alpha_1-\alpha_2+\alpha_1 k} \right) \bar{g} \\ s &= \bar{s} \end{aligned}$$

[26] In using a closed economy model, we are thinking about the early 1870s when France, Germany, and the United States all left bimetallism for a gold standard.

$$p = \left(\frac{\alpha_1}{1-\alpha_1-\alpha_2+\alpha_1 k}\right)\frac{e_g\bar{g}}{c}. \tag{22}$$

The equilibrium price of gold relative to silver is

$$\frac{u_g}{u_s} = \left(\frac{1-\alpha_1-\alpha_2+\alpha_1 k}{\alpha_2}\right)\frac{\bar{s}}{\bar{g}}. \tag{23}$$

Proposition 5.3. *Compare an interior equilibrium of the bimetallic monetary system with $m = 0$ described in proposition 5.1 with an equilibrium of the monometallic monetary system with $m = 0$ described in this subsection. A comparison of formula* (22) *for the price level under the gold-only economy with the price level* (18) *for the bimetallic economy shows that the price level is lower under the monometallic gold-only economy. The price of silver relative to gold is lower with the monometallic standard because more gold and less silver is used as money, changing the amounts of gold and silver allocated to the non monetary uses that determine the ratio of their marginal utilities.*

Friedman (1990b) identified two 1890s constituencies that urged the United States to return to bimetallism at a $\frac{u_g}{u_s} = \frac{e_s}{e_g}$ ratio of 16 to one: silver producers, who sought an increase in the relative price of silver, and debtors, who sought an increase in the price level.[27] Our model confirms the effects on both the price level and the relative price of silver asserted by adherents to the "Crime of '73" conspiracy story. But because it features a representative agent, it fails to model the diverse interests that would have led some agents to like and others to dislike those outcomes. Instead, as we see next, the representative agent in our model would view Congress's 1873 decision to demonetize silver as a blessing, not a crime.

5.6 Suboptimality of bimetallism

The monometallic equilibrium in subsection 5.5 evidently sets $\frac{e_g}{e_s}$ at or above the upper bound of the interval defined by inequalities (19) so that $s = \bar{s}$ and there are no silver coins. While continuing to assume that $m = 0$, we now seek a ϕ that respects the bounds (19)

[27] When the United States abandoned bimetallism in the 1870s, the price of silver relative to gold had been approximately 16 to one. Velde (2002) gathered additional data and then reassessed and extended Friedman's analysis in terms of the Velde and Weber (2000) model.

and that yields equilibrium outcomes with maximum welfare measured by $u(c, s, g)$.[28,29,30]

Proposition 5.4. *Minimum $u(c, s, g)$ is attained by setting $\phi = \tilde{\phi}$, the natural relative price defined in equation* (13)*. An interior equilibrium is dominated by both the $s = \bar{s}$ monometallic gold equilibrium at the upper bound of the interval* (19) *and the $g = \bar{g}$ monometallic silver equilibrium at the lower bound of the interval.*

Evidently, a benevolent government always prefers ϕ at a corner. This occurs because the government's choices of (g, s, ϕ) are restricted by "implementability" conditions in the form of equilibrium relationships, depicted by equations (16) or (25) and (17) or (26), that describe how a representative agent adjusts non-monetary consumptions of silver and gold in response to alternative government choices of the price of silver relative to gold ϕ. These implementability constraints confine the government's choice of (g, s) to an incentive-feasibility curve that is more concave than the representative consumer's indifference curve.[31] Figure 2 indicates why an optimum is monometallic. The solid curve is the locus of incentive-feasible (g, s) pairs, namely, (g, s) pairs that solve equations (16) or (25) and (17) or (26) and are swept out by the implicit function theorem as the government's choice variable $\phi = \frac{e_g}{e_s}$ moves from $\underline{\phi}$ to $\overline{\phi}$, causing g to fall and s to rise as ϕ increases. This is the set of (s, g) pairs from which the government chooses by setting $\phi \in [\underline{\phi}, \overline{\phi}]$. The three dotted lines are indifference curves, i.e., level curves of $U = Ac^{\alpha_1} s^{\alpha_2} g^{1-\alpha_1-\alpha_2}$. For the parameters set in the figure, the highest indifference curve is attained at $\overline{\phi}$ and is affiliated with a monometallic silver standard solution $g = \bar{g}$. The lowest indifference curve is attained by setting $\phi = \tilde{\phi}$, the natural relative price defined in (13).[32]

Whether a benevolent government wants gold or silver to be used as money depends on the representative consumer's preferences. When $\alpha_2 > 1 - \alpha_1 - \alpha_2$, i.e., when the

[28]This is our version of Proposition 2 of Velde and Weber (2000, p.1219). Our figure 2 is a counterpart to Velde and Weber's Figure 2.

[29]A classic case for preferring bimetallism over monometallism argued that an average of the price gold relative to other consumption goods and of the price of silver relative to consumption goods would be less volatile over time than either of those prices, providing a more stable price level. See Barro (1979) and Fisher (1911).

[30]Manuelli and Wallace (1984) analyze the efficiency of commodity money equilibria in another setting.

[31]This is thus an example of a Ramsey problem in which a "first-order approach" fails. Notice the role of third derivatives in the Velde and Weber (2000, p. 1220) explanation for corner solutions.

[32]At $\phi = \underline{\phi}$, $g = \bar{g}$, while at $\phi = \overline{\phi}$, $s = \bar{s}$. As ϕ increases in the interval $\phi \in [\underline{\phi}, \overline{\phi}]$, s increases and g decreases along the solid curve of incentive feasible (g, s) combinations. Because $\alpha_2 < 1 - \alpha_1 - \alpha_2$, a monometallic silver money outcome that sets $g = \bar{g}$ is optimal, given real balances of token money $\frac{m}{pc}$. Minimum utility is attained at the natural $\phi = \tilde{\phi}$ that occurs at the tangency of an indifference curve with the set of incentive-feasible (g, s).

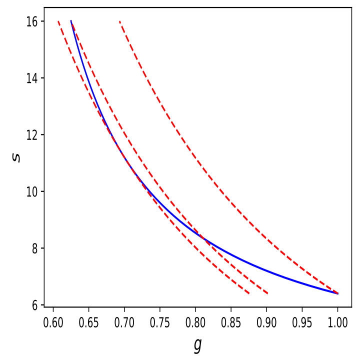

Figure 2: Level curves of utility (dashed lines) and (g, s) combinations available to a government that chooses $\phi = \frac{e_g}{e_s} \in [\underline{\phi}, \overline{\phi}]$ (solid line) for an economy with with $\alpha_1 = .3, \alpha_2 = .2, A = 1, c = 10, \bar{s} = 16, \bar{g} = 1, k = 1, \frac{m}{pc} = 0.0$.

power coefficient on silver in the utility function exceeds the power coefficient on gold, a monometallic gold standard dominates a monometallic silver standard. When $\alpha_2 < 1 - \alpha_1 - \alpha_2$, a monometallic silver standard dominates a monometallic gold standard. When $\alpha_2 = 1 - \alpha_1 - \alpha_2$, the government is indifferent between the two possible monometallic standards.

5.7 Gresham's law

Equations equations (16) or (25) and (17) or (26) imply that in the interval $(\underline{\phi}, \overline{\phi})$, s is an increasing function and g is a decreasing function of ϕ, the price of silver relative to gold set by the government. At $\underline{\phi}$, $g = \bar{g}$ and at $\overline{\phi}$, $s = \bar{s}$. If the government sets $\phi > \overline{\phi}$, the consequence is that silver is "undervalued at the mint", the market sets the price of silver relative to gold at $\overline{\phi}$, $s = \bar{s}$, and all coins are gold. If the government sets $\phi < \underline{\phi}$, gold is undervalued at the mint, the market sets the price of silver relative to gold at $\underline{\phi}$, $g = \bar{g}$, and all coins are silver. These statements are versions of Gresham's law, appropriately modified to incorporate Irving Fisher's response to critics of bimetallism.

6 Concluding remarks

The static models in this paper can help understand a variety of historical episodes and policy decisions in addition to those described in sections 3, 4, and 5. For example, the world price level stayed higher under the gold exchange standard after World War I than it had been under the regime with less paper money that had been in place before 1914. That outcome was an intended effect of Hawtrey's recommendation to institute a gold-exchange standard in the 1920s, an effect described by the analysis in section 4.2. Another example is that for many years until 1871 France and other large countries were able to sustain a bimetallic standard that kept the price of gold relative to silver pegged at about 15.5. The "Fisher wedge" described by equation (19) in section 5.3 and equation (20) tells how they could do that.[33]

The neglect of dynamics hard wired into our static models means that they have nothing to say about important aspects of the episodes in monetary history mentioned here. Salient among these are possibly interconnected issues about units of account and Phillips curve dynamics that caused observers like Friedman (1992, ch. 7) to want more sophisticated analysis of some of the episodes than we have provided.[34]

A Computing the price level when $m > 0$

A.1 Monometallic model

When m respects the bounds described in (8), namely, $0 \leqslant m \leqslant k\frac{\alpha}{1-\alpha}\bar{s}e$

$$s = \frac{1-\alpha}{1-\alpha+\alpha(k-\frac{m}{pc})}\bar{s}$$

and

$$\bar{s} - s = \frac{\alpha(k-\frac{m}{pc})}{1-\alpha+\alpha(k-\frac{m}{pc})}\bar{s}$$

Substituting this expression into (2) and rearranging gives the following quadratic equation in p:

$$[kc(1-\alpha+\alpha k)]p^2 + [-\alpha k e\bar{s} - m(1-\alpha+\alpha k) - \alpha k m]p$$

[33]The section 5.5 analysis explains how France's and Germany's 1871 abandoning of full bodied silver coins contributed to a rise in the price of gold relative to silver.

[34]Doepke and Schneider (2017) present a model of units of account that emphasizes important features of economic environments ignored in this paper.

$$+\left[\alpha\frac{m}{c}e\bar{s}+\alpha\frac{m^2}{c}\right]=0. \tag{24}$$

It can be verified that the larger root of (24) satisfies $k-\frac{m}{pc}\geqslant 0$ and that the smaller root violates it.

A.2 Bimetallic model

When $0<m<\bar{m}$

$$s=\frac{\alpha_2(\bar{s}+\frac{e_g}{e_s}\bar{g})}{1-\alpha_1-\alpha_1\frac{m}{pc}+\alpha_1 k} \tag{25}$$

$$g=\frac{e_s}{e_g}\frac{(1-\alpha_1-\alpha_2)(\bar{s}+\frac{e_g}{e_s}\bar{g})}{1-\alpha_1-\alpha_1\frac{m}{pc}+\alpha_1 k} \tag{26}$$

It follows that

$$\bar{s}-s=\frac{(1-\alpha_1-\alpha_2+\alpha_1(k-\frac{m}{pc}))\bar{s}-\alpha_2\frac{e_g}{e_s}\bar{g}}{1-\alpha_1+\alpha_1(k-\frac{m}{pc})}$$

and

$$\bar{g}-g=\frac{\frac{e_s}{e_g}(\alpha_1+\alpha_2-1)\bar{s}+(\alpha_2-\alpha_1(k-\frac{m}{pc}))\bar{g}}{1-\alpha_1+\alpha_1(k-\frac{m}{pc})}$$

Substituting these expressions into the money market equilibrium condition (12) and rearranging gives

$$\frac{\alpha_1(k-\frac{m}{pc})(e_s\bar{s}+e_g\bar{g})}{p(1-\alpha_1-\alpha_1(k-\frac{m}{pc}))}+\frac{m}{p}=kc$$

which can be rearranged to become the following quadratic equation in p:

$$kc(1-\alpha_1+\alpha_1 k)p^2+(-\alpha_1 k(e_s\bar{s}+e_g\bar{g})-m(1-\alpha_1+\alpha_1 k)-\alpha_1 km)p$$
$$+\left(\alpha_1\frac{m}{c}(e_s\bar{s}+e_g\bar{g})+\alpha_1\frac{m^2}{c}\right)=0 \tag{27}$$

B Proofs

Proposition 2.1:

Proof. We first consider the boundary at which $m = 0$. Here only silver is used as money, so

$$\frac{e(\check{s} - s)}{p} = k\check{c}. \tag{28}$$

Equation (2) implies $\frac{e}{p} = \frac{1-\alpha}{\alpha}\frac{\check{c}}{s}$, which, substituted into (28) and rearranged, implies

$$s = \frac{\check{s}}{1 + \frac{\alpha}{1-\alpha}k}. \tag{29}$$

It follows that

$$\frac{p}{e} = \left(\frac{\alpha}{1 - \alpha + \alpha k}\right)\frac{\check{s}}{\check{c}} \tag{30}$$

When $m = 0$, p is proportional to e with k being a determinant of the factor of proportionality. Here setting e is about units in which dollars are expressed: a change in e is just a change in a unit of measurement. When equation (30) prevails, an exogenous increase in the stock of silver $\check{s}$ increases the price level by decreasing the price of silver relative to the standard consumption good.

Next, we consider the other boundary at which $\frac{m}{p} = k\check{c}$. Here $s = \check{s}$, so no silver is used as money and all silver is enjoyed as an argument of $u(c, s)$:

$$\frac{m}{p} = k\check{c} \tag{31}$$

$$\frac{p}{e} = \frac{\alpha}{1 - \alpha}\frac{\check{s}}{\check{c}} \tag{32}$$

Use $p = \frac{\alpha}{1-\alpha}\frac{\check{s}}{\check{c}}$ and equation (31) to compute the following upper bound on token money m that is compatible with a fixed e:

$$m \leqslant k\check{c}\left(\frac{\alpha}{1 - \alpha}\right)\frac{\check{s}}{\check{c}}e.$$

So at a fixed e, m must respect the bounds (8). Within these bounds the amount of silver enjoyed is

$$s = \left(1 + \frac{k\alpha}{1 - \alpha}\right)^{-1}\left(\check{s} + \frac{m}{e}\right). \tag{33}$$

If m exceeds the upper bound in (33), the value of e set by the government becomes inoperative – meaning that no one exchanges silver for token money at that exchange rate and that traders instead exchange $\tilde{e} > e$ token dollars for an ounce of silver. Here $s = \check{s}$

and a system of structural equations determining $(p, \tilde{e})$ is

$$\begin{aligned} \frac{m}{p} &= k\check{c} \\ \tilde{e} &= \frac{1-\alpha}{\alpha}\frac{\check{c}}{\check{s}}p, \end{aligned} \tag{34}$$

the first being a pure quantity-theory equation that determines p as a function now purely of the stock of paper m and the second stating that the dollar price of silver is proportional to the price level.

□

Proposition 3.2:

Proof. Utility $u(c, s)$ is increasing in c and s. In the interval $0 \leqslant m \leqslant k\alpha\left[\frac{e}{p}\check{s} + \check{c}\right]$, equilibrium c and s described by (5) and (6), respectively, are both monotone increasing in m. □

Proposition 4.2:

Proof. Evidently, in the interval $0 \leqslant \frac{m}{p} \leqslant kc$, $s = \bar{s} - \frac{p}{e}(kc - \frac{m}{p})$ attains it maximum $\bar{s}$ by setting $\frac{m}{p} = kc$. □

Proposition 5.4:

Proof. Choose ϕ to maximize $u(c, s, g)$ subject to equilibrium formulas for s and g appearing in equations (16) and (17). The first-order necessary condition is $u_s \frac{\partial s}{\partial \phi} + u_g \frac{\partial g}{\partial \phi} = 0$, where the partial derivatives obtained from equations (16) and (17) are $\frac{\partial s}{\partial \phi} = \frac{\alpha_2}{1-\alpha_1+\alpha_1 k}\bar{g}$ and $\frac{\partial g}{\partial \phi} = \frac{-(1-\alpha_1-\alpha_2)}{1-\alpha_1+\alpha_1 k}\phi^{-2}\bar{s}$. Substituting these into the first-order condition, using formulas (25) and (26), and rearranging gives $\phi = \frac{1-\alpha_1-\alpha_2}{\alpha_2}\frac{\bar{s}}{\bar{g}}$, which equals the natural relative price $\tilde{\phi}$ in equation (13) and lies in the interior of the set defined by the inequalities (19). The second order condition verifies that this is a minimum. Moreover, where $\underline{\phi}$ is the lower bound in inequalities (19) and $\bar{\phi}$ is the upper bound, $u_s \frac{\partial s}{\partial \phi} + u_g \frac{\partial g}{\partial \phi} < 0$ when $\phi = \underline{\phi}$ where $g = \bar{g}, s < \bar{s}$ and $u_s \frac{\partial s}{\partial \phi} + u_g \frac{\partial g}{\partial \phi} > 0$ when $\phi = \bar{\phi}$ where $g < \bar{g}, s = \bar{s}$. □

C Cash-in-advance model

We can use the following specification of a cash-in-advance model along the lines of Lucas (1982) to derive a demand function for money of the form $kpc = m^d$, where m^d denotes the demand for dollars. Lucas's setup delivers (or builds in) $k = 1$, a unit velocity of money. We confine ourselves to a one-commodity money version of the model, and call that commodity silver. Like Lucas (1982) and Velde and Weber (2000), we assume that the standard consumption good is nondurable. To induce a direct connection to the static model presented in the text, we seek a stationary equilibrium and so confront the household with a time invariant price level and dollar-silver exchange rate pair (p, e).

Where $\beta \in (0, 1)$ is a discount factor, the representative household chooses sequences $\{c_t, h_t, s_{t+1}, m^d_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t, s_t) \tag{35}$$

subject to

$$\begin{aligned} s_{t+1} &= s_t + h_t \\ pc_t &\leqslant m^h_t \\ pc_t + eh_t + m^d_{t+1} &\leqslant m^d_t + p\check{c} \\ m^d_{t+1} &\geqslant 0 \end{aligned} \tag{36}$$

subject to initial conditions for s_0, m^d_0. The first equation of (36) is the law of motion for the stock of silver used as a consumption good; here h_t is purchases of silver at time t. This equation asserts that silver is perfectly durable. The second equation is a cash-in-advance constraint on purchases of the time t standard consumption good c; here m_t^d is the consumer's stock of cash carried over from period $t-1$. We do not impose a cash-in-advance constraint on purchases of silver. The third equation of (36) is the household's budget constraint. Associated with the household's problem is the Lagrangian

$$\sum_{t=0}^{\infty} \beta^t (u(c_t, s_t) + \lambda_t [m_t^d + pc_t - pc_t - eh_t - m_{t+1}^d] \tag{37}$$

$$+ \theta_t [s_t - h_t - s_{t+1}] + \phi_t [m_t^d - pc_t]) \tag{38}$$

where $\{\lambda_t, \theta_t, \phi_t\}$ are sequences of nonnegative Lagrange multipliers. First-order necessary conditions for an optimum include:

$$\begin{aligned}
c_t, t \geqslant 0 : \quad & u_{c,t} - \lambda_t p - \phi_t p = 0 \\
s_t, t \geqslant 1 : \quad & u_{s,t} + \theta_t - \beta^{-1} \theta_{t-1} = 0 \\
h_t, t \geqslant 0 : \quad & -\lambda_t e + \theta_t = 0 \\
m_t^d, t \geqslant 1 : \quad & -\beta^{-1} \lambda_{t-1} + (\lambda_t + \phi_t) = 0,
\end{aligned} \tag{39}$$

where $u_{c,t}, u_{s,t}$ denote partial derivatives of $u(c_t, s_t)$ with respect to c_t and s_t respectively at time t. If we solve these difference equations for steady-state values of $\lambda_t = \lambda, \theta_t = \theta, \phi_t = \phi \ \forall t$ and corresponding steady state values of u_{ct}, u_{st}, we deduce

$$u_c = \frac{p\phi}{1-\beta}$$

$$u_s = e\phi$$

and therefore

$$\frac{p}{e} = (1-\beta)\frac{u_c}{u_s}. \tag{40}$$

Equation (40) is a generalization of equation (2) in the text, one that collapses to it when $\beta = 0$. When $\beta \in (0,1)$, equation (40) asserts that it is the discounted present value of u_s whose ratio to u_c must equal $\frac{p}{e}$ in equilibrium, which makes sense. When the cash-in-advance constraint binds, as it will,

$$m_t^d = pc_t, \tag{41}$$

which is a $k = 1$ version of the demand function for money used in the text.

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