Stochastic Earnings Growth and Equilibrium Wealth Distributions*

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Abstract

The cross-section distribution of U.S. wealth is more skewed and fatter tailed than the distribution of labor earnings. Stachurski and Toda (2019) explain how plain vanilla Bewley-Aiyagari-Huggett (BAH) models with infinitely lived agents can't generate that pattern because the wealth process is bounded by an AR(1) process since the equilibrium risk-free rate is lower than the time rate of preference. Two modifications of a BAH model suffice to generate this pattern: (1) overlapping generations of agents who have low wealth at birth and pass through $N \geqslant 1$ life-stage transitions of stochastic lengths, and (2) labor-earnings processes that exhibit stochastic growth. With few parameters, our model does a good job of approximating the mapping from the Lorenz curve, Gini coefficient, and upper fat tail for cross-sections of labor earnings to their counterparts for cross sections of wealth. Three forces amplify wealth inequality relative to labor-earnings inequality: stochastic life-stage transitions; a strong precautionary savings motive for high wage earners especially after receiving positive permanent earnings shocks; and a life-cycle saving motive for agents who have low wealth at birth. The outcome that the equilibrium risk-free interest rate exceeds the time rate of preference fosters a fat-tailed wealth distribution.

Keywords: wealth inequality, income inequality, Bewley models, incomplete markets, borrowing constraint, power law, fat tail

JEL Classification: E2

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1 Introduction

We calibrate a sparsely parameterized continuous-time life-cycle model and use it to show how responses to permanent labor-earning shocks by households with high labor earnings widen its equilibrium distribution of wealth. Except for assuming a nonstationary labor earnings process and a stochastic multiple life-stage overlapping generations demographic structure, our model stays close to the discrete time Bewley-Aiyagari-Huggett (BAH) models with *stationary* labor earnings processes that have struggled to put sufficient mass at upper quantiles of equilibrium wealth distribution. That feature of BAH models led researchers to change assumptions in ways designed to make wealthier agents want to save more. Examples of such alterations include the warm-glow bequest and human capital motives of De Nardi (2004), very large earnings risk for high-earning households of Castañeda, Díaz-Giménez, and Ríos-Rull (2003), heterogenous preferences of Krusell and Smith (1998), and the importance of entrepreneurship of Quadrini (2000) and Cagetti and De Nardi (2006, 2009). We purposefully exclude these additional motivations to save because we want to determine how far nonstationary labor earnings processes and a stochastic life cycle by themselves go toward allowing a basic BAH's model to put enough mass in the upper end of an equilibrium wealth distribution. We show that by themselves, they do most of the job.

We use a pure counting process to model an agent's life cycle. At birth each agent has no wealth, the same initial labor earnings, and the same fixed number N of sequentially ordered life stages. Transition from life stage n to stage $(n+1)$ occurs at an exogenous constant probability per unit of time. At the final life stage N, the agent purchases an actuarially fairly priced (reverse) life annuity and dies with zero wealth. Our life-stage model is a continuous-time generalization of discrete-time life-cycle models used by Gertler (1999) and Castañeda, Díaz-Giménez, and Ríos-Rull (2003) and nests the "perpetual youth" model of Yaari (1965) and Blanchard (1985) as a special case. Agents are born owning little wealth. Our model's equilibrium interest rate *exceeds* the agent's time preference rate by enough to motivate sufficient savings to match the empirical aggregate capital-output ratio and to activate a force that helps make the cross-section distribution of wealth fatter than the distribution of labor earnings.

The exogenous labor-earnings process displays random growth within each life stage n, a feature that generates a cross-section fat-tailed earnings distribution via a mechanism similar to ones in Gabaix (1999), Luttmer (2007, 2011), Toda and Walsh (2015), and Jones and Kim (2018). In our quantitative analysis, we economize on parameters by assuming that the labor-earnings process remain unchanged over the agent's life cycle. Gabaix (1999) shows that the distribution of city populations is well described by a Pareto distribution, also known as Zipf's law. Luttmer (2007, 2011) constructs models that generate fat-tailed firm size distributions. Toda and Walsh (2015) show that cross-section distributions of US consumption and its growth rate obey the double power law.[1] Jones and Kim (2018) generate an endogenous cross-section fat-tailed earnings distribution in a Schumpeterian creative-destruction model with heterogeneous entrepreneurs. We build on an insight of Gabaix, Lasry, Lions, and Moll (2016) and Jones and Kim (2018) that a properly tweaked random earnings growth model implies that earnings inequality is fractal.[2]

In conjunction with discounted constant-relative-risk-averse (CRRA) preferences, the random growth with drift labor-earnings process implies decision rules that induce wealthier agents to save enough to generate an equilibrium wealth distribution whose upper quantiles approximate US data well. Our model's analytic tractability allows us to unveil basic forces that shape saving decision rules and equilibrium outcomes.

De Nardi (2015) points out that the heart of the problem with BAH-style models is that they predict that "rich people are not nearly rich enough, middle-class people are too rich, and poor people are too poor, compared with the actual data." This is because "the nature of precautionary savings implies that households save to self-insure against earnings risk but that, as a result, the saving rate decreases and then turns negative when a person's net worth is large enough relative to her labor earnings. Hence, the saving rate

[1] Gabaix (2009) and Luttmer (2010) survey these mechanisms. A key insight in the power-law literature is that random growth (properly modified to account for stationarity) and *ex ante* heterogeneity naturally generate Pareto distributions and that a parameter fixing the random growth rate governs the fatness of the tail. For early classics on Pareto distributions, see Champernowne (1953), Simon (1955), and Mandelbrot (1960).

[2] Here is an example of (constant) fractal inequality: Jones and Kim (2018) write, "What fraction of the income going to the top 10 percent of earners accrues to the top 1 percent? What fraction of the income going to the top 1 percent of earners accrues to the top 0.1 percent? What fraction of the income going to the top 0.1 percent of earners accrues to the top 0.01 percent? The answer to each of these questions – which turns out to be around 40 percent in the United States today – is a simple function of the parameter that characterizes the power law."

of the wealthy in these models is negative." She concludes that "basic Bewley models, whether featuring infinitely-lived agents or life-cycle agents with more realistic patterns of earnings and savings over the life cycle, are far from doing a good job of matching the observed distribution of wealth ... While in the data wealth is concentrated in the hands of a small number of rich people and the saving rate of the rich is high, many models used for quantitative policy evaluation fail to match these facts."

In our model, *permanent* shocks to levels of their labor earnings make rich people keep saving at high rates, as they do in U.S. data. This happens because precautionary savings motives of those with high earnings stay strong even after a long sequence of positive earnings shocks. Even though earnings are expected to grow and shocks are permanent, the marginal propensity to consume (MPC) out of permanent shocks to earnings stays lower than one except for very large wealth-earnings ratios.[3] Because they have little wealth at birth, young agents also have strong incentives to save. Strong saving motives are promoted by an equilibrium interest rate that exceeds a representative agent's subjective discount rate, something that does not occur in BAH models with infinitely-lived agents. A combination of permanent earnings shocks and a high equilibrium interest rate makes strong savings motives persist throughout even a wealthy person's life. That leads to big wealth inequality.

We capitalize on the tractability of continuous-time stochastic modeling techniques that also underly mean field game theory. We solve Hamilton-Jacobi-Bellman equations "almost by hand". We use the optimal decision rules to solve appropriate instances of Kolmogorov forward equations that restrict a stationary joint distribution of labor earnings and wealth.[4] Optimal saving rules at different stages of life indicate how permanent earnings shocks ignite precautionary savings motives that affect even wealthier people of all ages and that enable our model to generate a cross-section wealth distribution that has a fatter tail than

[3]We describe an equilibrium in which an agent's MPC out of permanent earnings shocks approaches zero as her wealth-earnings ratio x approaches zero, either because her financial wealth approaches zero or because her earnings are extremely high.

[4]There is a recent surge of interest in using continuous-time models via Kolmogorov forward equations (also knowns as Fokker-Planck equations) to analyze equilibrium distributions of economic objects including city size, firm size, income, and wealth. A partial list includes Gabaix (1999), Wang (2003), Luttmer (2007, 2011), Toda and Walsh (2015), Benhabib, Bisin, and Zhu (2016), Gabaix, Lasry, Lions, and Moll (2016), Achdou, Han, Lasry, Lions, and Moll (2017), and Jones and Kim (2018).

cross-section earnings. We report both the Gini coefficient and Lorenz curve as Castañeda, Díaz-Giménez, and Ríos-Rull (2003), and De Nardi (2004) have also done. When possible, we also provide a closed-form characterization for the upper tails of wealth and earnings with exponents of Pareto distributions.

Concavity of optimal consumption decision in the wealth-earnings *ratio* x reflects an agent's enduring precautionary saving motive and fosters wealth inequality. An agent with high labor earnings can also have a *low* wealth-earnings ratio, x, making it optimal to save a lot. Furthermore, when an agent with high labor earnings receives a sequence of positive earnings shocks, its motive to save becomes even stronger, providing a force that contributes to high equilibrium wealth inequality.

A typical BAH model's joint cross-section distribution of wealth and labor earnings also describes the fraction of time that each individual spends in each set of wealth, labor earnings states. Equality between these two probability distributions in BAH models is an essential ingredient of Stachurski and Toda (2019)'s finding that wealth cannot have a fatter tail than labor earnings in BAH models with infinitely-lived agents. Our model decouples those two joint distributions: an equilibrium cross section distribution of wealth and labor earnings does *not* describe life-time fractions that each individual spends in possible wealth, labor earnings pairs. That disarms the Stachurski-Toda mechanism and makes the equilibrium joint cross-section distribution of wealth and earnings have fatter tails for wealth than for labor earnings.[5]

Research papers that generate endogenous Pareto distributions for wealth include Benhabib, Bisin, and Zhu (2011, 2015, 2016), Toda (2014), Hubmer, Krusell, and Smith (2016), and Nirei and Aoki (2016). The mechanisms that produce those Pareto distributions operate via either an asset accumulation equation (random growth models) or a capital accumulation equation in a neoclassical growth model. In contrast, we start with an empirically plausible fat-tailed cross-section earnings distribution and use the standard BAH consumption-smoothing mechanism endogenously to generate a cross-section distribution for wealth that has a fatter tail than earnings.

Because they do not start with exogenous earnings and don't allow for endogenous

[5]Stachurski and Toda (2019, sec. 4) describe modifications of canonical BAH models that disarm their impossibility theorem.

savings, most continuous-time wealth distribution models are not in the BAH tradition. But there are notable exceptions. Achdou, Han, Lasry, Lions, and Moll (2017) formulate BAH-style models in continuous time. Unlike our model, they retain the assumption that labor earnings are governed by a stationary stochastic process.

2 Setup

Time and an agent's age $t \in [0, +\infty)$ are both continuous. Equal measures of agents are born and die over each small interval of time. Markets are incomplete. Agents are identical at birth but differentiated afterwards by their luck. Each agent receives statistically independent realization of an exogenous stochastic labor earnings stream over a stochastic life time that is almost surely finite.

An agent's life stage $\{S_t\}$ is a non-decreasing integer-valued stochastic process that at age t takes a value inside a set of integers $\{1, 2, \ldots, N\}$, where $N \geqslant 1$ is finite. An agent begins life in stage $n = 1$ at age $t = 0$. Conditional on being in life stage n at age t, over a small age interval $(t, t + dt)$, an agent remains in life stage n with probability $1 - \lambda_n dt$ and advances to life stage $(n + 1)$ with probability $\lambda_n dt$. This structure induces a sequence $\{\tau_n\}_{n=1}^{N}$ of random ages at which an agent moves from life stage n to life stage $(n+1)$, so that $\tau_n = \inf\{t : S_t = n+1\}$. An agent is exposed to mortality risk only during life stage $S_t = N$. Wang (2003) uses this stochastic life-cycle model to study equilibrium wealth distribution with negative exponential utility and an affine labor-earnings process.[6] Luttmer (2011) uses a closely related stochastic process to model dynamics of firms' blueprints.

Interpretations of Life Stages. Two interpretations of S_t are plausible. One is that life-stage S_t indexes a single person's age-t health status. Here we would calibrate stage-dependent labor-earnings processes to make health-related productivity be correlated with age t.

An alternative interpretation is that the entity being modeled is a family dynasty with parents who are altruistic. Parents in stage n want to leave bequests to heirs in stage $(n+1)$

[6]See Duffie (2010) for an introduction to applications of affine processes to term structure of interest rates and credit risk models in Finance.

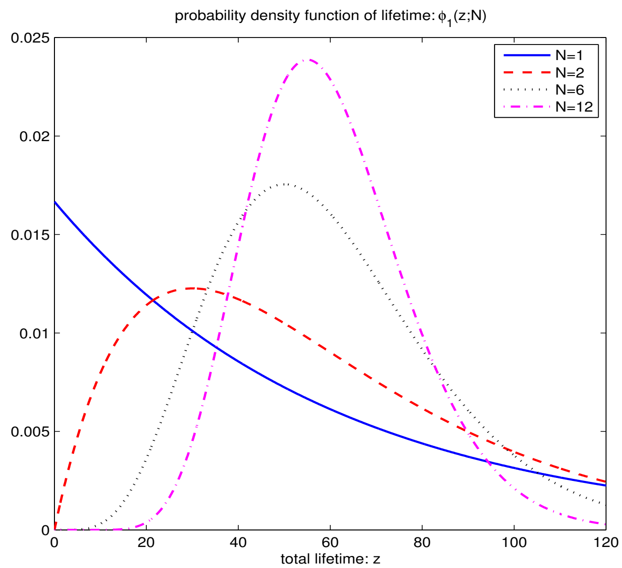

Figure 1: PROBABILITY DENSITIES OF LIFE LENGTHS z AT BIRTH IN FOUR MODELS WITH $N = 1, 3, 6, 12$ LIFE STAGES. Transition intensities $\lambda_n = \lambda$ for all n in each of four models but $N/\lambda = 60$ across the four models.

and cannot fully hedge their own death risk until the dynasty reaches its terminal stage N. A dynasty stochastically transitions from one generation to the next and eventually becomes extinct. By adopting this interpretation, we could account for accidental bequests. Thus, there is a sense in which the mechanism of De Nardi (2004) and other bequest models is also at work in our model.

Benhabib, Bisin, and Luo (2019) quantified an equilibrium model of the U.S. wealth distribution and social mobility. We can use a family dynasty instance of our model to study inter-generational economic mobility together with a cross-section wealth distribution.

Let z be the *remaining* length of life of an agent now in stage n who has $(N - n + 1)$ remaining life stages. The random variable z is the sum of $(N - n + 1)$ independently and identically distributed exponentially random variables each with the rate parameter λ (and hence a mean of $1/\lambda$). It has the following probability density function:

$$\phi_n(z; N) = \frac{\lambda e^{-\lambda z}(\lambda z)^{N-n}}{(N-n)!} \, . \tag{1}$$

This instance of a Gamma function generates an Erlang distribution with two parameters:

the shape parameter k equals $(N - n + 1)$, the number of remaining life stages, and the rate parameter is λ. When $n = 1$, z is also the random length of life for a new born whose distribution is given by equation (1) in a model with N life stages.

Figure 1 plots density functions $\phi_1(z; N)$ of life lengths z and also of the *remaining* lengths of life for an agent in stage 1 with N remaining life stages in models with $N = 1, 3, 6, 12$. To show how models with $N > 1$ can provide more realistic mortality with few parameters, we set transition intensities $\lambda_n = \lambda$ for all n in each of four models and set $N/\lambda = 60$ to deliver the same average life lengths of 60 years for each N. An $N = 1$ perpetual youth model generates too many very old people. Thus, if λ is calibrated to yield a realistic average (working) life span of $\lambda = 1/60$ years,[7] then the probability of living longer than 120 years is $e^{-120/60} = 13.5\%$. The probability of living longer than 120 years is 9.2%, 2.0%, and 0.3%, for $N = 2$, $N = 6$, and $N = 12$ models. Evidently, increasing the number of life-stages N while holding average age fixed at N/λ delivers thinner and thinner right tails for life lengths. In an $N > 1$ model, an older agent is more likely to be in a later than an early life-stage n.

An agent ranks consumption processes $\{C_t\}_{t=0}^{\infty}$ by discounted expected utilities:[8]

$$\mathbb{E}\left[\int_0^{\tau_{N+1}} e^{-\rho t}\, U(C_t)\, dt\right], \tag{2}$$

where $\rho > 0$ is a discount rate and $\mathbb{E}[\,\cdot\,]$ is a mathematical expectation with respect to probability distributions of the stage of life process $\{S_t\}$ and of the labor-earnings process $\{Y_t\}$. We assume a constant relative-risk-aversion instantaneous utility function

$$U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \text{if } \gamma \geqslant 0,\ \gamma \neq 1 \\ \ln(C) & \text{if } \gamma = 1 \end{cases}$$

A representative firm operates a production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$, where $A > 0$, $\alpha \in (0, 1)$, K is the aggregate capital stock, and L is the aggregate labor stock. Physical capital depreciates at a constant rate δ. The firm rents capital and labor in competitive markets. The firm's optimization problem implies that a competitive equilibrium interest

[7]We interpret an age of 0 in our model as the beginning of a living individual's age of 18 in real life.

[8]For the agent's objective function, without loss of generality, we set the agent's birth time τ_1 to 0.

rate r and wage w satisfy:

$$r = F_K(K, L) - \delta \text{ and } w = F_L(K, L). \tag{3}$$

Although many BAH models assume a stationary labor-earnings process, econometric studies have often estimated nonstationary processes that include permanent shocks.[9] For that reason, we assume that labor earnings $\{Y_t\}$ follow diffusion processes with permanent shocks only. Thus, each agent has the following labor earnings process during life stage S_t:

$$dY_t = \mu_{S_t} Y_t\, dt + \sigma_{S_t} Y_t\, d\mathcal{B}_t\,, \qquad 0 \leqslant t < \tau_{N+1}\,, \tag{4}$$

where $\mathcal{B}$ is a standard Brownian motion, $Y_0 > 0$ is initial labor earnings at birth, and μ_{S_t} and σ_{S_t} are stage-S_t-dependent growth rates and volatilities of labor earnings, respectively. Process (4) asserts that within each life stage S_t, the growth rate of labor earnings, dY_t/Y_t, is independently and identically distributed. Therefore, shocks to labor earnings are permanent in *levels*. Specification (4) lets labor earnings growth and volatility both depend on stage of life S_t, a random variable that is correlated with age and that lets us approximate plausible age-earnings profiles. Although details differ, our labor-earnings process has both permanent shocks and some life-cycle features similar to those used by Zeldes (1989), Deaton (1991), Carroll (1997), and Gourinchas and Parker (2002).

Random earnings growth models with adjustments to ensure stationarity generate Pareto distributions with fat tails as demonstrated by Gabaix (1999), Luttmer (2007), Gabaix, Lasry, Lions, and Moll (2016), and Jones and Kim (2018).[10] We recognize that there is persuasive evidence that the earnings process has other interesting features such as skewness (see Guvenen, Ozkan, and Song (2014) and De Nardi, Fella, and Paz-Pardo (2020)). We choose a simple earnings model in order to focus on the channel through which

[9] For example, see MaCurdy (1982), Abowd and Card (1989), and Meghir and Pistaferri (2004), and Blundell, Pistaferri, and Preston (2008). Here we ignore important fixed effects such as education and gender, as well as other life-cycle variations across agents. Our labor-earnings process could be extended to also feature a transitory component. For example, see Section 10 in Wang, Wang, and Yang (2016) for one such generalization.

[10] We can generalize our earnings model to allow for jumps as in Gabaix, Lasry, Lions, and Moll (2016) and Section 9 in Wang, Wang, and Yang (2016), but we omit jumps because our diffusion model is sufficient to deliver our key results that cross-section wealth is more skewed and fat tailed than earnings.

we generate a fatter tailed distribution for wealth than for earnings while acknowledging that our simple random growth model neglects how labor earnings respond to transient shocks.

Applying Ito's formula to equation (4) verifies that the dynamics of $\ln Y$ during life stage n are:

$$d \ln Y_t = g_n dt + \sigma_n d\mathcal{B}_t \,, \tag{5}$$

where the expected change of log income during life stage n, i.e., the drift in (5), equals

$$g_n = \mu_n - \frac{\sigma_n^2}{2} \,. \tag{6}$$

where $\sigma_n^2/2$ is a Jensen's inequality correction term at stage n.

The arithmetic Brownian motion (5) implies the following discrete-time process:

$$\ln Y_{t+1} - \ln Y_t = g_n + \sigma_n \epsilon_{t+1} \,, \tag{7}$$

where the time-t conditional distribution of ϵ_{t+1} is a standard normal random variable. Thus, during life stage n, $\ln Y$, is a unit-root process whose first difference is independently and normally distributed with mean g_n and volatility σ_n. The Ito correction term can make the expected labor earnings growth rate in logarithms g_n differ substantially from the growth rate of labor earnings Y in levels, μ_n. For example, at an annual frequency, with $\mu_n = 1.5\%$ and $\sigma_n = 10\%$, we have $g_n = 1\%$, which is one third lower than the growth rate $\mu_n = 1.5\%$ due to the Jensen's inequality term, $\sigma_n^2/2 = 0.5\%$. Because labor earnings growth shocks are i.i.d., shocks to *levels* of Y are permanent.

Let X denote an agent's wealth process and set initial wealth X_0 to zero. During each stage of life, an agent can trade a risk-free financial asset that offers a constant rate of return r. At age t and life stage $S_t < N$, over a small increment $(t, t + dt)$, the agent faces zero mortality risk. Therefore, whenever $S_t < N$, or equivalently when $0 \leqslant t < \tau_N$, where $\tau_N = \inf\{u : S_u = N\}$, wealth evolves as:

$$dX_t = (rX_t + Y_t - C_t)dt, \quad 0 \leqslant t < \tau_N \,. \tag{8}$$

During end-of-life stage N an agent purchases an actuarially fair "reverse-life-insurance" contract that provides a flow of life-time payments in exchange for having agreed to transfer end-of-life wealth $X_{\tau_{N+1}}$ to the insurance company. Preferences of an agent in life stage N are the same as those of a perpetual youth with a discount rate ρ that is augmented by a mortality hazard rate $\lambda_N > 0$ to become an effective discount rate $\rho + \lambda_N$. When $\tau_N < t < \tau_{N+1}$, an agent is in life stage N and her wealth evolves as:

$$dX_t = (rX_{t-} + \lambda_N X_{t-} + Y_{t-} - C_{t-})dt - X_{t-}dS_t\,, \qquad \tau_N < t < \tau_{N+1}\,. \tag{9}$$

Thus, during life stage N two new terms augment the saving rates $(rX_{t-} + Y_{t-} - C_{t-})$ during life stages $n < N$: (a) an actuarially fair payment rate $\lambda_N X_{t-}$ from the insurance company to the agent; and (b) a one-time transfer of wealth $X_{\tau_{N+1}-}$ from the agent to the insurance company at the stochastic death moment $t = \tau_{N+1}$ when $dS_t = dS_{\tau_{N+1}} = 1$.

An agent cannot borrow against future labor earnings, i.e.,

$$X_t \geqslant 0, \quad \text{for all} \quad t \geqslant 0\,, \tag{10}$$

but she can dissave when her assets are positive. *Financial income* consists of interest income rX_t and also, but only during end-of-life stage N, reverse life insurance payments $\lambda_N X_t$. *Non-financial* income equals labor earnings Y_t.

3 Saving Choices

We compute optimal decision rules and an object that we call "certainty equivalent wealth" as functions of wealth, labor earnings, and life stage in closed forms up to some interconnected ordinary differential equations with economically interpretable boundary conditions for each life stage.

3.1 Recursions

We work backwards from stage N to stage 1. An agent in the final stage N acts as a perpetual Yaari-Blanchard youth so her value function satisfies the Hamilton-Jacobi-

Bellman (HJB) equation:

$$\begin{aligned}(\rho + \lambda_N)\, V_N(X,Y) \;&=\; \max_{C>0}\; U(C) + ((r+\lambda_N)X + Y - C)V_{N,X}(X,Y) \\ &\quad + \mu_N Y V_{N,Y}(X,Y) + \frac{\sigma_N^2 Y^2}{2} V_{N,YY}(X,Y)\,. \qquad (11)\end{aligned}$$

The left side of HJB equation (11) discounts "flow" value by $\rho + \lambda_N$ in order to account for the probability of death per unit of time. The coefficient on $V_{N,X}$ on the right side of (11) sets the rate of return on savings at $r + \lambda_N$ rather than r because the agent has purchased reverse life insurance. The agent optimally sets C to equate the two sides of (11).

Value functions for life stages $n \in (1, N-1)$ satisfy HJB equations:

$$\begin{aligned}\rho V_n \;&=\; \max_{C>0}\; U(C) + (rX + Y - C)V_{n,X}(X,Y) + \mu_n Y V_{n,Y}(X,Y) + \frac{\sigma_n^2 Y^2}{2} V_{n,YY}(X,Y) \\ &\quad + \lambda_n (V_{n+1}(X,Y) - V_n(X,Y))\;. \qquad (12)\end{aligned}$$

When life-stage $S_t \leqslant N-1$, an agent's death probability is zero over all small time intervals. An agent earns a rate of return on savings X equal to the risk-free rate r. The last term in (12) comes from the stochastic transition probability from stage n to stage $(n+1)$.

Value functions have a homogeneity property that lets us write them as

$$V_n(X,Y) = \frac{(b_n P_n(X,Y))^{1-\gamma}}{1-\gamma} \qquad 1 \leqslant n \leqslant N\,, \qquad (13)$$

where $P_n(X,Y)$ is an agent's "certainty equivalent wealth" at life stage n, an object interpretable as a welfare measure expressed in units of the consumption good. Thus, imagine that at some stage of life, an agent has two options: either (1) adhering to the saving plan prescribed by the model; or (2) surrendering both her savings X and her continuation life-stage-dependent labor earnings processes Y in exchange for retiring immediately with wealth level Ω, from which she can either consume or else save and earn the risk-free rate r for the rest of life. Wealth $\Omega = P_n(X,Y)$ makes the agent indifferent between these two options. From knowing $P_n(X,Y)$, we can uniquely pin down b_n.

To compute the $\{b_n\}$ sequence, we can start from the following formula for the coefficient

b_N at stage N

$$b_N = [\rho + \lambda_N + (1 - 1/\gamma)(r - \rho)]^{\frac{1}{1-1/\gamma}} , \quad (14)$$

and work backwards to compute the coefficient b_n in (13) at stage n via the recursion:

$$\frac{\gamma b_n^{1-1/\gamma} - \rho}{1-\gamma} + \frac{\lambda_n}{1-\gamma}\left[\left(\frac{b_{n+1}}{b_n}\right)^{1-\gamma} - 1\right] + r = 0 . \quad (15)$$

We restrict parameters so that things make economic sense. For example, we set $\rho + \lambda_N + (1 - 1/\gamma)(r - \rho) > 0$.

The $P_n(X, Y)$ functions allow us to characterize optimal consumption rules. The homogeneity property of $V_n(X, Y)$ depicted in equation (13) generates policy functions and other important objects that scale by labor earnings. The wealth-earnings ratio $x = X/Y$ becomes a state variable that lets us express optimized utility in terms of a function $p_n(x) = P_n(X, Y)/Y$ and the optimal consumption rule in terms of a function $c_n(x) = C_n(X, Y)/Y$.

First-order conditions for consumption associated with HJB equations (11) and (12) imply

$$c_n(x) = m_n \, p_n(x) \, (p_n'(x))^{-1/\gamma} , \quad (16)$$

where

$$m_n = b_n^{1-1/\gamma} . \quad (17)$$

An important result is that incomplete markets make $p'(x) > 1$ for all finite values of x, which means that financial wealth is valuable beyond its pure purchasing value. Certainty-equivalent wealth scaled by labor earnings Y for life stage $n = N$, $p_n(x)$, satisfies the ODE:

$$\begin{aligned} 0 \;=\; & \left(\frac{\gamma(b_N p_N'(x))^{1-1/\gamma} - (\rho + \lambda_N)}{1-\gamma} + \mu_N - \frac{\gamma\sigma_N^2}{2}\right) p_N(x) + p_N'(x) \\ & + (r + \lambda_N - \mu_N + \gamma\sigma_N^2) x p_N'(x) + \frac{\sigma_N^2 x^2}{2}\left(p_N''(x) - \gamma\frac{(p_N'(x))^2}{p_N(x)}\right) . \end{aligned} \quad (18)$$

For earlier life stages $S_t = n \leqslant N-1$, $p_n(x)$ satisfies the ODE:

$$0 = \left(\frac{\gamma(b_n p_n'(x))^{1-1/\gamma} - \rho}{1-\gamma} + \mu_n - \frac{\gamma\sigma_n^2}{2}\right) p_n(x) + p_n'(x) + (r - \mu_n + \gamma\sigma_n^2) x p_n'(x)$$
$$+\frac{\sigma_n^2 x^2}{2}\left(p_n''(x) - \gamma\frac{(p_n'(x))^2}{p_n(x)}\right) + \frac{\lambda_n p_n(x)}{1-\gamma}\left[\left(\frac{b_{n+1}p_{n+1}(x)}{b_n p_n(x)}\right)^{1-\gamma} - 1\right]. \qquad (19)$$

When wealth $X = 0$, the no-borrowing constraint (10) implies that consumption C cannot exceed labor earnings ($C \leqslant Y$). We can express (10) in terms of scaled variables as:

$$c_n(0) \leqslant 1\,, \qquad \text{for} \quad 1 \leqslant n \leqslant N\,, \qquad (20)$$

a constraint that may or may not bind. If $c_n(0) < 1$, the agent's saving motive is strong enough to keep wealth X always strictly positive. In this case, relaxing constraint (20) has no value, so a Lagrange multiplier on constraint $X \geqslant 0$ is zero.

If $c_n(0) = 1$ and constraint (20) binds, then zero wealth $X = 0$ is an absorbing state. Campbell and Mankiw (1990) and Kaplan and Violante (2014) refer to consumers with zero wealth who set $C = Y$ as hand-to-mouth consumers and document that they constitute a sizable proportion of consumers. For such consumers, $c_n(0) = 1$. This condition and the optimal consumption rule (16) jointly imply that certainty equivalent wealth $p_n(0)$ and its first derivative $p_n'(0)$ are linked via $m_n\, p_n(0)\, (p_n'(0))^{-1/\gamma} = 1$, a boundary condition on the function p_n at $x = 0$.

To find another boundary condition for $p_n(x)$, we note that as x approaches infinity the agent uses holdings of the single risk-free asset completely to buffer all idiosyncratic labor-earnings shocks, but stage-of-life shocks remain uninsurable. We can show that as $x \to \infty$, $p_n(x)$ satisfies the condition:

$$\lim_{x\to\infty} p_n(x) = x + q_n\,, \qquad \text{for} \quad 1 \leqslant n \leqslant N\,, \qquad (21)$$

where scaled certainty-equivalent values of labor earnings defined as $\{q_n : 1 \leqslant n \leqslant N\}$ satisfy

$$q_n = \frac{b_n^{1-\gamma} + \lambda_n b_{n+1}^{1-\gamma} q_{n+1}}{b_n^{1-\gamma}(r - \mu_n) + \lambda_n b_{n+1}^{1-\gamma}}\,, \quad 1 \leqslant n < N\,, \qquad (22)$$

and

$$q_N = \frac{1}{r + \lambda_N - \mu_N} \,. \tag{23}$$

Having computed the sequence of $\{b_n : 1 \leqslant n \leqslant N\}$ from the recursion defined by (14) and (15), we can solve (22) recursively for $\{q_n\}$ by starting from (23) at stage N.

We have thus established that an agent's optimal consumption rule is (16) and that the scaled certainty equivalent wealth $p_n(x)$ satisfies (19) at life stages $n \leqslant N-1$ and (18) at life stage N, subject to boundary conditions (20) and (21).

Dynamics of Scaled Wealth x. By using Ito's Lemma, we express the dynamics for agent's scaled wealth x_t when $0 \leqslant t < \tau_N$:

$$dx_t = \left[1 + \left(r - \mu_n + \sigma_n^2\right) x_t - c_n(x_t)\right] dt - \sigma_n x_t \, d\mathcal{B}_t \,, \quad 0 \leqslant t < \tau_N \,. \tag{24}$$

During life's final stage N, scaled wealth evolves as

$$dx_t = \left[1 + \left(r + \lambda_N - \mu_N + \sigma_N^2\right) x_{t-} - c_N(x_{t-})\right] dt - \sigma_N x_{t-} \, d\mathcal{B}_t - x_{t-} dS_t \,. \tag{25}$$

3.2 Optimal Value Functions and Decision Rules

For parameter values described in Section 5, Figures 2 and 3 portray scaled certainty equivalent wealth $p_n(x)$ and the optimal consumption-earnings ratios $c_n(x)$ at stages n for our $N = 1$ and the $N = 2$ models.

3.2.1 The $N = 1$ Model

Figure 2 plots $N = 1$ objects. Panels A and B show that net scaled certainty-equivalent wealth, $p(x) - x$, is increasing and concave in the wealth-earnings ratio x and that $p'(x) - 1 \geqslant 0$. The dashed lines in Panels A and B depict $p(x) - x = q = 15.24$ and $p'(x) = 1$, the solution under a complete markets in which earnings and life-stage shocks are both insurable. The wedge between $p(x) - x$ and $q = 15.24$ captures the loss of the certainty equivalent wealth that comes from incomplete markets. For a penniless agent, certainty equivalent wealth $p(0) = 13.37$ of labor earnings is 12.3% lower than $q = 15.24$ under complete markets. Thus, an agent values a marginal unit of wealth at a premium of about

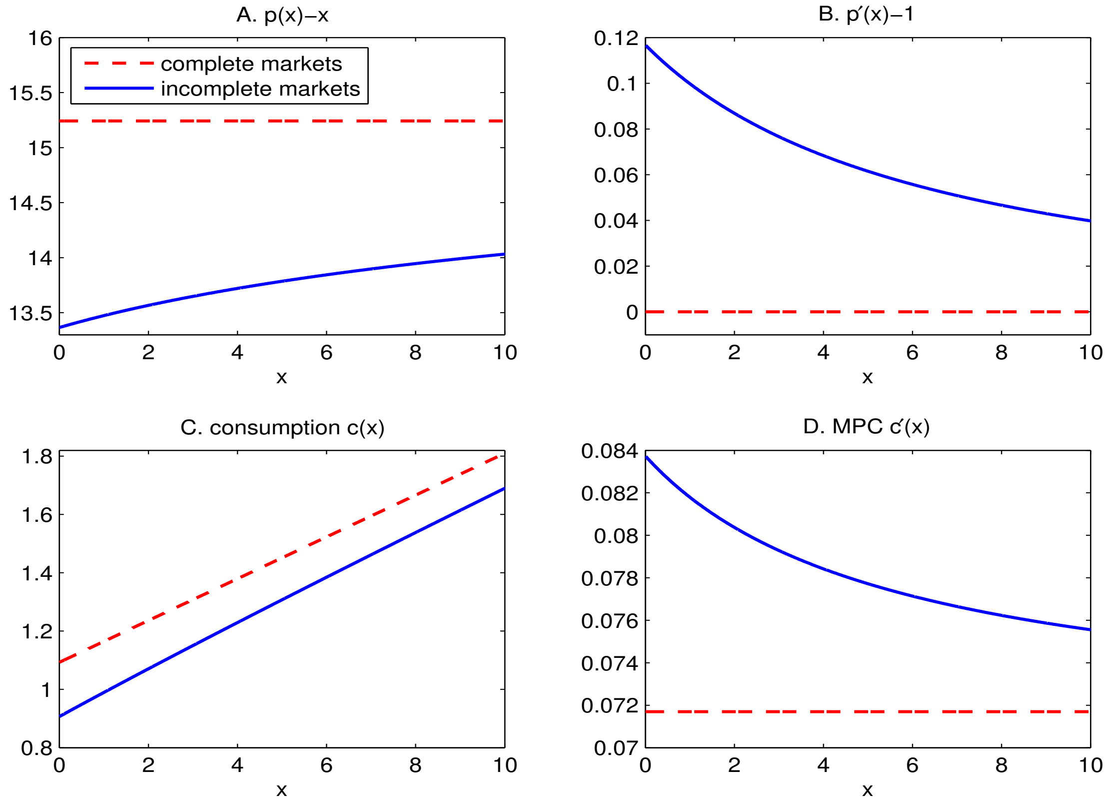

Figure 2: Net scaled certainty-equivalent wealth $p(x) - x$, marginal certainty-equivalent value of wealth $p'(x) - 1$, consumption-earnings ratio $c(x)$, and MPC $c'(x)$ for the $N = 1$ model. Dashed red and solid blue lines delineate outcomes for our complete and incomplete markets models, respectively. Under complete markets, $p(x) - x = q = 15.24$ and $c(x) = mp(x)$, where the MPC is given by $m = 7.2\%$. Parameter values are reported in Table 1.

12%, i.e., $p'(0) = 1.12$. Even when $x = 10$, $p(10) - 10 = 14$, which is still 8% lower than $q = 15.24$. Thus, the wedge between $p(x) - x$ and q remains substantial even for very large values of x. Evidently, incomplete-markets have first-order effects on an agent's welfare as measured by certainty equivalent wealth.

Panels C and D of Figure 2 show that an agent's consumption-earnings ratio, $c(x)$, is increasing and concave in the wealth-earnings ratio x. The MPC $c'(x)$ starts at $c'(0) = 8.4\%$ and slowly decreases towards the CM benchmark value, $m = 7.2\%$ as $x \to \infty$, indicating that the rich want to save much more than the poor, as they indeed do in US data. With complete markets, $c(x) = m(x + q)$. As measured by reduced consumption, the wedge

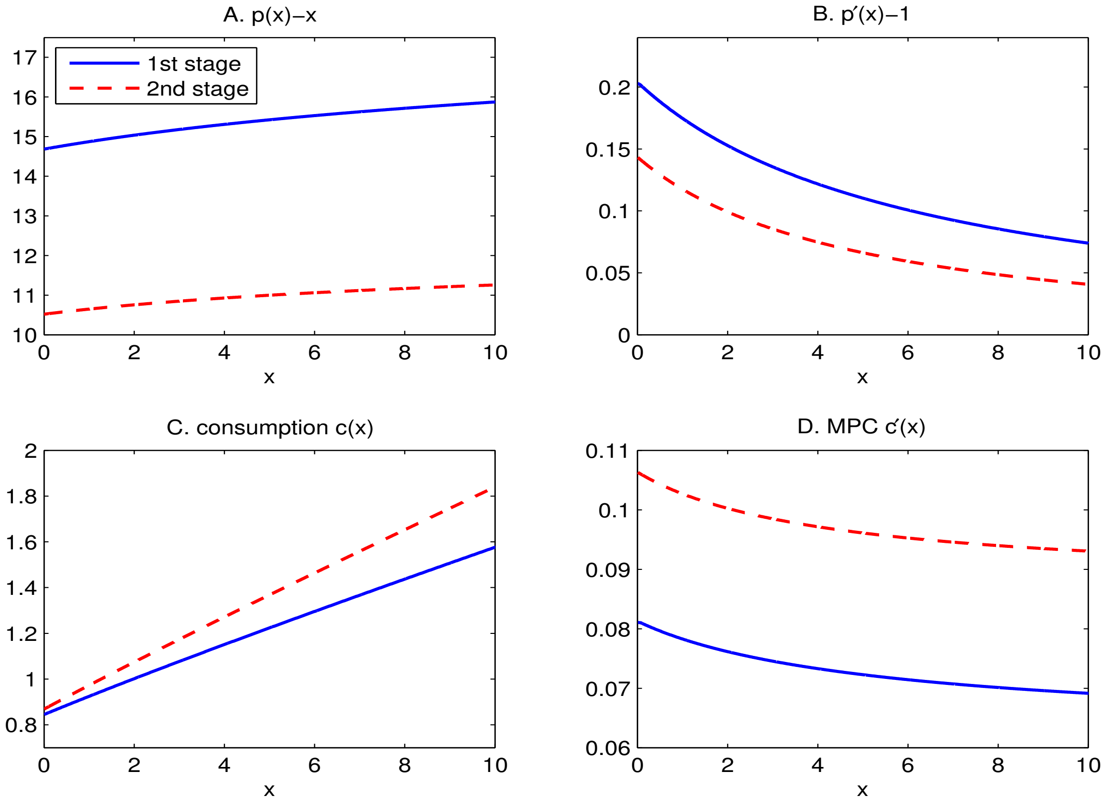

Figure 3: NET SCALED CERTAINTY-EQUIVALENT WEALTH $p(x) - x$, MARGINAL CERTAINTY-EQUIVALENT VALUE OF WEALTH $p'(x) - 1$, CONSUMPTION-EARNINGS RATIO $c(x)$, AND THE MPC $c'(x)$ FOR THE $N = 2$ MODEL. Scaled certainty-equivalent values of labor earnings are $q_1 = 18.70$ and $q_2 = 12.35$, for stages 1 and 2, respectively, while $m_1 = 6.31\%$ and $m_2 = 8.83\%$. Parameter values are reported in Table 1.

between the two lines in Panel C describes the loss of utility that comes from markets being incomplete.

3.2.2 The $N = 2$ Model

Figure 3 plots features of our $N = 2$ model. Panels A and B again show that net scaled certainty-equivalent wealth, $p(x) - x$, is increasing and concave in the wealth-earnings ratio x, as it also is in Figure 2 for the $N = 1$ model. Evidently, $p(x) - x$ and its derivative $p'(x) - 1$ are both higher in life stage 1 than in life stage 2. This makes sense because an agent with the same levels of X and Y in her earlier life stage is relatively wealthier in terms of certainty equivalent wealth, $p_1(x) > p_2(x)$, and therefore is relatively poor in terms of liquid financial wealth, i.e., is more "liquidity constrained", which leads to a higher marginal valuation for a unit increase of wealth X, i.e., $p_1'(x) > p_2'(x)$. For example, a penniless agent values a dollar windfall at a 14.4% premium in stage 2 ($p_2'(0) - 1 = 0.144$), while she would assign a 20.3% premium to the same windfall in life stage 1 ($p_1'(0) - 1 = 0.203$).

Panels C and D show that an agent's consumption is increasing and concave in the wealth-earnings ratio x in both stages due to incomplete markets as also occurs for the $N = 1$ model in Figure 2. The results for consumption are less obvious than for $p(x)$. Why does an agent consume more in stage 2 than in stage 1 at a given level of x, as Panel C shows? This outcome might seem peculiar because certainty-equivalent wealth is lower in stage 2 than in stage 1 for a fixed level of (X, Y), i.e., $p_1(x) > p_2(x)$ as depicted in Panel A. We call this outcome a "certainty-equivalent wealth" effect and impute it to forces that end up causing $c_2(x)$ to exceed $c_1(x)$ and that we now turn to explain.

First, because there is no bequest motive, the consumption motive is stronger in later life stages. Second, in our model the agent uses the reverse annuity market in the final life stage, stage 2 in this case, to exchange her end-of-life wealth for higher consumption. Indeed, that the MPC in stage 2 in the limit as $x \to \infty$, m_2, exceeds the MPC m_1 in stage 1, i.e., $m_2 = 8.83\% > m_1 = 6.31\%$, reflects these two forces. Third, uninsurable labor-earnings shocks induce smaller distortions to an agent's consumption in her last life stage because her shorter expected life span weakens her precautionary saving motive. For that reason, $p_n'(x)$ falls with advancing life stage n, as we see in Panel B. These three forces encourage an agent to consume more in stage 2 than in stage 1. Together, these three forces induce a (highly nonlinear) intertemporal substitution that, since both $m_2 > m_1$ and $p_2'(x) < p_1'(x)$, make $c_2(x)$ exceed $c_1(x)$. Thus, the optimal consumption rule (16) teaches us that the "intertemporal substitution" effect dominates the "certainty-equivalent wealth" effect and causes an agent to consume more at a given x when in stage 2 than when in stage 1.

An agent's consumption increases as she moves into later stages of life, a force that weakens our model's ability to generate high wealth accumulation for the rich. Nevertheless, our model can still generate a large wealth concentration, as we show in Section 5.

We note that the MPC increases with stage n. For example, the MPC for a penniless

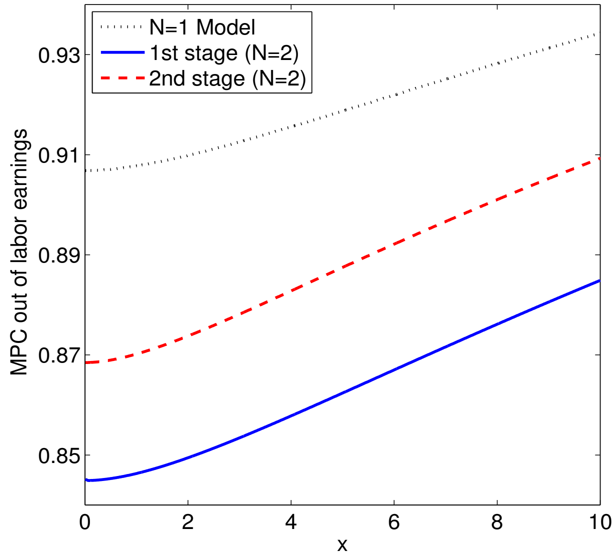

Figure 4: MPCs out of permanent earnings $C_Y(X,Y)$ for both the $N=1$ model (black dotted line) and the $N=2$ model in stage 1 (blue solid line) and stage 2 (red dashed line). Parameter values are reported in Table 1.

$(x=0)$ agent is 10.63% in life stage 2, which is larger than 8.11%, her MPC in life stage 1.

MPC out of (permanent) earnings. Although earnings grow $(\mu > 0)$ and earnings shocks are permanent, a precautionary savings motive often causes $C_Y(X,Y)$, the MPC out of earnings, to be below one (especially in an empirically plausible range). The homogeneity property implies $C_Y(X,Y) = c(x) - c'(x)x$ and hence C_Y equals $c(0)$ in all stages when $x = 0$. For the $N=1$ model, when $x=0$, $C_Y = 0.91$ and in the limit as $x \to \infty$, C_Y approaches the complete-markets level: $mq = 1.1$ (recall that $m = \rho + \lambda + (1 - 1/\gamma)(r - \rho) = 7.2\%$ is $q = 1/(r + \lambda - \mu) = 15.24$.) For the $N=2$ model, when $x=0$, $C_Y = c(0) = 0.85$ in stage 1 and 0.87 in stage 2, respectively.[11]

In Figure 4, we plot the MPC $C_Y = c(x) - xc'(x)$ as a function of x for the $N=1$ and $N=2$ models (both stages for the latter.) We see that the MPC out of earnings increases with x. This follows from $C_{Yx} = xc''(x)X/Y^2 = x^2c''(x)/Y < 0$, as optimal

[11]That the MPC out of earnings is lower than one at $x=0$ follows from $C_Y(0,Y) = c(0) - c'(0) \times 0 = c(0) \leqslant 1$, which follows from the no-borrowing constraint, $c(0) \leqslant 1$. In equilibrium, $c(0) < 1$. This is because if $c(0) = 1$, there could be not be a positive aggregate capital stock.

consumption $c(x)$ is concave in x. An agent can self-insure better the higher is the value of x. Accordingly, for a given level of wealth X, an agent who has a higher level of Y or receives a positive earnings shock is less self-insured than desired, fostering higher saving. This generates a force that contributes to fattening the tail of the distribution of wealth relative to the distribution of earnings. Also, note that C_Y is higher in stage 2 than in stage 1 since a stage-2 agent has a shorter life horizon and is able to hedge mortality risk.

4 Stationary Equilibrium

By assuming no aggregate shocks and a continuum of agents, we follow Aiyagari (1994) and focus on steady-state equilibria. Agents have identical but statistically independent labor-earnings processes.[12]

Stationary demographics. Let Π_n denote the measure of agents in life stage n and normalize the measure of the living agents (in all stages) to unity so that $\sum_{n=1}^{N} \Pi_n = 1$. Stationarity requires that measures of agents in each stage are constant over time and that flows into stage $(n+1)$ from stage n occur at the same rates as flows into stage n from stage $(n-1)$, so that

$$\Pi_n \, \lambda_n = \Pi_{n-1} \, \lambda_{n-1} \, . \tag{26}$$

Because $\sum_{n=1}^{N} \Pi_n = 1$ and equation (26) holds for $n = 2, \ldots, N$, we obtain

$$\Pi_n = \frac{\lambda_n^{-1}}{\sum_{n=1}^{N} \lambda_n^{-1}} . \tag{27}$$

Market clearing for capital and labor. Equality of aggregate demand and supply of capital requires:

$$K = \mathbb{E}(X) \equiv \int_0^\infty X \phi_X(X) dX \, , \tag{28}$$

where $\phi_X(X)$ is the cross-section stationary probability density of wealth X.

[12] Krusell and Smith (1998) analyze how the joint distribution of income and wealth responds to aggregate shocks.

Let H denote an agent's endowed labor units (e.g., hours). Each agent supplies labor inelastically. In equilibrium labor demand equals labor supply: $L = H$. Let $w = \mathbb{E}(Y)/H$ denote the average wage rate across all agents. Because aggregate labor cost for production wL equals aggregate labor earnings for all agents, using a law of large numbers,[13] we have

$$wL = wH = \mathbb{E}(Y) \equiv \int_0^\infty Y\phi_Y(Y)dY\,, \tag{29}$$

where $\phi_Y(Y)$ is the cross-section stationary distribution of labor earnings across all ages: $\phi_Y(Y) = \sum_{n=1}^{N}\Pi_n\phi_{n,Y}(Y)$ and $\phi_{n,Y}(Y)$ is the cross-section stationary distribution of labor earnings Y for agents in life stage n. Therefore, an agent's labor earnings Y_t exceeds the average level $\mathbb{E}(Y)$ if and only if her wage rate Y_t/H at t exceeds w.

The steady-state equilibrium average wage rate (which is also the wage rate received by an agent with average labor efficiency) and interest rate r satisfy

$$w \;=\; F_L(K,L) = A(1-\alpha)(K/L)^\alpha = A(1-\alpha)\left(\frac{\mathbb{E}(X)}{H}\right)^\alpha\,, \tag{30}$$

$$r \;=\; F_K(K,L) - \delta = A\alpha(K/L)^{\alpha-1} - \delta = \frac{\alpha}{1-\alpha}\frac{wH}{K} - \delta = \frac{\alpha}{1-\alpha}\frac{\mathbb{E}(Y)}{\mathbb{E}(X)} - \delta\,. \tag{31}$$

Stationary distribution of earnings and wealth. To calculate the cross-section stationary distribution of labor earnings, starting from stage 1, we recursively solve the following Kolmogorov Forward (Fokker-Planck) equations:

$$0 = -\frac{\partial(\mu_n Y\phi_{n,Y}(Y))}{\partial Y} + \frac{1}{2}\frac{\partial^2(\sigma_n^2 Y^2\phi_{n,Y}(Y))}{\partial Y^2} - \lambda_n\phi_{n,Y}(Y) + \lambda_{n-1}\phi_{n-1,Y}(Y) \tag{32}$$

for stages $2 \leqslant n \leqslant N$ and

$$0 = -\frac{\partial(\mu_1 Y\phi_{1,Y}(Y))}{\partial Y} + \frac{1}{2}\frac{\partial^2(\sigma_1^2 Y^2\phi_{1,Y}(Y))}{\partial Y^2} - \lambda_1\phi_{1,Y}(Y) \tag{33}$$

for stage 1. For any stage n, computing $\phi_{n,Y}(Y)$ involves solving a one-dimensional ordinary differential equation.

We can calculate the cross-section stationary distribution of wealth by first computing

[13]See Sun (2006) for technical conditions under which we can construct the associated probability and agent measures that allow invoking a law of large numbers.

the cross-section joint distribution of wealth and earnings. Let $\phi_{n,XY}(X,Y)$ denote this cross-section joint distribution in stage n. The following Kolmogorov Forward (Fokker-Planck) equations hold:

$$\lambda_1\phi_{1,XY} = -\frac{\partial\left(\mu_{1,X}(X,Y)\phi_{1,XY}\right)}{\partial X} - \frac{\partial\left(\mu_1 Y\phi_{1,XY}\right)}{\partial Y} + \frac{1}{2}\frac{\partial^2\left(\sigma_1^2 Y^2\phi_{1,XY}\right)}{\partial Y^2}\,, \tag{34}$$

$$\lambda_n\phi_{n,XY} = -\frac{\partial(\mu_{n,X}(X,Y)\phi_{n,XY})}{\partial X} - \frac{\partial(\mu_n Y\phi_{n,XY})}{\partial Y} + \frac{1}{2}\frac{\partial^2(\sigma_n^2 Y^2\phi_{n,XY})}{\partial Y^2} + \lambda_{n-1}\phi_{n-1,XY}\,, \tag{35}$$

where $\mu_{n,X}(X,Y)$ is the drift of wealth X is given by

$$\mu_{n,X}(X,Y) = rX + Y - C_n(X,Y), \qquad 1 \leqslant n \leqslant N-1\,, \tag{36}$$

in stage $n \leqslant N-1$ and by

$$\mu_{N,X}(X,Y) = (r+\lambda_N)X + Y - C_N(X,Y) \tag{37}$$

in stage N. After obtaining $\phi_{n,XY}(X,Y)$, we can compute the cross-section stationary distribution of wealth by integrating over Y: $\phi_{n,X}(X) = \int_0^\infty \phi_{n,XY}(X,Y)dY$.

Our model's homogeneity property simplifies computing the cross-section equilibrium wealth distribution. It can be accomplished as follows. First, we simulate a path of the standard Brownian motion $\mathcal{B}_t$ starting with $\mathcal{B}_0 = 0$. Second, we obtain the corresponding sample path for Y by substituting the simulated path of $\mathcal{B}_t$ into the dynamics (4) for Y with the initial condition $Y_0 = 1$. Third, we use the process for x_t given in (24) for stage $n < N$ and (25) for stage N together with the optimal scaled-consumption rule $c(x_t)$ given in (16) to obtain the paths for x_t and c_t starting with $x_0 = X_0/Y_0 = 0$. Finally, we obtain X_t by multiplying the two paths x_t and Y_t at each t. When an agent dies, we bring in a new agent with no wealth and Y_0. We continue this process until we reach a very high number of years, e.g., $t = 10^8$.

Next, we introduce widely used measures of inequality.

Lorenz curve, Gini coefficient, and fractal inequality. For a nonnegative random variable W with cumulative distribution function $G_W(\,\cdot\,)$, the *Lorenz curve* $\mathcal{L}_W(z)$ is defined on $0 \leqslant z \leqslant 1$ as:

$$\mathcal{L}_W(z) \quad = \quad \frac{\int_0^z G_W^{-1}(u)du}{\int_0^1 G_W^{-1}(u)du}\,, \tag{38}$$

where $G_W^{-1}(\,\cdot\,)$ denotes the inverse of $G_W(\,\cdot\,)$. Evidently, $\mathcal{L}_W(z)$ is the proportion of total W owned by the bottom z percent of people. The Gini coefficient for W is a widely used measure of wealth inequality. It equals twice the area between the 45% line of equality and the Lorenz curve $\mathcal{L}_W(z)$:[14]

$$\Gamma_W = 2\int_0^1 (z - \mathcal{L}_W(z))dz\,. \tag{39}$$

To describe fat right tails, we use both power-law exponents and "fractal inequality" (FI) as in Jones and Kim (2018). For a given random variable W, fractal inequality $FI_W(u)$ is defined as the fraction of W that goes to the top $(10 \times u)$ percent of agents divided by the fraction of W that goes to the top u percent:

$$FI_W(u) \equiv \frac{1 - \mathcal{L}_W(1 - 0.01 \times u)}{1 - \mathcal{L}_W(1 - 0.1 \times u)}\,. \tag{40}$$

Stationary Equilibrium. A competitive equilibrium consists of value functions (or, alternatively, certainty equivalent wealth functions) and optimal saving functions at all stages n; the interest rate r, the wage rate w for an agent with average productivity, stationary population demographics, and a stationary distribution for the cross-section distribution for wealth and earnings (X, Y) that satisfy

1. Given r and the stochastic labor-earnings process $\{Y_s : s \geqslant 0\}$ and X_0, value functions and optimal policies satisfy and attain, respectively, the HJB equations described in Section 3.

2. The interest rate r and w satisfy (30) and (31), respectively.

3. Equations (28) and (29) hold so that markets for capital and labor clear.

[14]See Figure 5 as an example.

4. The cross-section distribution of wealth and earnings (X, Y) is invariant over time and characterized by (34) and (35).

5 Quantitative Analysis and Economic Mechanism

After setting parameter values, we describe properties of an equilibrium cross-section wealth distribution as manifested in Lorenz curves, Gini coefficients, and power-law exponents.

5.1 Imported and Newly Calibrated Parameters

Table 1 describes parameter values that we shall use to compute equilibria for $N = 1$ and $N = 2$ instances of our model. Panel A.1 reports parameters that we intentionally import from prominent BAH papers. Panel A.2 reports parameters that we set to hit expected life length targets of 60 years for both the $N = 1$ and $N = 2$ models. Panel B reports parameters calibrated specifically for this study, namely, drifts and volatilities governing labor-earnings processes.

Panel A.1 describes a suite of parameters set at consensus values in BAH papers. We adopted these consensus values purposefully in order to help us isolate sources of new findings about the equilibrium wealth distribution that our model brings. We set preference and production function parameters to values used by Huggett (1996) and De Nardi (2004). Following Prescott (1986) and Cooley and Prescott (1995), we set the capital share of income, α, to 0.36. We set an annual depreciation rate of capital, δ, to 6% to match an estimate of the US depreciation-output ratio reported by Stokey and Rebelo (1995). We want an aggregate capital-output ratio to 3 as in Castañeda, Díaz-Giménez, and Ríos-Rull (2003) and De Nardi (2004), which in light of equation (31) leads to an equilibrium interest rate r equals 6% per annum as in Huggett (1996) and De Nardi (2004). We set the productivity parameter A to 0.9, so that the wage rate w for an agent with the average labor efficiency equals unity. We set the coefficient of relative risk aversion at $\gamma = 2$, a commonly used value.

Panel A.2 of Table 1 reports how we set parameters that governing life-stage transitions to make life expectancies under the $N = 1$ and $N = 2$ versions of the model be equal.[15] For

[15] Our model with $N = 2$ corresponds to the discrete-time version of the stochastic life-cycle model used

Table 1: PARAMETER SETTINGS AND CALIBRATION.

Panel A.1 Assigned			
Parameters	Symbol	$N = 1, 2$	
Risk aversion	γ	2	
Subjective discount rate	ρ	5%	
Capital share	α	0.36	
Capital depreciation rate	δ	6%	
Productivty	A	0.896	
Panel A.2 Life-stage			
Parameters	Symbol	$N = 1$	$N = 2$
Transition intensity	λ	0.0167	0.033
Panel B. Calibration			
Parameters	Symbol	$N = 1$	$N = 2$
Earnings growth volatility	σ	9.9%	12.7%
Expected earnings growth	μ	1.11%	1.26%

Targets: (labor-earnings Gini Γ_Y, capital-output ratio $K/F(K, L)) = (0.63, 3)$

our $N = 1$ model, we set the hazard parameter $\lambda_1 = 0.0167$ in order to target an agent's expected lifetime at $1/\lambda_N = 60$ years, as in Castañeda, Díaz-Giménez and Ríos-Rull (2003). For our $N = 2$ model, we set $\lambda_2 = \lambda_1$ and target expected durations of $1/\lambda_1 = 30$ years for both life-stages so that we obtain the same expected total lifetime of $60 = 30 + 30$ years as for our $N = 1$ model. In this way, we approximate a setting in which mortality risk is lower for most younger people and higher for most older people.

Panel B of Table 1 reports outcomes from jointly calibrating the expected labor earnings growth μ and labor earnings growth volatility σ by targeting a pair of quantities: a Gini coefficient for the cross-section labor earnings of 0.63 and a capital-output ratio of 3, as in Castañeda, Díaz-Giménez, and Ríos-Rull (2003). Similarly, De Nardi (2004) uses labor earnings growth volatility to match the Gini coefficient of labor earnings. The calibrated

by Castañeda, Díaz-Giménez, and Ríos-Rull (2003). Gertler (1999) used a discrete-time version of an $N = 2$ version of our model to study social security. Heathcote, Storesletten, and Violante (2017) use an $N = 1$ model in their study of optimal tax progressivity.

Table 2: Cross-section distributions of earnings and wealth. The parameter values for both the $N = 1$ and $N = 2$ models are reported in Table 1.

Panel A. Percentage earnings in the top						
	Gini	1%	5%	20%	40%	60%
U.S. data	0.63	15	31	61	84	97
$N = 1$	0.63	33	49	69	81	89
$N = 2$	0.63	29	46	67	81	90
Panel B. Percentage wealth in the top						
	Gini	1%	5%	20%	40%	60%
U.S. data	0.78	30	54	79	93	98
$N = 1$	**0.77**	39	58	79	91	96
$N = 2$	**0.72**	34	53	75	88	95

values are $\mu = 1.11\%$ and $\sigma = 9.9\%$ for the $N = 1$ model and are $\mu = 1.26\%$ and $\sigma = 12.7\%$ for the $N = 2$ model. While we have calibrated earnings expected growth rate μ and growth volatility σ to target two macro moments, these values are broadly in line with micro estimates reported in the literature (see Meghir and Pistaferri (2011) for a survey.)

5.2 Implications for Cross-Section Earnings and Wealth Distributions

Table 2 reports the model-implied Lorenz curves for exogenous labor earnings and endogenous wealth in Panels A and B, respectively, for both the $N = 1$ and $N = 2$ models in addition to the empirical distributions of earnings and wealth in the U.S.[16] For the $N = 1$ and $N = 2$ models separately, we calibrate λ to a Gini coefficient target of 0.63 for cross-section earnings that characterizes the data.

Let's look at exogenous labor earnings first. In Appendix B, we report that the cross-

[16]Data for distributions of earnings and wealth in the U.S. Economy are borrowed from Castañeda, Díaz-Giménez, and Ríos-Rull (2003).

section labor earnings Y follows a double Pareto distribution, also used in Luttmer (2007), Gabaix (2009), and Toda and Walsh (2015).

The $N = 1$ model. Although our simple model of labor earnings model neglects responses to transient shocks, the implied earnings distribution is able to capture key features of the empirical Lorenz curve. In the $N = 1$ model, the top 1% receive 33% of the total earnings while in the data they receive 15% of the total earnings. This aspect of our model contrasts with properties of classic BAH models that generate too little earnings concentration at the top. For example, the model-implied Gini coefficient for cross-section earnings in Aiyagari (1994) is 0.1 and the top 1% earnings-rich receive only 6.8% of total earnings.

Next, we turn to the endogenous wealth distribution reported in Panel B for our $N = 1$ model. Our $N = 1$ model delivers a Gini coefficient for cross-section wealth of 0.77 that closely approximates the wealth Gini coefficient 0.78 in the US data.

A successful feature of our model is that it predicts that the Gini coefficient for wealth is larger than that for earnings (0.63). The model generates an endogenous wealth distribution that has a fatter tail than exogenous earnings distribution, something that classic BAH with stationary exogenous labor earnings processes don't do.

Our model is too stingy with free parameters to approximate the entire wealth Lorenz curve well. Compared with observed wealth concentration, our model generates more concentrated wealth holdings for the rich. For example, the top 1% wealth-rich owns about 39% of the total wealth, while the top 1% earnings-rich makes about 33% of the total earnings in the model. As they were also for the earnings distribution, our models predictions here differ qualitatively from those of classic BAH models with stationary labor earning processes in which model-implied wealth concentration at the top is much lower than what is observed. For example, in Aiyagari (1994), the Gini coefficient for cross-section wealth is 0.38 and the top 1% only owns about 3.2% of the aggregate wealth as opposed to about 30% in the data. Thus, relative to the data, our model with non-stationary labor earnings generates *too much* wealth concentration at the top, reversing a salient finding from BAH models with stationary labor earnings. To help our model match the observed upper tail of the wealth distribution, we would somehow have to attenuate forces that push wealth toward the wealthiest, not strengthen them as has been done in BAH models with stationary labor earnings processes.

What happens when we move from the $N = 1$ model to the $N = 2$ model? We shall see that qualitative features of key predictions (e.g., fatter tailed distribution for wealth than earnings) continue to hold while fits improve.

The $N = 2$ model. Evidently, calibrating our $N = 2$ earnings model by setting the model-implied Gini coefficient at 0.63 as we do for the $N = 1$ model yields a better fit with the empirical Lorenz curve. While the $N = 2$ model still generates too much earnings concentration at the top, it gets closer to the observations than does the $N = 1$ model.

The Gini coefficient for cross-section wealth equals 0.72, which is further away from the 0.78 in the data than is the $N = 1$ model. But the Lorenz curve for the $N = 2$ model is closer to the data than is the $N = 1$ model. For example, the top 1% wealth-rich owns about 34% of total wealth. compared to 30% in the data, and the top 5% wealth-rich owns 53% of aggregate wealth, which agrees with the data. Overall, our $N = 2$ model generates a cross-section wealth distribution that is reasonably close to the empirical distribution.

In summary, with the caveat that our earnings model generates too much concentration of earnings at the top, our $N = 1$ and $N = 2$ models both generate cross-section wealth distributions with large wealth concentrations at the top, broadly consistent with U.S data.

5.3 $N = 1$ Model with No Earnings Shocks

We avail ourselves of analytic formulas deployed by Moll, Rachel, and Restrepo (2019) to isolate forces that determine outcomes within a streamlined $N = 1$ version of our model with deterministic earnings growth. We can drop subscript 1 from all relevant parameters and variables, for example, by writing $\phi_{1,Y}(Y)$ as $\phi_Y(Y)$. To assure existence of the equilibrium objects in play, we assume

$$\lambda > \mu \geqslant 0, \tag{41}$$

so that the death rate exceeds the earnings growth rate which exceeds zero. To compute a stationary equilibrium of our model in the BAH tradition, we proceed as follows. We start from an exogenous cross-section distribution of labor earnings that satisfies a power law that was derived in several influential papers that we mention below. Next, for a given interest rate within an admissible set that is consistent with positive aggregate savings, we use the optimal decision rule for consumption together with budget constraints to deduce the dynamics of a single individual's wealth and an implied cross-section distribution of wealth. Then we compute an interest rate that equates aggregate supplies of labor and capital to aggregate quantities that firms demand.

Cross-section earnings distribution. When $\sigma = 0$ and $X_0 = 0$ for all agents, length of life is the only source of heterogeneity across agents. Labor earnings are determined by an agent's age: $Y_t = Y_0 \, e^{\mu t}$. Along with condition (41), a constant mortality rate λ implies that the cumulative distribution function of the cross-section of earnings is

$$\Phi_Y(Y) = 1 - \left(\frac{Y}{Y_0}\right)^{-\lambda/\mu}, \tag{42}$$

with mean

$$\mathbb{E}(Y) = \int_0^\infty Y d\Phi_Y(Y) = \frac{\lambda}{\lambda - \mu} Y_0 \, . \tag{43}$$

Thus, the cross-section earnings distribution has a fat tail with a power-law exponent $\lambda/\mu > 1$. By using equation (42) and the definition of a Lorenz curve provided in (38), we obtain the following expression for the Lorenz curve of labor earnings:

$$\mathcal{L}_Y(z) = \frac{\int_0^z \Phi_Y^{-1}(u) du}{\int_0^1 \Phi_Y^{-1}(u) du} = 1 - (1 - z)^{\frac{\lambda - \mu}{\lambda}} \, . \tag{44}$$

The fraction of labor earnings earned by the top $(10 \times u)$ percent of agents that goes to the top u percent, defined as $FI_Y(u)$ in (40), is

$$FI_Y(u) = \frac{1 - \mathcal{L}_Y(1 - 0.01 \times u)}{1 - \mathcal{L}_Y(1 - 0.1 \times u)} = \frac{(0.01 \times u)^{\frac{\lambda - \mu}{\lambda}}}{(0.1 \times u)^{\frac{\lambda - \mu}{\lambda}}} = 10^{-\frac{\lambda - \mu}{\lambda}} < 1 \, . \tag{45}$$

Condition (41) ($\lambda > \mu$) implies $0.1 < FI_Y(u) < 1$, which means earnings has a fat right tail with a constant FI for all admissible levels of u. These results were obtained by Gabaix (1999), Luttmer (2007, 2011), Benhabib and Bisin (2018), Jones and Kim (2018), and others who combined exponential growth with a constant exit rate.

Finally, by using equation (44) and the definition given in (39), we then obtain the following formula for the Gini coefficient of labor earnings:

$$\Gamma_Y = 2\int_0^1 (z - \mathcal{L}_Y(z))dz = \frac{\mu}{2\lambda - \mu}\,. \tag{46}$$

Condition (41) implies $0 \leqslant \Gamma_Y < 1$. When $\mu = 0$ and $\sigma = 0$, $\Gamma_Y = 0$.

Optimal consumption. The scalar $q = 1/(r + \lambda - \mu)$ converts a unit of labor-earnings Y_t into the human wealth qY_t concept of Friedman (1957) so that $P_t = X_t + qY_t$ is total wealth, the sum of financial and human wealth. Total wealth $\{P_t; t \geqslant 0\}$ serves as the single state variable that determines a living agent's life-time utility if and only if $X_t > 0$ at all $t > 0$ before death. We shall verify that in equilibrium $X_t > 0$ for this range of ages. Because the market structure allows agents to hedge the only risk they face, namely mortality risk, the solution of ODE (18) is $p(x) = x + q$ and the optimal consumption rule is linear in total wealth P_t:

$$C_t = mP_t, \tag{47}$$

where $m = \rho + \lambda + (1 - \gamma^{-1})(r - \rho)$. By substituting decision rule (47) into (4) and (9) and setting $\sigma = 0$, we verify that during an agent's life time:

$$\frac{dC_t}{C_t} = \frac{dP_t}{P_t} = \left(\frac{r-\rho}{\gamma}\right) dt \tag{48}$$

and that $P_t = X_t + qY_t = P_0\, e^{(r-\rho)t/\gamma}$ and $C_t = X_t + qY_t = mP_0\, e^{(r-\rho)t/\gamma}$. Since Y_t evolves deterministically, an agent's age t is tied to her earnings by $t = \frac{\ln(Y_t/Y_0)}{\mu}$ and wealth X_t at age t is the following function of earnings Y_t:

$$X(Y_t) = X_t = (X_0 + qY_0)e^{\frac{r-\rho}{\gamma}t} - qY_t = qY_0 e^{\frac{r-\rho}{\gamma}t} - qY_t = qY_0\left[\left(\frac{Y_t}{Y_0}\right)^{\frac{r-\rho}{\gamma\mu}} - \frac{Y_t}{Y_0}\right] \tag{49}$$

that we want to be positive and increasing in Y_t. For $X(Y) > 0$ and $\frac{d}{dY}X(Y) > 0$ on $(0, +\infty)$ it is necessary that

$$\frac{r-\rho}{\gamma\mu} > 1 \iff r > \rho + \gamma\mu \tag{50}$$

so that the equilibrium interest rate r has to exceed ρ, the agent's discount rate.[17] That condition is violated by the $r < \rho$ equilibrium outcome at the heart of the BAH models with infinitely-lived agents. Indeed, (50) asserts something even stronger, namely, that the interest rate r must exceed $\rho+\gamma\mu$, the interest rate affiliated with an *augmented golden rule* for the classic Ramsey non-stochastic optimal growth model. Inequality (50) also implies that the growth rate of consumption exceeds the growth rate of earnings, a consequence of a constant marginal propensity to consume out of total wealth. Since inequality (50) hold, equation (49) implies that wealth is a convex function of earnings at all levels, a shape that amplifies wealth inequality relative to earnings inequality.

Cross-section wealth distribution. The inverse function $X(Y)$ presented in equation (49) is increasing in Y when $\frac{r-\rho}{\gamma\mu} > 1$, as is true in a stationary equilibrium, so it follows that $\Phi_X(X_t) = \Phi_Y(Y_t)$ and

$$\Phi_X(X) = 1 - \left(\frac{Y(X)}{Y_0}\right)^{-\frac{\lambda}{\mu}} . \tag{51}$$

Therefore the mean of X is

$$\mathbb{E}(X) = \frac{\lambda(r-\rho-\gamma\mu)}{(\lambda\gamma-(r-\rho))(\lambda-\mu)} \frac{1}{r+\lambda-\mu} Y_0 \,. \tag{52}$$

The cross-section distribution of wealth X is asymptotically fat tailed with a power-law exponent of

$$\zeta = \frac{\gamma\lambda}{r-\rho} \,. \tag{53}$$

This follows from

$$\lim_{X\to\infty} \frac{1-\Phi_X(X)}{\left(\frac{X}{qY_0}\right)^{-\zeta}} = \lim_{Y\to\infty} \frac{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{\mu}}}{\left(\frac{X(Y)}{qY_0}\right)^{-\zeta}} = \lim_{Y\to\infty} \frac{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{\mu}}}{\left(\frac{Y}{Y_0}\right)^{-\zeta\cdot\frac{r-\rho}{\gamma\mu}}} = \lim_{Y\to\infty} \frac{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{\mu}}}{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{\mu}}} = 1 \,, \tag{54}$$

[17] Mortality risk is fully hedged via an actuarially fair reverse annuity and so is not the source of the $r > \rho$ inequality result.

where the first equality uses (51), the second equality uses (49) and the inequality given in (50), and the third equality follows from (53).

Unlike the cross-section earnings distribution, which satisfies a power law over the entire support of Y, the cross-section wealth distribution satisfies a power law only in the limit as $X \to \infty$. For very wealthy people who have low values of u, the fraction of wealth owned by the top $10 \times u$ percent that goes to the top u percent of people, defined as $FI_X(u)$ in (40), is

$$FI_X(u) = 10^{\zeta^{-1}-1}\,, \tag{55}$$

where ζ is given in equation (53). In modeling how automation affects earnings and wealth inequality, Moll, Rachel, and Restrepo (2019) obtain exactly the same power-law exponent. In our model, but not theirs, agents hedge mortality risk.

Inspecting the mechanism: from micro to macro and from earnings to wealth distributions. Taken together, inequality (50) and equation (53) imply that cross-section wealth has a fatter right tail than earnings, as the power-law exponent for cross-section wealth is smaller than that for cross-section earnings: $\zeta < \lambda/\mu$.

By using equation (C.9) in Appendix C and the definitions given in (38) and (39), we obtain the following formula for the Lorenz curve of wealth, $\mathcal{L}_X(z)$:

$$\begin{aligned}\mathcal{L}_X(z) &= \frac{\int_0^z \Phi_X^{-1}(u)du}{\int_0^1 \Phi_X^{-1}(u)du} \\ &= \frac{\gamma(\lambda-\mu)}{r-(\rho+\gamma\mu)}\left(1-(1-z)^{\frac{(\rho+\gamma\lambda)-r}{\lambda\gamma}}\right) - \frac{(\rho+\gamma\lambda)-r}{r-(\rho+\gamma\mu)}\left(1-(1-z)^{\frac{\lambda-\mu}{\lambda}}\right)\end{aligned} \tag{56}$$

and the following formula for Gini coefficient of wealth:

$$\Gamma_X = 2\int_0^1 (z-\mathcal{L}_X(z))dz \;=\; \frac{2\gamma\lambda^2+\mu(\rho-r)}{(\rho-r+2\gamma\lambda)(2\lambda-\mu)}\,. \tag{57}$$

Our condition (41) that $\lambda > \mu$ implies that $\Gamma_X > \Gamma_Y$.

Figure 5 portrays the mechanism that generates a fatter tailed distribution for cross-section wealth than for earnings. Panels A and B show that earnings Y grows at a constant

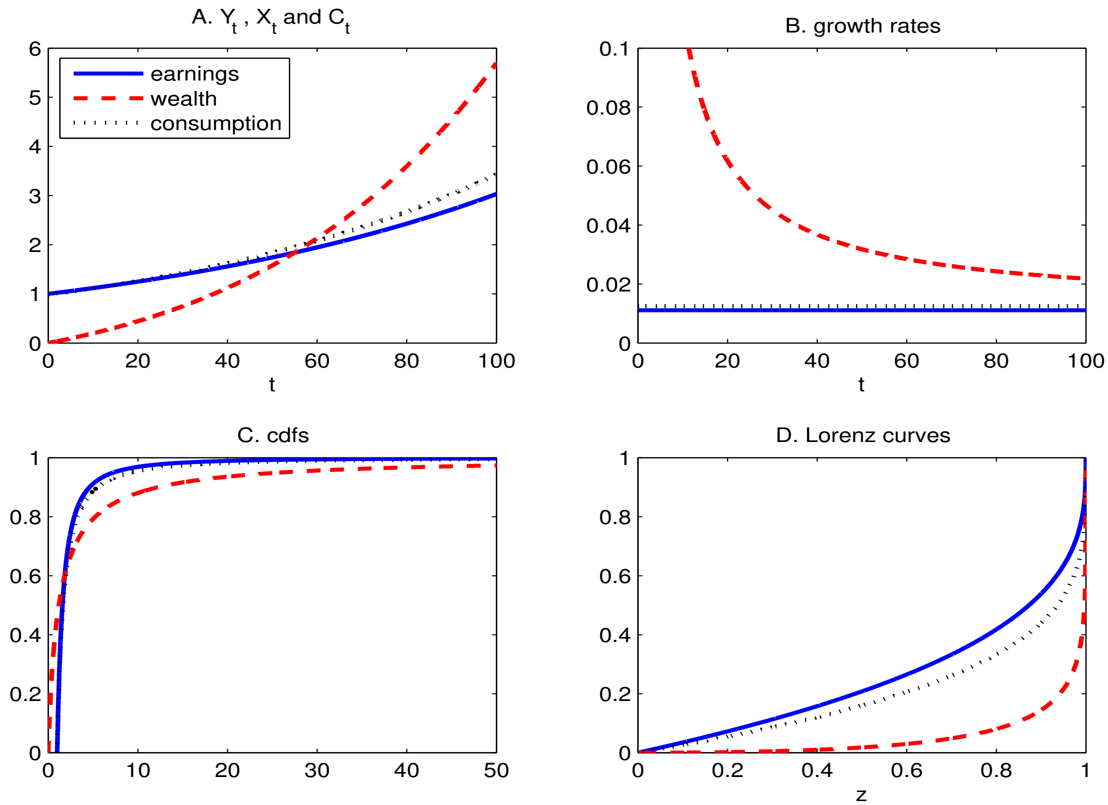

Figure 5: Earnings, wealth, and consumption: micro dynamics and macro cross-section distribution for the deterministic $N = 1$ model. Panel A plots the levels of Y_t, X_t, and C_t. Panel B plots the corresponding growth rate of change over time: $\dot{X}_t/X_t$, $\dot{Y}_t/Y_t$ and $\dot{C}_t/C_t$. Panel C plots the CDFs of Y, X, and C. Panel D plots the equilibrium stationary cross-section Lorenz curves for Y, X and C. Parameter values are reported in Table 1 with $\sigma = 0$.

rate μ, which is lower than the consumption growth rate $(r-\rho)/\gamma$. This is because X grows at a nonlinear rate greater than consumption and earnings growth rates. While the growth rate of wealth decreases with age starting from ∞ at $X_0 = 0$, dX_t increases with age.

Panel C plots the cross-section distributions of Y, X, and C. Panel D plots the corresponding Lorenz curves. The CDFs for both earnings and consumption are described globally by power laws with different exponents. Interestingly, the distribution of consumption is fatter tailed than that of earnings. This result may appear surprising as agents prefer consumption smoothing over time. The reason for this seemingly counterintuitive result is that consumption grows faster than earnings in equilibrium for all living agents, i.e., condition (50). The distribution of wealth is not globally Pareto as earnings and consumption are but instead asymptotically approaches the shape of a Pareto distribution with the same power-law exponent as consumption distribution at the right tail (e.g., $X \to \infty$). Panel D shows that wealth has a significantly steeper/convex Lorenz curve than consumption does, which in turn has a steeper/convex Lorenz curve than earnings Y does. Therefore, the Gini coefficient for X is larger than that for consumption C, which is larger than that for earnings Y.

Aggregate earnings and wealth and interest rate r. An individual agent inelastically supplies her labor and accumulates wealth according to

$$dX_t = ((r+\lambda)X_{t-} + Y_{t-} - C_{t-})\,dt - X_{t-}dS_t\,, \tag{58}$$

where the last term describes the individual's wealth transfer to the insurance company at the instant of death. Summing over all agents and using the law of large numbers (Sun (2006)), we obtain

$$\begin{aligned} d\mathbb{E}(X_t) &= \left[(r+\lambda)\mathbb{E}(X_{t-}) + \mathbb{E}(Y_{t-}) - \mathbb{E}(C_{t-})\right]dt - \lambda\mathbb{E}(X_{t-})dt\,, && (59)\\ &= \left[r\mathbb{E}(X) + \mathbb{E}(Y) - \mathbb{E}(C)\right]dt\,, && (60)\end{aligned}$$

where the last term in equation (59) is gross receipts collected by the insurance company. Because there is no aggregate demographic risk, so the death rate of λ always balances the birth rate of λ to stabilize total population mass at unity. As in Blanchard (1985), the insurance company makes zero profits and the net aggregate transfers from deceased to living is zero.

In a stationary equilibrium, $dK_t = d\mathbb{E}(X_t) = 0$ so equation (60) implies that aggregate consumption satisfies

$$\mathbb{E}(C) = r\mathbb{E}(X) + \mathbb{E}(Y)\,. \tag{61}$$

We have the National Income and Product Accounts typically associated with BAH models:

$$F(K,L) = F_K K + F_L L = (r+\delta)K + wL = (r+\delta)\mathbb{E}(X) + \mathbb{E}(Y) = \mathbb{E}(C) + \delta K\,. \tag{62}$$

The first equality follows from the Euler's theorem applied to the aggregate Cobb-Douglas production function; the second equality uses the firm's first-order conditions for factors of production: $F_K(K,L) = (r+\delta)$ and $F_L(K,L) = w$; the third equality follows from the market clearing conditions $K = \mathbb{E}(X_t)$ and $wL = \mathbb{E}(Y_t)$; the fourth equality follows (61).

To compute the stationary equilibrium interest rate, take the firm's first-order conditions for capital and labor, $r + \delta = F_K(K,L)$ and $w = F_L(K,L)$ and then substitute (43) and (52) for $\mathbb{E}(Y)$ and $\mathbb{E}(X)$, respectively, to obtain

$$r = \frac{\alpha}{1-\alpha}\frac{wL}{K} = \frac{\alpha}{1-\alpha}\frac{\mathbb{E}(Y)}{\mathbb{E}(X)} = \frac{\alpha}{1-\alpha}\frac{(\rho+\lambda\gamma-r)(r+\lambda-\mu)}{r-(\rho+\gamma\mu)} - \delta. \tag{63}$$

The string of equalities implies the following restriction on the stationary equilibrium r:

$$\begin{aligned}\Psi(r) \equiv r^2 &- [\rho + (1-\alpha)(\gamma\mu-\delta) + \alpha(\gamma\lambda - (\lambda-\mu))]\, r \\ &- [(1-\alpha)\delta(\rho+\gamma\mu) + \alpha(\lambda-\mu)(\rho+\gamma\lambda)] = 0\,. \end{aligned} \tag{64}$$

One root of this quadratic equation is positive, the other negative. The positive root is the equilibrium interest rate.[18] Evidently

$$\Psi(0) = -((1-\alpha)\delta(\rho+\gamma\mu) + \alpha(\lambda-\mu)(\rho+\gamma\lambda)) < 0\,, \tag{65}$$

$$\Psi(\rho) = -(1-\alpha)(\rho+\delta)\gamma\mu - \alpha\gamma\lambda(\rho+\lambda-\mu) < 0\,, \tag{66}$$

$$\Psi(\rho+\gamma\mu) = -(\lambda-\mu)\alpha\gamma(\rho+\gamma\mu+\lambda-\mu) < 0\,, \tag{67}$$

$$\Psi(\rho+\gamma\lambda) = (\lambda-\mu)(1-\alpha)\gamma(\rho+\gamma\lambda+\delta) > 0\,. \tag{68}$$

Figure 6 reveals that the equilibrium interest rate satisfies:

$$\rho \leqslant \rho + \gamma\mu < r < \rho + \gamma\lambda. \tag{69}$$

Role of earnings growth μ. In Table 3, we conduct a comparative static exercise with respect to the earnings growth rate μ. For all rows, $\sigma = 0$. All other parameter values are reported in Table 1. In addition to the equilibrium interest rate r, we report Gini

[18]Equation (48) implies that a negative r together with $X_0 = 0$ (and hence $P_0 = Y_0$) would not cohere with the requirement that $\mathbb{E}(X) = K > 0$.

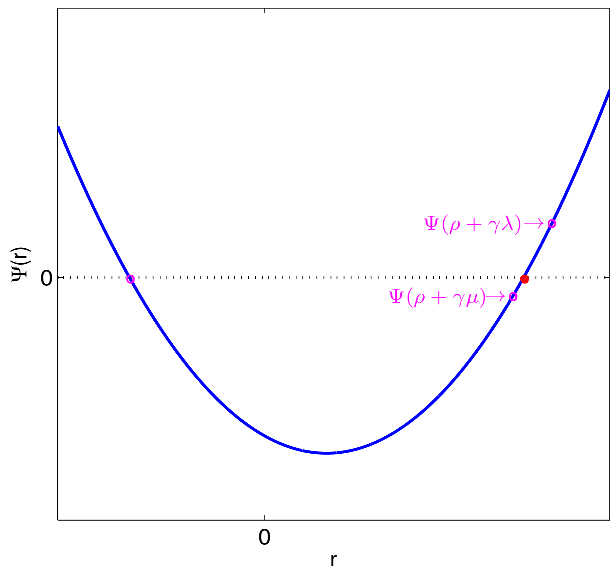

Figure 6: The quadratic function $\Psi(r)$. The equilibrium interest rate satisfies $\Psi(r^*) = 0$ and $(\rho + \gamma\mu) < r^* < (\rho + \gamma\lambda)$ when $\lambda > \mu \geqslant 0$ as assumed in condition (41).

coefficients for earnings and wealth (Γ_Y and Γ_X), power-law exponents (ξ_Y and ξ_X), and fractal inequalities ($FI_Y(z)$ and $FI_X(z)$). Because distributions of X and Y both obey power laws with constant exponents for all z, we can show that $FI_Y(z)$ and $FI_X(z)$ are constant at all z. That is, the comparison in terms of FIs holds at all levels of z and hence we can omit z from the table.

First consider the case in which $\mu = 0$. There is zero cross-section earnings inequality (hence $\Gamma_Y = 0$, $\xi_Y = \infty$ and $FI_Y = 0.1$.) An equilibrium interest rate $r = 5.88\%$ that exceeds the annual rate of time preference $\rho = 5\%$ makes young people want to save. Our analytical formulas indicate that wealth has a power-law exponent of $\zeta = \gamma\lambda/(r - \rho) = 2 \times (1/60)/(0.0588 - 0.05) = 3.79$, a Gini coefficient of $\Gamma_X = 1/(2 - \zeta^{-1}) = 0.58$, and fractal inequality $FI_X(z) = 10^{1/\zeta - 1} = 0.18$ for all z. The fraction of wealth earned by the top $10 \times z$ percent of people that goes to the top z percent, is 18%, which is more than the population ratio, 10%, indicating wealth is fat tailed. The corresponding row in Table 3 shows that in equilibrium the pure life-cycle savings motive for young people, all of whom are born with no (or small) wealth, can generate a fat-tailed wealth distribution even when their labor earnings are perfectly equal and wealth inequality is entirely driven by how long

Table 3: EFFECTS OF μ FOR THE $N = 1$ MODEL WITH DETERMINISTIC EARNINGS GROWTH ($\sigma = 0$). Γ_Y and Γ_X are the Gini coefficient for earnings and wealth, respectively. For all levels of Y, the power-law exponent for earnings is $\xi_Y = \lambda/\mu$ and for wealth approaches $\xi_X = \zeta = \gamma\lambda/(r-\rho)$ as $X \to \infty$. All other parameter values are reported in Table 1.

μ	r	Γ_Y	Γ_X	ξ_Y	ξ_X	FI_Y	FI_X
0	*5.88%*	*0*	*0.58*	∞	*3.79*	0.10	0.18
0.5%	6.60%	0.18	0.72	3.34	2.08	0.20	0.30
0.71%	6.91%	0.27	0.78	2.35	1.75	0.27	0.37
1%	7.34%	0.43	0.87	1.67	1.43	0.40	0.50
1.11%	**7.50%**	**0.50**	**0.90**	**1.51**	**1.34**	**0.46**	**0.56**
1.29%	7.77%	0.63	0.95	1.29	1.21	0.59	0.68

different agents live.

At a given interest rate r, a higher labor earnings growth rate μ strengthens incentives to borrow against future income to finance current consumption. To encourage savings and clear the asset market, the equilibrium r increases with μ.

For example, as μ increases from zero to 1.11% (the value in our baseline calibration), cross-section earnings inequality increases because older people have higher earnings: the Gini coefficient for cross-section earnings increases from zero to $\Gamma_Y = 1.11\%/(2 \times (1/60) - 1.11\%) = 0.5$ and the earnings tail becomes fatter (with the power-law exponent ξ_Y decreasing from ∞ to $\lambda/\mu = (1/60)/1.11\% = 1.51$.) As a result, the fraction of earnings received by the top $10 \times z$ percent of agents that goes to the top z percent, $FI_Y(z) = 10^{1/\xi_Y - 1} = 10^{(1/1.51)-1} = 46.3\%$ for all z. This number is substantial larger than the corresponding 10% population and indicates substantial earnings inequality.

In order to elicit saving, the equilibrium interest rate increases from 5.88% to 7.5%. Because the equilibrium interest rate is higher, the returns from savings when $\mu = 1.11\%$ are much greater than when $\mu = 0$. As a result, the cross-section wealth inequality increases substantially. The Gini coefficient for wealth Γ_X increases to 0.9 from 0.58 and the wealth tail becomes fatter with the power-law exponent $\xi_X = \zeta = \gamma\lambda/(r-\rho)$ decreasing to $2 \times (1/60)/(7.5\% - 5\%) = 1.34$ from 3.79, and the fraction of wealth owned by the top $10 \times z$

percent of agents owned by the top z percent, the FI measure increases to $FI_X = 10^{1/\zeta - 1} = 10^{1/1.34-1} = 56\%$ from 18%.

Table 3 confirms two key results about wealth inequality. First, a higher growth rate of earnings increases Gini coefficients and fattens right tails of both earnings and wealth. Second, for all levels of μ, wealth inequality is larger than earnings inequality whether we measure them with Gini coefficients ($\Gamma_X > \Gamma_Y$) or the power-law exponents for right tails ($\xi_X < \xi_Y$) or fractal inequality ($FI_X > FI_Y$). This occurs because in our model the aged are both earnings-rich and wealth-rich. The longer someone lives, the wealthier she becomes.

6 Concluding Remarks

We have demonstrated how putting multi-stage stochastic life cycles and permanent labor-earnings shocks into an otherwise standard BAH model unleashes forces that create substantial wealth inequality as measured by widely accepted criteria like Gini coefficients and Lorenz curves. To allow us to understand what drives our results, we have kept our model ruthlessly parsimonious in terms of parameters. By being less stingy with parameters, we can extend the model to do more. To capture important features that we have ignored, for example, transients shocks to labor earnings, we anticipate that we can apply similar analytical techniques to the ones we have deployed here.

Our model generates a distribution of marginal propensities to consume (MPCs), an object of interest for a number of topics.[19] To take one example of substantial contemporary interest, our model environment is one in which taxing wealth and transferring it to the very young can have substantial effects on welfare (e.g., based on a utilitarian welfare criterion) as well as on wealth and the interest rate, effects that would be transmitted through the shape of the consumption function analyzed in Subsection 3.2. We anticipate using our model to study this and other policy experiments in future research, even while acknowledging the limits of the present model in addressing taxation questions by its having

[19] Kaplan and Violante (2014) study consequences of fiscal stimuli. Other papers study transition mechanisms of monetary policy (Auclert (2019); Kaplan, Moll, and Violante (2018), effects of a credit crunch or house price movements on consumer spending (e.g. Guerrieri and Lorenzoni (2017); and how inequality affects aggregate demand, e.g., Auclert and Rognlie (2018).

taken the labor earnings process as exogenous as in the BAH tradition.

Finally, we have excluded aggregate shocks and focused on a stochastic steady-state analysis. By deploying techniques from mean field game theory, we hope to explore how to be able to adapt the model to incorporate aggregate shocks. Gabaix, Lasry, Lions, and Moll (2016) use mean field games to analyze the dynamics of inequality. They show that standard random-growth-based models generate transition dynamics that are too slow relative to those observed in the data. Guvenen, Karahan, Ozkan, and Song (2015) and De Nardi, Fella, and Paz-Pardo (2020) document that logarithmic earnings innovations are very fat-tailed. We aspire to include richer earnings processes and aggregate transition dynamics in future work.

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Appendices

Appendix A sketches proofs of main results in Section 3. Appendix B summarizes the cross-section earnings distribution and provides proofs for the $N = 1$ model. Appendix C provides details for the equilibrium analysis of a deterministic version of the $N = 1$ model. Appendix D provides additional details about how we calculate aggregate variables.

A Proofs for Solutions in Section 3

First, by using the HJB equations given in (11) and (12), we obtain the following FOC for consumption:

$$U'(C_n) = V_{n,X}(X,Y)\,, \tag{A.1}$$

which equates the marginal benefit of consumption $U'(C_n)$ with the marginal utility of savings $V_{n,X}(X,Y)$. Using the value function given in (13) and the homogeneity property $P_n(X,Y) = p_n(x)Y$, we obtain the optimal scaled consumption rule $c_n(x)$ given in (16). Substituting (13), $P_n(X,Y) = p_n(x)Y$, and (16) into the HJB equations (11) and (12), we obtain the ODE (18) for $p_N(x)$ and the ODE (19) for $p_n(x)$ where $n \leqslant N-1$. Substituting $p_N(x) = x + q_N$ into (18) and letting $x \to \infty$, we obtain

$$\begin{aligned} 0 &= \left(\frac{\gamma b_N^{1-1/\gamma} - (\rho+\lambda_N)}{1-\gamma} + \mu_N\right)(x+q_N) + 1 + (r+\lambda_N-\mu_N)x \\ &= \left(\frac{\gamma b_N^{1-1/\gamma} - (\rho+\lambda_N)}{1-\gamma} + r + \lambda_N\right)x + \left(\frac{\gamma b_N^{1-1/\gamma} - (\rho+\lambda_N)}{1-\gamma} + \mu_N\right)q_N + 1\,. \end{aligned} \tag{A.2}$$

As (A.2) must hold for all x, we obtain $b_N = \left[\rho + \lambda_N + (1-1/\gamma)(r-\rho)\right]^{\frac{1}{1-1/\gamma}}$ as given by (14). And then substituting (14) into (A.2), we obtain (23) for q_N. Similarly, substituting $p_n(x) = x + q_n$ into (19) and letting $x \to \infty$, we obtain

$$\begin{aligned} 0 \;=\; & \left(\frac{\gamma b_n^{1-1/\gamma} - \rho}{1-\gamma} + \frac{\lambda_n}{1-\gamma}\left[\left(\frac{b_{n+1}}{b_n}\right)^{1-\gamma} - 1\right] + \mu_n\right)(x+q_n) + 1 + (r-\mu_n)x \\ & + \lambda_n \left(\frac{b_{n+1}}{b_n}\right)^{1-\gamma}(q_{n+1}-q_n) \end{aligned}$$

$$
\begin{aligned}
= & \left(\frac{\gamma b_n^{1-1/\gamma}-\rho}{1-\gamma}+\frac{\lambda_n}{1-\gamma}\left[\left(\frac{b_{n+1}}{b_n}\right)^{1-\gamma}-1\right]+r\right)x+1 \\
& +\left(\frac{\gamma b_n^{1-1/\gamma}-\rho}{1-\gamma}-\frac{\lambda_n}{1-\gamma}-\mu_n\right)q_n+\lambda_n\left(\frac{b_{n+1}}{b_n}\right)^{1-\gamma}q_{n+1}\,. \qquad \text{(A.3)}
\end{aligned}
$$

Since (A.3) must hold for all x, we obtain b_n as given by (15). Then substituting (15) into (A.3) gives equation (22) for q_n.

B Cross-section Earnings Distribution for the $N = 1$ Model

B.1 Closed-Form Solutions

Proposition B.1. *The cumulative distribution function of labor earnings Y is given by:*

$$
\Phi_Y(Y) = \begin{cases} \frac{\beta_2}{\beta_2-\beta_1}\left(\frac{Y}{Y_0}\right)^{\beta_1}, & Y < Y_0\,, \\ 1-\frac{\beta_1}{\beta_1-\beta_2}\left(\frac{Y}{Y_0}\right)^{\beta_2}, & Y \geqslant Y_0\,. \end{cases} \qquad \text{(B.1)}
$$

where $\beta_1 > 0$ and $\beta_2 < -1$ are the two roots of following the quadratic equation for β:[20]

$$
0 = \lambda + \left(\mu - \frac{\sigma^2}{2}\right)\beta - \frac{\sigma^2\beta^2}{2}\,. \qquad \text{(B.2)}
$$

The distribution function $\Phi_Y(Y)$ in (B.1) is known as the double Pareto distribution and has been studied in Luttmer (2007), Gabaix (2009), and Toda and Walsh (2015). The right tail is governed by a power law: for large $\widehat{Y}$, $\text{Prob}[Y \geqslant \widehat{Y}] = \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{\widehat{Y}}{Y_0}\right)^{1/\xi_Y}$ where ξ_Y is the power-law exponent

$$
\xi_Y = -\beta_2 > 1 > 0\,, \qquad \text{(B.3)}
$$

where $\beta_2 < -1$ is the negative root for (B.2).

[20] When $\sigma \neq 0$, equation (B.2) has two roots, $\beta_1 > 0$ and $\beta_2 < -1$, as $\lambda > 0$ and $\lambda - \mu > 0$. When the earnings process is deterministic ($\sigma = 0$), an important special case, equation (B.2) is linear with only one root: $\beta_2 = -\lambda/\mu$.

The cross-section average of labor earnings is

$$\mathbb{E}(Y) = \frac{\beta_1\beta_2}{(\beta_1+1)(\beta_2+1)}Y_0 = \frac{\lambda}{\lambda-\mu}Y_0 \,. \tag{B.4}$$

The Lorenz curve $\mathcal{L}_Y(\,\cdot\,)$ of labor earnings Y is:

$$\mathcal{L}_Y(z) = \begin{cases} \frac{\beta_2+1}{\beta_2}\left(\frac{\beta_2-\beta_1}{\beta_2}\right)^{\frac{1}{\beta_1}} z^{\frac{\beta_1+1}{\beta_1}} \,, & 0 \leqslant z < \frac{\beta_2}{\beta_2-\beta_1} \,, \\ 1 - \frac{\beta_1+1}{\beta_1}\left(\frac{\beta_1-\beta_2}{\beta_1}\right)^{\frac{1}{\beta_2}} (1-z)^{\frac{\beta_2+1}{\beta_2}} \,, & \frac{\beta_2}{\beta_2-\beta_1} \leqslant z \leqslant 1 \,. \end{cases} \tag{B.5}$$

Using the definition of the Gini coefficient Γ_Y in (39), we find that the Gini coefficient of labor earnings is

$$\Gamma_Y \;=\; \frac{2\beta_2^2 + 2\beta_1^2 - \beta_1\beta_2 + \beta_2 + \beta_1}{(\beta_2-\beta_1)(2\beta_1+1)(2\beta_2+1)} \,. \tag{B.6}$$

The "fractal inequality" $FI_Y(z)$ for earnings is

$$FI_Y(z) = \frac{1-\mathcal{L}_{1,Y}(1-z)}{1-\mathcal{L}_{1,Y}(1-10\times z)} = \left(\frac{1}{10}\right)^{\frac{\beta_2+1}{\beta_2}} \,, \tag{B.7}$$

provided that $z \leqslant \frac{\beta_1}{\beta_1-\beta_2}$.

B.2 Proofs

Using the Kolmogorov Forward equation (33) for the case with $N=1$ we obtain

$$0 = (\sigma^2-\mu)\phi_Y(Y) + (2\sigma^2-\mu)Y\phi_Y'(Y) + \frac{\sigma^2Y^2}{2}\phi_Y''(Y) - \lambda\phi_Y(Y) \,. \tag{B.8}$$

The density function $\phi_Y(Y)$ takes the form of a double Pareto (power law) distribution:

$$\phi_Y(Y) = \begin{cases} \kappa_1 Y^{\beta_1-1} \,, & Y < Y_0 \,, \\ \kappa_2 Y^{\beta_2-1} \,, & Y \geqslant Y_0 \,, \end{cases} \tag{B.9}$$

where $\beta_1 > 1$ and $\beta_2 < -1$ are roots of the quadratic equation

$$0 \;=\; (\lambda + \mu - \sigma^2) + (\mu - 2\sigma^2)(\beta - 1) - \frac{\sigma^2(\beta-1)(\beta-2)}{2}\,, \tag{B.10}$$

which implies (B.2).

Because $\phi_Y(Y)$ must be continuous at Y_0, we have

$$\kappa_1 Y_0^{\beta_1 - 1} = \kappa_2 Y_0^{\beta_2 - 1}\,. \tag{B.11}$$

By integrating the density, we obtain:

$$1 = \int_0^{Y_0} (\kappa_1 Y^{\beta_1 - 1}) dY + \int_{Y_0}^{\infty} (\kappa_2 Y^{\beta_2 - 1}) dY = \frac{\kappa_1 Y_0^{\beta_1}}{\beta_1} - \frac{\kappa_2 Y_0^{\beta_2}}{\beta_2}\,. \tag{B.12}$$

Jointly solving (B.11) and (B.12), we obtain:

$$\kappa_1 = \frac{\beta_1\beta_2}{\beta_2 - \beta_1} Y_0^{-\beta_1} = \frac{\lambda_1}{\sqrt{(\mu_1 - \sigma_1^2/2)^2 + 2\lambda_1\sigma_1^2}} Y_0^{-\beta_1}\,, \tag{B.13}$$

$$\kappa_2 = \frac{\beta_1\beta_2}{\beta_2 - \beta_1} Y_0^{-\beta_2} = \frac{\lambda_1}{\sqrt{(\mu_1 - \sigma_1^2/2)^2 + 2\lambda_1\sigma_1^2}} Y_0^{-\beta_2}\,. \tag{B.14}$$

Substituting (B.13) for κ_1 and (B.14) for κ_2 into (B.9), we obtain the cross-section stationary distribution of earnings $\phi_Y(Y)$

$$\phi_Y(Y) = \begin{cases} \frac{\lambda}{\sqrt{(\mu-\sigma^2/2)^2 + 2\lambda\sigma^2}} Y_0^{-\beta_1} Y^{\beta_1 - 1}\,, & Y < Y_0\,, \\ \frac{\lambda}{\sqrt{(\mu-\sigma^2/2)^2 + 2\lambda\sigma^2}} Y_0^{-\beta_2} Y^{\beta_2 - 1}\,, & Y \geqslant Y_0\,, \end{cases} \tag{B.15}$$

By integrating $\phi_Y(Y)$, we obtain $\Phi_Y(Y)$ is given by (B.15). Let $\Phi_Y^{-1}(\cdot)$ denote the inverse distribution function of Y. We can show that

$$\Phi_Y^{-1}(u) = \begin{cases} \left(\frac{\beta_2 - \beta_1}{\beta_2} u\right)^{\frac{1}{\beta_1}} Y_0\,, & 0 \leqslant u < \frac{\beta_2}{\beta_2 - \beta_1}\,, \\ \left(\frac{\beta_1 - \beta_2}{\beta_1} (1-u)\right)^{\frac{1}{\beta_2}} Y_0\,, & \frac{\beta_2}{\beta_2 - \beta_1} \leqslant u \leqslant 1\,. \end{cases} \tag{B.16}$$

By integrating $\Phi_Y^{-1}(\cdot)$, we obtain

$$\int_0^z \Phi_Y^{-1}(u)du \tag{B.17}$$
$$= \begin{cases} \frac{\beta_1}{\beta_1+1}\left(\frac{\beta_2-\beta_1}{\beta_2}\right)^{\frac{1}{\beta_1}} Y_0 z^{\frac{\beta_1+1}{\beta_1}}\,, & 0 \leqslant z < \frac{\beta_2}{\beta_2-\beta_1}\,, \\ \frac{\beta_2}{\beta_2+1}\left(\frac{\beta_1-\beta_2}{\beta_1}\right)^{\frac{1}{\beta_2}} Y_0 \left[\left(\frac{\beta_1}{\beta_1-\beta_2}\right)^{\frac{\beta_2+1}{\beta_2}} - (1-z)^{\frac{\beta_2+1}{\beta_2}}\right] + \frac{\beta_1\beta_2}{(\beta_1+1)(\beta_2-\beta_1)}Y_0\,, & \frac{\beta_2}{\beta_2-\beta_1} \leqslant z \leqslant 1\,. \end{cases}$$

Finally, by using $\mathcal{L}_Y(\cdot) = \frac{\int_0^z \Phi_Y^{-1}(u)du}{\int_0^1 \Phi_Y^{-1}(u)du}$, we obtain the Lorenz curve (B.5) for earnings.

C A Deterministic $N = 1$ Model ($\sigma = 0$)

Cross-section earnings distribution. When $\sigma = 0$, the Kolmogorov Forward equation for $\phi_Y(Y)$ is

$$0 = -\left(\frac{\partial \mu Y \phi_Y(Y)}{\partial Y}\right) - \lambda \phi_Y(Y)\,. \tag{C.1}$$

The density function implied by (C.1) is

$$\phi_Y(Y) = \frac{\lambda}{\mu Y_0}\left(\frac{Y}{Y_0}\right)^{-\frac{\mu+\lambda}{\mu}}\,, \tag{C.2}$$

which implies the CDF given in (42). (With $\sigma = 0$, equation (B.2) is linear and the only root is $\beta = -\lambda/\mu$.)

Cross-section wealth distribution. Since $P_t = 0$ at the stochastic death moment $t = \tau_{N+1}$, we can rewrite (48) as follows for $t \leqslant \tau_{N+1}$:

$$dP_t = \left(\frac{r-\rho}{\gamma}\right) P_{t-}dt - P_{t-}dS_t\,. \tag{C.3}$$

Applying the Kolmogorov Forward Equation to $P_t = P(X_t, Y_t)$, we obtain

$$0 = -\frac{\partial}{\partial P}\left[\left(\frac{r-\rho}{\gamma}P\right)\phi_P(P)\right] - \lambda\phi_P(P) = -\left(\frac{r-\rho}{\gamma} + \lambda\right)\phi_P(P) - \frac{r-\rho}{\gamma}P\phi_P'(P)\,. \tag{C.4}$$

By solving (C.4), we obtain the following cross-section stationary distribution of P:

$$\phi_P(P) = \zeta \frac{1}{P_0^{-\zeta}} P^{-\zeta-1}, \tag{C.5}$$

where $P_0 = qY_0 = \left(\frac{1}{r+\lambda-\mu}\right) Y_0$ and ζ is given by (53). Equation (C.5) implies the following CDF for P:

$$\Phi_P(P) = 1 - \left(\frac{P}{P_0}\right)^{-\zeta}. \tag{C.6}$$

Next compute the inverse of the CDF $\Phi(X)$ for wealth X. Re-writing (51) yields

$$\frac{Y(X)}{Y_0} = (1 - \Phi_X(X))^{-\frac{\mu}{\lambda}}. \tag{C.7}$$

Substituting (C.7) into (49), we obtain

$$\begin{aligned} X &= \left[\left((1 - \Phi_X(X))^{-\frac{\mu}{\lambda}}\right)^{\frac{r-\rho}{\gamma\mu}} - (1 - \Phi_X(X))^{-\frac{\mu}{\lambda}}\right] qY_0 \\ &= \left[(1 - \Phi_X(X))^{\frac{\rho-r}{\lambda\gamma}} - (1 - \Phi_X(X))^{-\frac{\mu}{\lambda}}\right] qY_0 . \end{aligned} \tag{C.8}$$

Let $u = \Phi_X(X)$. Then, $X = \Phi_X^{-1}(u)$ and we can rewrite (C.8) as

$$\Phi_X^{-1}(u) = \left((1-u)^{\frac{\rho-r}{\lambda\gamma}} - (1-u)^{-\frac{\mu}{\lambda}}\right) qY_0 . \tag{C.9}$$

Integrating $\Phi_X^{-1}(\cdot)$ from 0 to z yields

$$\int_0^z \Phi_X^{-1}(u)du = \left[\frac{\lambda\gamma\left(1-(1-z)^{\frac{\rho-r+\lambda\gamma}{\lambda\gamma}}\right)}{\rho-r+\lambda\gamma} - \frac{\lambda\left(1-(1-z)^{\frac{\lambda-\mu}{\lambda}}\right)}{\lambda-\mu}\right] qY_0 . \tag{C.10}$$

We use equation (C.10) when calculating the wealth Lorenz curve and Gini coefficient.

D Computing Aggregates

We compute equilibrium objects by iterating over candidate interest rates. First, for a given r, we compute total savings $\mathbb{E}(X)$ by aggregating over individual's optimal savings demand. Second, equations (30) and (31) imply that the wage rate w can be deduced from the factor price frontier

$$w = A(1-\alpha)\left(\frac{r+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}} . \tag{D.1}$$

Third, endowed labor units H are exogenous and the agent does not value leisure. Thus, the total wage payment to labor equals total labor earnings: $wH = \mathbb{E}(Y)$. Since we fix μ and σ when we perform comparative static analyses, we infer the value of Y_0 from $wH = \mathbb{E}(Y)$.

Fourth, we solve for the aggregate capital stock K by using the equilibrium increasing relation between K and w given in (30). Finally, we check whether the aggregate K obtained in step 4 equals the aggregate savings $\mathbb{E}(X)$ obtained in step 1. If so, we have found a fixed point. Otherwise, we continue the iteration process until we find one. From a fixed point, we obtain equilibrium objects, r, w, Y_0, K, with the implied aggregate capital-output ratio $K/F(K,L) = \frac{(K/H)^{1-\alpha}}{A}$.

For the deterministic $N = 1$ model, we have closed-form solutions. The equilibrium interest rate is the positive root in equation (64), the equilibrium wage rate w satisfies (D.1), and

$$\begin{aligned} Y_0 &= \frac{wH}{\frac{\lambda}{\lambda-\mu}} = \frac{(\lambda-\mu)wH}{\lambda} = \frac{A(1-\alpha)(\lambda-\mu)H}{\lambda}\left(\frac{r+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}} , \\ K &= \mathbb{E}(X) = \frac{r-\rho-\gamma\mu}{\lambda\gamma-(r-\rho)}\frac{A(1-\alpha)H}{r+\lambda-\mu}\left(\frac{r+\delta}{A\alpha}\right)^{\frac{\alpha}{\alpha-1}} . \end{aligned}$$