Optimal Taxation without State-Contingent Debt

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Abstract

In Lucas and Stokey's (1983) economy, tax rates inherit the serial correlation structure of government expenditures, belying Barro's (1979) result that taxes should be a random walk for any stochastic process of government expenditures. To recover a version of Barro's 'random walk' tax-smoothing outcome, we modify Lucas and Stokey's (1983) economy to permit only risk-free debt. Having only risk-free debt confronts the Ramsey planner with additional constraints on equilibrium allocations beyond those imposed under Lucas and Stokey's assumption of complete markets. We formulate the Ramsey problem under incomplete markets in terms of a Lagrangian and solve the associated first-order conditions. An analytical example and some numerical impulse response functions partially affirm Barro's analysis. Putting incomplete markets into Lucas and Stokey's environment imparts near unit root behavior to government debt, independently of the government expenditure process, an outcome like Barro's.

Optimal Taxation without State-Contingent Debt

"It appears to have been the common practice of antiquity, to make provision, during peace, for the necessities of war, and to hoard up treasures before-hand, as the instruments either of conquest or defence; without trusting to extraordinary impositions, much less to borrowing, in times of disorder and confusion." David Hume, 'Of Public Credit,' 1777.

1. Introduction

Robert Barro (1979) embraced an analogy with a permanent income model of consumption to conjecture that debt and taxes should follow random walks, regardless of the serial correlations of government expenditures.[1] Lucas and Stokey (1983) lost Barro's intuition when they formulated a Ramsey problem for a model with complete markets, no capital, exogenous Markov government expenditures, and state-contingent taxes and government debt. They discovered that optimal tax rates and government debt are not random walks, and that the serial correlations of optimal tax rates are tied closely to those for government expenditures. Lucas and Stokey make taxes smooth in the sense of having small variances, not being random walks.

However, the consumption model that inspired Barro assumes a consumer who faces incomplete markets and adjusts holdings of a risk-free asset to smooth consumption across time and states. By assuming complete markets, Lucas and Stokey disrupted Barro's analogy.[2]

This paper recasts the optimal taxation problem in an incomplete markets setting. By permitting only risk-free government borrowing, we revitalize parts of Barro's consumption-smoothing analogy. Work after Barro (summarized and extended by Chamberlain and Wilson (2000)) has taught us much about the consumption-smoothing model. Under some restrictions on preferences and the quantities of risk-free claims that the government can own, the consumption-smoothing model allows us to reaffirm Barro's random walk characterization of optimal taxation. But dropping the restriction on government asset holdings or modifying preferences causes the results to diverge in important ways from Barro's.

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[1] Hansen, Roberds, and Sargent (1991) describe the testable implications of various models like Barro's.

[2] We have heard V.V. Chari and Nancy Stokey conjecture that results closer to Barro's would emerge in a model that eliminates complete markets and permits only risk-free borrowing. An impediment to evaluating this conjecture has been that the optimal taxation problem with only risk-free borrowing is difficult because complicated additional constraints restrict competitive allocations (see Chari, Christiano, and Kehoe (1995, p.366)).

Our interest in formalizing the conjecture about Barro's model originates partly from historical episodes that pit Barro's model against Lucas and Stokey's. For example, see the descriptions of French and British 18th century public finance cited in Sargent and Velde (1995). Time series graphs of Great Britain's debt resemble a martingale with drift and are much smoother than graphs of government expenditures, which show large temporary increases associated with wars. Barro's model implies behavior like those graphs while Lucas and Stokey's model does not.[3] Our adaptation of Lucas and Stokey's model to rule out state-contingent debt is capable of generating behavior like Britain's. Section 6 illustrates this claim by displaying impulse responses to government expenditure innovations for both Lucas and Stokey's original model and our modification of it.

The remainder of this paper is organized as follows. Section 2 describes our basic model. It modifies Lucas and Stokey's environment by having the government buy and sell only risk-free one period debt. We show how confining the government to risk-free borrowing retains all of Lucas and Stokey's restrictions on an equilibrium allocation, and also adds a *sequence* of restrictions. These restrictions emanate from the requirement that the government's debt be risk-free. We formulate a Lagrangian for the Ramsey problem and show how the additional constraints introduce two new state variables: the government debt level and a variable dependent on past Lagrange multipliers. The addition of these state variables to Lucas and Stokey's model allows taxes and government debt to behave more like they do in Barro's model. First order conditions associated with the saddle point of this Lagrangian form a system of expectational difference equations whose solution determines the Ramsey outcome under incomplete markets. These equations are difficult to solve in general. Therefore, section 3 analyzes a special case with utility linear in consumption, but concave in leisure. This specification comes as close as possible to fulfilling Barro's intuition, but requires additional restrictions on the government expenditure process and the government debt in order fully to align with Barro's conclusions. In particular, we show that if the government's *asset* level is not restricted, the Ramsey plan under incomplete markets will eventually set the tax rate to zero and finance all expenditures from a war chest.[4] However, if we arbitrarily put a binding upper limit on the government's asset level, the Ramsey plan will set taxes and government debt exactly as Barro suggested.

Without the binding upper bound on government assets, in the example of section 3, the multiplier determining the tax rate converges. Section 4 introduces another example, one with an absorbing state for government expenditures, in which that multiplier also converges, but now to a nonzero value for the multiplier, implying a positive tax rate. Sections 4 and 5 then study analytically how general is the result that the multiplier determining the tax rate converges. Together these sections show that the result is not true for general preferences and specifications of the government expenditure process.

[3] Perhaps Lucas and Stokey's model does better at explaining France's behavior, with its recurrent defaults, which might be interpreted as occasionally low state-contingent payoffs.

[4] See the first section of David Hume (1777), where Hume commends such a policy.

Section 4 studies how far the martingale convergence approach used in the consumption-smoothing literature can take us. Section 5 takes a more direct approach to studying the limiting behavior of that multiplier in general versions of our model. Under a condition that the government expenditure process remains sufficiently random we show that, in general, the multiplier will not converge to a non-zero value, meaning that the allocation cannot converge to that for a complete market Ramsey equilibrium. That result establishes the sense in which the previous examples are both special. Section 6 briefly describes linear impulse response functions of numerically approximated equilibrium allocations. The computed example shows tax rates that combine a feature of Barro (by displaying a unit-root component) with those of the Lucas and Stokey's Ramsey plan (with higher dependence of taxes on current shocks).

Throughout this paper, we assume that the government binds itself to the Ramsey plan. Therefore, we say nothing about Lucas and Stokey's discussions of time consistency and the structure of government debt.

2. The economy

Technology and preferences are those specified by Lucas and Stokey. Let c_t, x_t, g_t denote consumption, leisure, and government purchases at time t. The technology is

$$c_t + x_t + g_t = 1. \tag{1}$$

Government purchases g_t follow a Markov process, with transition density $P(g'|g)$ and initial distribution π. We assume that (P, π) is such that $g \in [g_{\min}, g_{\max}]$. Except for some special examples, we also assume that P has a unique invariant distribution with full support $[g_{\min}, g_{\max}]$.

A representative household ranks consumption streams according to

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, x_t), \tag{2}$$

where $\beta \in (0,1)$, and E_0 denotes the mathematical expectation conditioned on time 0 information.

The government raises all revenues through a time-varying flat rate tax τ_t on labor at time t. Households and the government make decisions whose time t components are functions of the history of government expenditures $g^t = (g_t, g_{t-1}, \ldots, g_0)$, and of initial government indebtedness b^g_{-1}.

Incomplete markets with debt limits

Let $\omega_t \equiv \tau_t(1-x_t)-g_t$ denote the time t net-of-interest government surplus. Households and the government borrow and lend only in the form of risk-free one-period debt. The government's budget and debt limit constraints are:

$$b^g_{t-1} \geq \omega_t + p^b_t b^g_t, \quad t \geq 0 \tag{3}$$

$$\underline{M} \leq b^g_t \leq \overline{M}, \quad t \geq 0. \tag{4}$$

Here p^b_t is the price in units of time-t consumption of a risk-free bond paying one unit of consumption in period $t+1$ for sure; b^g_t represents the number of units of time $t+1$ consumption that at time t the government promises to deliver. When (3) holds with strict inequality, we let the left side minus the right side be a nonnegative level of lump sum transfers T_t to the household. The upper and lower debt limits $\overline{M}, \underline{M}$ in (4) influence the optimal government plan. We discuss alternative possible settings for $\overline{M}, \underline{M}$ below.

The household's problem is to choose stochastic processes $\{c_t, b^g_t\}_{t=0}^{\infty}$ to maximize (2) subject to the sequence of budget constraints

$$p^b_t b^g_t + c_t \leq (1-\tau_t)(1-x_t) + b^g_{t-1} + T_t, \ t \geq 0, \tag{5}$$

where b^g_t here denotes the household's holdings of government debt. The household also faces debt limits analogous to (4), which we assume are less stringent (in both directions) than those faced by the government. Therefore, in equilibrium, the household's problem always has an interior solution. The household's first-order conditions require that the price of risk-free debt satisfies

$$p^b_t = E_t \beta \frac{u_{c,t+1}}{u_{c,t}}, \forall t \geq 0, \tag{6}$$

and that taxes satisfy

$$\frac{u_{x,t}}{u_{c,t}(1-\tau_t)} = 1. \tag{7}$$

Debt limits

By analogy with Chamberlain and Wilson's (2000) and Aiyagari's (1994) analyses of a household savings problem, we shall analyze two kinds of debt limits, called 'natural' and 'ad hoc'. Natural debt limits come from taking seriously the risk-free status of government debt and finding the maximum debt that could be repaid almost surely under an optimal tax policy. We call a debt or asset limit ad hoc if it is more stringent than a natural one.

In our model, the natural asset and debt limits are difficult to compute in general. We compute and discuss them for an important special case in section 3.

Definitions

We use the following definitions.

DEFINITIONS 1: A feasible *allocation* is a stochastic process $\{c_t, x_t, g_t\}$ satisfying (1). A *bond price process* is a stochastic process $\{p_t^b\}$ whose time t element is measurable with respect to $\{g^t, b_{-1}^g\}$. Given b_{-1}^g and a stochastic process $\{g_t\}$, a *government policy* is a stochastic process for $\{\tau_t, b_t^g\}$ whose time t element is measurable with respect to (g^t, b_{-1}^g).

DEFINITION 2: Given b_{-1}^g and a stochastic process $\{g_t\}$, a *competitive equilibrium* is an allocation, a government policy, and a bond price process that solves the household's optimization problem and that satisfies the government's budget constraints (3) and (4).

Because we have made enough assumptions to guarantee an interior solution of the consumer's problem, a competitive equilibrium is fully characterized by (1), (3), (4), (7), (6).

DEFINITION 3: The *Ramsey problem* is to maximize (2) over competitive equilibria. A *Ramsey outcome* is a competitive equilibrium that attains the maximum of (2).

We use a standard strategy of casting the Ramsey problem in terms of a constrained choice of allocation. We use (6) and (7) to eliminate asset prices and taxes from the government's budget and debt constraints, and thereby deduce sequences of restrictions on the government's allocation in *any* competitive equilibrium with incomplete markets. Lucas and Stokey showed that under complete markets, competitive equilibrium imposes a single intertemporal constraint on allocations. We shall show that with incomplete markets competitive equilibrium allocations must satisfy that one restriction from Lucas and Stokey, and also others that impose that the government only purchase or sell risk-free debt.

From now on, we use (7) to represent the government surplus in terms of the allocation as $\omega_t \equiv \omega(c_t, g_t) = (1 - u_{x,t}/u_{c,t})(c_t + g_t) - g_t$. The following proposition characterizes the restrictions that the government's budget and behavior of households place on competitive equilibrium allocations:[5]

[5] This proposition extends remarks of Chari, Christiano, and Kehoe (1995, p. 366), by reformulating the measurability constraint from a 'difference equation' to an 'isoperimetric' form.

PROPOSITION 1: Take the case $T_t = 0$.[6] Given b^g_{-1}, a stochastic process for $\{c_t, g_t, x_t\}$ is a competitive equilibrium if and only if the following constraints are satisfied:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \omega_t = b^g_{-1} \tag{8}$$

$$\underline{M} \leq E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} \leq \overline{M} \quad \forall t \geq 0, \forall g^t \in [g_{\min}, g_{\max}]^{t+1} \tag{9}$$

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} \text{ is measurable with respect to } g^{t-1} \quad \forall t \geq 0, \forall g^t \in [g_{\min}, g_{\max}]^{t+1}. \tag{10}$$

PROOF: We relegate the proof to appendix A.

In the complete markets setting of Lucas and Stokey, (8) is the *sole* 'implementability' condition that government budget balance and competitive household behavior impose on the equilibrium allocation. The incomplete markets setup leaves this restriction intact, but adds three *sequences* of constraints. Constraint (10) requires that the allocation be such that at each date $t \geq 0$, $B_t \equiv E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j}$, the present value of the surplus (evaluated at date t Arrow-Debreu prices), be known one period ahead.[7,8] Condition (9) requires that the debt limits be respected. Condition (8) is but the time 0 version of constraint (10).

We approach the task of characterizing the Ramsey allocation by composing a Lagrangian for the Ramsey problem, and attaching a Lagrange multiplier to each measurability constraint.[9] We use the convention that variables dated t are measurable with respect to the history of shocks up to t. We attach stochastic processes $\{\nu_{1t}, \nu_{2t}\}_{t=0}^{\infty}$ of

[6] We take the case of zero lump sum transfers for simplicity. It is trivial to introduce lump sum transfers.

[7] There is a parallel 'constraint' in the complete markets case, since B_t needs to be measurable with respect to g^t in that case. But this constraint is trivially satisfied by the definition of $E_t(\cdot)$.

[8] This proposition is reminiscent of Duffie and Shafer's (1985) characterization of incomplete markets equilibrium in terms of 'effective equilibria' that, relative to complete markets allocations, require next-period allocations to lie in subspaces determined by the menu of assets. In particular, see the argument leading to Proposition 1 in Duffie (1992, p. 216-217).

[9] The Ramsey problem with incomplete markets has been called a 'computationally difficult exercise' (Chari, Christiano, and Kehoe (1995, p. 366)) because imposing the sequence of measurability constraints (10) seems daunting. For a class of special examples sharing features with the one in section 3, Hansen, Roberds, and Sargent (1991) focus on the empirical implications of the measurability constraints (10).

Lagrange multipliers to the inequality constraints on the left and right of (9), respectively. We incorporate condition (10) by writing it as $b^g_{t-1} = E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j}$, multiplying it by u_{ct}, and attaching a Lagrange multiplier $\beta^t \gamma_t$ to the resulting time t component. Then the Lagrangian for the Ramsey problem can be represented, after applying the law of iterated expectations and Abel's summation formula (see Apostol (1975, p 194)), as

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t, 1 - c_t - g_t) - \psi_t u_{c,t}\omega_t + u_{c,t}(\nu_{1t}\overline{M} - \nu_{2t}\underline{M} + \gamma_t b^g_{t-1}) \Big\} \tag{11}$$

where

$$\psi_t = \psi_{t-1} + \nu_{1t} - \nu_{2t} + \gamma_t. \tag{12}$$

Here $\gamma_0 \leq 0$ with strict inequality except where government expenditures are identically zero. The multipliers $\psi_t \leq 0$ for $t \geq 0$, while $\psi_{-1} = 0$; γ_t can be either positive or negative. To see why $\gamma_0 < 0$, differentiate the Lagrangian with respect to b^g_{-1}, and notice that $u_{c,0}\gamma_0$ can be regarded as the effect on the welfare of the representative household of an increase in the present value of government purchases. The nonpositive random multiplier ψ_t measures the effect on the representative household's welfare of an increase in the present value of government expenditures from time t onward. The multiplier γ_t measures the marginal impact of news about the present value of government expenditures on the maximum utility attained by the planner.[10]

The Ramsey problem under complete markets can be interpreted as a special case in which $\gamma_{t+1} = \nu_{1t} = \nu_{2t} \equiv 0 \; \forall t \geq 0$, and γ_0 is the (scalar) multiplier on the time 0 present value government budget constraint: these specifications imply that $\psi_t = \psi_0 = \gamma_0$ for complete markets. Relative to the complete markets case, the incomplete markets case augments the Lagrangian with the appearances of $b^g_{t-1}, \gamma_t, \; \forall t \geq 1$, and $\underline{M}, \overline{M}$ in the Lagrangian, and the effects of $\gamma_t, \nu_{1t}, \nu_{2t}$ on ψ_t in (12).[11]

We want to investigate whether the additional constraints on the Ramsey allocation move us toward Barro's tax-smoothing outcome. For $t \geq 1$, the first-order condition with respect to c_t can be expressed as

$$u_{c,t} - u_{x,t} - \psi_t \kappa_t + (u_{cc,t} - u_{cx,t})(\nu_{1t}\overline{M} - \nu_{2t}\underline{M} + \gamma_t b^g_{t-1}) = 0, \tag{13}$$

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[10] The present value is evaluated at Arrow-Debreu prices for markets for markets that are reopened at time t after g_t is observed.

[11] The Ramsey problem is not recursive in the natural state variables (b^g_{t-1}, g_t): because future control variables appear in the measurability constraints, the optimal choice at time t is *not* a time invariant function of the natural state variables. Nevertheless, the Lagrangian in (11) and the constraint (12) suggest that a recursive formulation can be recovered if ψ_{t-1} is included in the state variables. Indeed, a 'recursive contracts' approach described in Appendix B can be used to show formally that the optimal choice at time t is a time invariant function of state variables $(\psi_{t-1}, b^g_{t-1}, g_t)$.

where[12]

$$\kappa_t = (u_{cc,t} - u_{cx,t})\omega_t + u_{ct}\omega_{c,t}. \tag{14}$$

It is useful to study this condition under both complete and incomplete markets.

Complete markets

With complete markets, $\nu_{1t} = \nu_{2t} = \gamma_{t+1} = 0 \ \forall t \geq 0$ causes (13) to collapse to

$$u_{c,t} - u_{x,t} - \gamma_0 \kappa_t = 0, \tag{15}$$

which is a version of Lucas and Stokey's condition (2.9) for $t \geq 1$. From its definition (14), κ_t depends on the level of government purchases only at t. Therefore, given the multiplier γ_0, (15) determines the allocation and associated tax rate τ_t as a time-invariant function of g_t only. Past g's do not affect today's allocation. The *sole* intertemporal link is through the requirement that γ_0 must take a value to satisfy the time 0 present value government budget constraint. Equation (15) implies that, to a linear approximation, τ_t and all other endogenous variables mirror the serial correlation properties of the g_t process. The 'tax-smoothing' that occurs in this complete markets model is 'across states' and is reflected in the diminished variability of tax rates and revenues relative to government purchases, but *not* in any propagation mechanism imparting more pronounced serial correlation to tax rates than to government purchases. Evidently, in the complete markets model, the government debt B_t also inherits its serial correlation properties entirely from g_t.

Incomplete markets

In the incomplete markets case, equation (12) suggests that ψ_t changes (permanently) each period because γ_t is non-zero in all periods. Being of either sign, γ_t causes ψ_t either to increase or to decrease permanently. This statement can be made formally by showing that, when debt limits don't bind, the multiplier ψ_t is a risk-adjusted martingale, imparting a unit-root component to the solution of (13). Taking the derivative of the Lagrangian with respect to b_t^g we get

$$E_t[u_{c,t+1}\gamma_{t+1}] = 0. \tag{16}$$

This implies that γ_t can be positive or negative, and that ψ_t can rise or fall in a stochastic steady state. Assuming that the debt constraints don't bind in equilibrium next period (formally, assuming that $\nu_{1,t+1}, \nu_{2,t+1} = 0$ with probability one conditional on information at t) and using (12) gives

$$\psi_t = (E_t[u_{c,t+1}])^{-1} E_t[u_{c,t+1}\psi_{t+1}]. \tag{17}$$

[12] In the definition of κ_t, it is understood that total differentiation of the function $u = u(c, 1-c-g)$ with respect to c is occurring. Evidently, $\kappa_t = (u_{ct} - u_{xt}) + c_t(u_{cc,t} - 2u_{cx,t} + u_{xx,t}) + g_t(u_{xx,t} - u_{cx,t})$.

Using the definition of conditional covariance, the above equation can be further decomposed as

$$\psi_t = E_t[\psi_{t+1}] + (E_t[u_{c,t+1}])^{-1}\mathrm{cov}_t[u_{c,t+1}, \psi_{t+1}].$$

Equation (13) shows that this approximate martingale result is not precisely Barro's: first because ψ is not a pure martingale, and second because (13) makes τ_t depend also on $\gamma_t b^g_{t-1}$, and so distorts the pure martingale outcome. In section 4, we pursue how much information can be extracted from (17).

Example 1: Serially uncorrelated government purchases

The case in which government expenditures are i.i.d. provides a good laboratory for bringing out the implications of prohibiting state-contingent debt. With complete markets, the one-period state contingent debt falling due at t satisfies $m_{t-1}(g_t) = B_t = \omega_t + \sum_{j=1}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j}$ where $m_{t-1}(g)$ means the quantity of claims purchased at $t-1$ contingent on $g_t = g$. With a serially independent g_t process, and since both consumption and ω are time-invariant functions of g_t, the expectation conditional on g_t equals an unconditional expectation, constant through time, implying

$$u_{c,t} m_{t-1}(g_t) = u_{c,t}\omega_t + \beta E u_c B, \tag{18}$$

where $E u_c B = \frac{E u_c \omega}{1-\beta}$. Equation (18) states that, measured in marginal utility units, the gross payoff on government debt equals a constant plus the time t surplus, which is serially uncorrelated. In marginal utility units, the time t value of the state contingent debt with which the government leaves every period is a constant, namely, $\beta E u_c B$. The one-period rate of return on this debt is high in states when the surplus ω_t is pushed up because g_t is low, and it is low in states when high government expenditures drive the surplus down. There is no propagation mechanism from government purchases to the value of debt with which the government leaves each period, which is constant.

With incomplete markets, the situation is very different. Government debt evolves according to

$$B_{t+1} = R_t[B_t - \omega_t], \tag{19}$$

where $R_t \equiv (p^b_t)^{-1}$ and B_{t+1} is denominated in units of time $t+1$ consumption goods. Since the gross real interest rate is a random variable exceeding one, this equation describes a propagation mechanism by which even a serially independent government surplus process ω_t would impart close to unit root behavior to the debt level. Of course, even with i.i.d. government expenditures, the absence of complete markets causes the surplus process itself to be serially correlated, as described above.[13]

[13] Two polar cases make the tax rate depend only on current g_t: the case of optimal taxes with complete markets and the case of a balanced budget (i.e., $b^g_t = 0$ and taxes set to balance the budget for all t).

Reason for examples

So far, we have shown that the optimal tax is determined jointly by g_t, b^g_{t-1}, and a state variable that resembles a martingale, namely ψ_t. Dependence on g_t induces effects like those found by Lucas and Stokey. Dependence on ψ_t impels a martingale component, like those found by Barro. It is impossible to determine which effect dominates at this level of generality. To learn more, we now restrict the curvature of the one period utility function to create a workable special example.

3. *An example affirming Barro*

In the Ramsey problem, the government simultaneously chooses taxes and manipulates intertemporal prices. Manipulating prices substantially complicates the problem, especially with incomplete markets. We can simplify by adopting a specification of preferences that eliminates the government's ability to manipulate prices.[14] This brings the model into the form of a consumption-smoothing model (e.g., Chamberlain and Wilson (2000) and Aiyagari (1994)) and allows us to adapt results for that model to the Ramsey problem. We shall establish a martingale result for tax rates under an arbitrary restriction on the level of risk-free assets that the government can *acquire*.

EXAMPLE 2: CONSTANT MARGINAL UTILITY OF CONSUMPTION

We assume that $u(c,x) = c + H(x)$, where H is an increasing, strictly concave, three times continuously differentiable function. We assume $H'(0) = \infty$ and $H'(1) < 1$ to guarantee that the first best has an interior solution for leisure, and $H'''(x)(1-x) > 2H''(x)$ for all $x \in (0,1)$ to guarantee existence of a unique maximum level of revenue.[15]

Making preferences linear in consumption ties down intertemporal prices. Then (6) and (7) become

$$p^b_t = \beta \tag{20a}$$
$$H'(x_t) = 1 - \tau_t. \tag{20b}$$

Equation (20a) makes the price system independent of the allocation.

These are polar cases because, in the first, the possibilities for intertemporal and inter-state smoothing of taxes are as large as possible, while in the second there is no room for any kind of tax smoothing. In both polar cases, the tax rate is a function of g_t only. Though it lies between these polar cases, our incomplete markets model, with assets that allow intertemporal but no inter-state insurance, imparts an additional martingale-like component to the dynamics.

[14] This example was suggested by Rao Aiyagari.

[15] The latter assumption is satisfied, for example, if $H''' > 0$.

Government revenue is $R(x) = (1 - H'(x))(1 - x)$ with derivatives

$$R'(x) = -H''(x)(1-x) - (1 - H'(x)) \tag{21}$$

$$R''(x) = -H'''(x)(1-x) + 2H''(x). \tag{22}$$

Our assumptions on H guarantee that $R'' < 0$. Hence R is strictly concave. Letting x_1 be the first best choice for leisure satisfying $H'(x_1) = 1$, we know that $x_1 < 1$. Since $R'(x_1) > 0$ and $R(x_1) = R(1) = 0$, strict concavity of R implies that there is a unique $x_2 \in (x_1, 1)$ that maximizes the revenue and satisfies $R'(x_2) = 0$. The government wants to confine x_t to the interval $[x_1, x_2]$. Concavity of R implies that R' is monotone and, therefore, that R is monotone increasing on $[x_1, x_2]$.

Natural debt limits

Aiyagari and others define an agent's 'natural debt limit' to be the maximum level of indebtedness for which the debt can be repaid almost surely, given the agent's income process. Here the natural debt limit for the government is evidently

$$\overline{M} = \frac{1}{1-\beta}(R(x_2) - g_{\max}).$$

To discover a natural *asset* limit, we write the government budget constraint with zero revenues and transfers at the maximum government expenditure level as

$$b^g_{t-1} = -g_{\max} + p^b b^g_t,$$

where $p^b = \beta$. Evidently the natural asset limit for the government is

$$\underline{M} = -\frac{g_{\max}}{1-\beta}.$$

The government has no use for more assets because it can finance all expenditures from interest on its assets even in the highest government expenditure state.

Imposing $c_t \geq 0$ gives a natural borrowing limit for the consumer

$$\bar{b}^c \leq \frac{H'(x_2)(1-x_2)}{1-\beta},$$

where the numerator is the lowest after-tax income of the household.

We assume that parameters are such that

$$\bar{b}^c > -\underline{M}.$$

Ramsey problem and an associated permanent income model

Under this specification, the Ramsey problem acquires the form of the consumption-smoothing problem. Because the revenue function is monotone on $[x_1, x_2]$, we can invert it to get the function $x = x(R)$ for $R \in \mathcal{R} \equiv [0, R(x_2)]$. The fact that $R'' < 0$ implies that $x'' > 0$. Since the term $1 - g_t$ is exogenous, it can be dropped from the objective of the government. We represent the government's one-period return function as $W(R) = -x(R) + H(x(R))$. Notice that $W(R)$ equals minus the dead weight loss from raising revenues R, and thus matches Barro's one-period return function.

With the above assumptions $W(R)$ is a twice continuously differentiable, strictly concave function on $\mathcal{R}$. To see this, note that

$$W'(R) = -x'(R) + H'(x(R))\, x'(R)$$

$$W''(R) = -(1 - H'(x(R)))\, x''(R) + H''(x(R))\, (x'(R))^2$$

Since $x''(R) > 0$ and since H is concave, the above formula for W'' implies that W is concave. Furthermore, $W(R)$ has a strict maximum at $R = 0$, associated with the first-best tax rate of $x_1 = 0$.

Then the Ramsey problem can be expressed

$$\max_{\{R_t, b_t^g\}} E_0 \sum_{t=0}^{\infty} \beta^t W(R_t) \tag{23}$$

subject to

$$b_t^g \leq \beta^{-1} \left[g_t + b_{t-1}^g - R_t\right] \tag{24a}$$

$$b_t^g \in [\underline{M}, \overline{M}]. \tag{24b}$$

We restrict revenues to be in $\mathcal{R}$ and the sequence of revenues to be in the infinite Cartesian product $\mathcal{R}^\infty$.

We can map our model into the consumption problem by letting R play the role of consumption, $W(R)$ the one-period utility function of the consumer, g_t exogenous labor income, and b_t^g the household's debt.[16], [17]

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[16] See Chamberlain and Wilson (2000), Aiyagari (1994), and their references for treatments of this problem. Hansen, Roberds, and Sargent (1991) pursue the analogy between the consumption- and tax-smoothing problems.

[17] Though it agrees in mathematical form with the consumption-smoothing model, there are differences of sign and location. Thus, higher g_t lowers the value of the problem, while higher labor or endowment income raises the value of the consumption problem. Also, bliss occurs at zero R, while the bliss level of consumption, if it exists, is positive.

As Chamberlain and Wilson (2000) describe, the solution of the consumption problem depends on the utility function, the relation of the interest rate to the discount factor, and whether there persists sufficient randomness in the income process. Problem (23), (24) corresponds to a special consumption problem with a finite bliss level of consumption and the gross interest rate times the discount factor identically equal to unity. For such a problem, if income remains sufficiently stochastic, then under the natural debt limits, consumption converges to bliss consumption and assets converge to a level sufficient to support that consumption.

As we shall see in the next subsection, there is a related result for the Ramsey plan under incomplete markets: tax revenues converge to zero and government assets converge to a level always sufficient to support government purchases from interest earnings alone, with lump sum transfers being used to dispose of surplus earnings. To sustain randomly fluctuating tax rates in the limit requires arresting such convergence. Putting a binding upper limit on assets prevents convergence, as we shall show by applying results from the previous section to the special utility function of this section.

Incomplete markets, 'natural asset limit'

For example 2, the definition of κ_t in (14) implies

$$\kappa_t = -R'(x_t) \leq 0 \quad \text{for } x_t \in [x_1, x_2]. \tag{25}$$

The variables (τ_t, x_t, ψ_t) are then determined by (12), which we repeat for convenience, and the following specialization of (13):

$$\psi_t = \psi_{t-1} + \nu_{1t} - \nu_{2t} + \gamma_t$$
$$\tau_t = 1 - H'(x_t) = -\psi_t R_t.$$

Under the natural asset limit and the ability to make positive lump sum transfers, $\nu_{2t} \equiv 0$. Then (12), $u_{ct} = 1$ and (16) imply that

$$E_{t-1}\psi_t \geq \psi_{t-1}. \tag{26}$$

Inequality (26) asserts that the nonpositive stochastic process ψ_t is a submartingale. It is bounded above by zero. Therefore, the submartingale convergence theorem (see Loeve (1977))) asserts that ψ_t converges almost surely to a nonpositive random variable. There are two possibilities:

1. If the Markov process for g has a unique nontrivial invariant distribution, then our Lemma 3 below shows that ψ_t converges almost surely to zero. In that case, (13) implies that τ_t converges to the first-best tax rate $\tau_t = 0$, and leisure converges to the first best x_1. The level of government assets converges to the level $\frac{g_{\max}}{1-\beta}$ sufficient to

finance $g_{\max}$ from interest earnings. Transfers are eventually zero when $g_t = g_{\max}$ but positive when $g_t < g_{\max}$.

2. If the Markov process for g has an absorbing state, then ψ_t can converge to a strictly negative value; ψ converges when g_t enters the absorbing state. From then on, taxes and all other variables in the model are constant.

Barro's result under an ad hoc asset limit

Thus, under the natural asset limit, this example nearly sustains Barro's martingale characterization for the tax rate, since ψ_t is a submartingale and taxes are a function of ψ_t. But the government accumulates assets and, in the limit, the allocation is first best and taxes are zero. We now show that imposing an ad hoc asset limit makes outcomes align with Barro's even in the limit as t grows, at least away from corners. When $\underline{M} > -\frac{g_{\max}}{(1-\beta)}$, the lower limit on debt occasionally binds. This puts a non-negative multiplier ν_{2t} in (12) and invalidates the submartingale implication (26). This markedly alters the limiting behavior of the model in the case that the Markov process for g_t has a unique invariant distribution. In particular, rather than converging almost surely, ψ_t can continue to fluctuate randomly if randomness in g persists sufficiently. Off corners (i.e., $\nu_{2t} = \nu_{1t} = 0$), ψ_t fluctuates like a martingale. But on the corners, it will not. If we impose time-invariant ad hoc debt limits $\underline{M}, \overline{M}$, the distribution of government debt will have a nontrivial distribution with randomness that does not disappear even in the limit. Also, ψ will have the following type of 'inward pointing' behavior at the boundaries.[18] If government assets are at the lower bound and $g_{t+1} = g_{\max}$, then taxes are set at x_2 and government assets stay at the lower bound. If $g_{t+1} < g_{\max}$, then taxes will be lower and government assets will drift up. If government assets are at the upper bound and $g_{t+1} = g_{\min}$, then just enough taxes are collected to keep assets at the upper bound; while if $g_{t+1} > g_{\min}$, then assets will drift downward.

Complete markets: constant tax rates

For comparison, it is useful to describe what the allocation and taxes would be under complete markets in Example 2. In the complete markets case, restrictions (24) are replaced by the following version of Lucas and Stokey's single implementability constraint:

$$b_{-1}^g = E_0 \sum_{t=0}^{\infty} \beta^t (R_t - g_t). \tag{27}$$

The policy that maximizes (23) subject to (27) sets revenues and tax rates equal to constants, and transfers to zero. This can be shown directly, but it is instructive to show it

[18] Rao Aiyagari pointed out this pattern to us in detail.

simply by applying the results of the previous section. Then equations (25) and (15) imply

$$\tau_t = 1 - H'(x_t) = -\gamma_0 R'(x_t). \tag{28}$$

Recall that $R'(x) \geq 0$ for $x \in [x_1, x_2)$ and that $\gamma_0 \leq 0$. The restrictions on $R(x)$ on $[x_1, x_2]$ derived above imply that there is a unique $x_t = x^{CM}$ that solves (28). Thus, under complete markets the tax rate and leisure are constant over time and across states.

Example 2 ties down u_{ct} by assuming linear utility. The next two sections study whether taxes can be expected to converge under more general utility specifications.

4. Non convergence of ψ_t

Example 2 showed how a submartingale property under the natural debt and asset limits guaranteed that ψ converges a.s. Furthermore, in that case, the limit would often be zero.

In this section we explore whether it is possible to obtain a general result about convergence by exploiting the martingale property of ψ_t. We study the interaction of the convergence of ψ_t and $u_{c,t}$ under more general preferences. We will show that if we can determine the asymptotic behavior of the predictability of $u_{c,t}$, then we can also show convergence of ψ_t. We proceed to ask whether ψ_t can converge when $u_{c,t}$ does not. We show that, in general, if $u_{c,t}$ does not converge, as happens in most models, then we can say very little about convergence of ψ_t.

We already argued that if the debt limits can bind, then ψ_t should not be expected to converge. Throughout this section we assume that the natural debt and asset limits are imposed, so that the asset and debt limits never bind and (16) implies

$$\psi_t \leq E_t \left(\frac{u_{c,t+1}}{E_t(u_{c,t+1})} \; \psi_{t+1} \right). \tag{29}$$

We also assume throughout this section that $u_c(c, x) > 0$ for all feasible c, x.

Using terminology common in finance, (29) and the fact that $E_t \left(\frac{u_{c,t+1}}{E_t(u_{c,t+1})} \right) = 1$ makes ψ a 'risk-adjusted (sub)martingale'. Risk adjusted martingales converge under suitable conditions. One strategy to prove convergence involves finding an equivalent measure that satisfies a particular boundary condition.[19] We follow a related approach of Chamberlain and Wilson (2000), and give an example where the required boundary condition is satisfied. Unfortunately, we will also show that the standard boundary conditions are violated in the general case.

[19] See, for example, Duffie (1996, chapter 4).

Submartingale convergence

We begin with what seems like an encouraging result. Let $\theta_t \equiv \prod_{\tau=1}^{t} \frac{u_{c,\tau}}{E_{\tau-1}(u_{c,\tau})}$.

LEMMA 1: $\{\theta_t \psi_t\}$ is a submartingale. Therefore, it converges a.s. to a random variable $\overline{\theta\psi}$ that is finite with probability one.

PROOF: By assumption, the debt limits are never binding and (29) holds for all periods with probability one. Since $\theta_t \geq 0$, multiplying both sides of (29) by θ_t, we have

$$\theta_t \psi_t \leq E_t(\theta_{t+1}\psi_{t+1}) \quad \text{a.s.} \tag{30}$$

Since $\theta_t \psi_t \leq 0$, this product converges a.s. to a finite variable by Theorem A, page 59 of Loeve (1977). ∎

Lemma 1 implies convergence of ψ_t only if we can say something about the asymptotic behavior of θ_t. In particular, if θ_t converges to a non-zero limit, then Lemma 1 allows us to conclude that ψ_t converges.[20] This can be guaranteed in an interesting special case:

EXAMPLE 3: ABSORBING STATES IMPLY ψ_t CONVERGES

Assume that $\{g_t\}$ has absorbing states in the sense that $g_t = g_{t-1}$ a.s. for t large enough, so that fluctuations cease and $u_{c,t} = E_{t-1}(u_{c,t}(\omega))$.

In this case, since $0 < u_{c,t} < \infty$, it is clear that θ_t converges to a positive number almost surely. Then Lemma 1 implies that $\psi_t \to \psi_\infty$ a.s. and, therefore, the tail of the allocation is a Ramsey equilibrium with complete markets. The limiting random variable ψ_∞, plays the role of Lucas and Stokey's single multiplier for that tail allocation. The value of ψ_∞ depends on the realization of the government expenditure path. One can state sufficient conditions to guarantee that there is a positive probability that the absorbing state is reached before the first best is attained, so that $\psi_\infty < 0$ with positive probability (for example, if the initial level of debt is sufficiently high and if there is a positive probability of reaching the absorbing state in one period). But even with an absorbing state, a Markov process (P, π) can put positive probability on an arbitrarily long sequence of random government expenditures that gives the government the time and incentive to accumulate enough assets to reach the first best.

Therefore, in example 3 taxes always converge. It is easy to construct examples in which there is a positive probability of converging to a Ramsey (Lucas and Stokey) equilibrium

[20] This is same the proof strategy of Chamberlain and Wilson (2000). Our Lemma 1 is analogous to Theorem 1 of Chamberlain and Wilson. However, in their model, θ_t is exogenous (it depends on the exogenous interest rate process), so that these authors can proceed by making alternative assumptions about the interest rate process that guarantee that θ_t converges either to a positive number or to infinity. We cannot make assumptions directly about c because this variable is determined within the model.

with non-zero taxes and also a positive probability of converging to the first best where taxes are zero.

But if θ_t converges to zero, Lemma 1 cannot be used to conclude anything about convergence of ψ_t.[21] So our next task is to say something about the asymptotic behavior of θ_t. The following lemma gives us a negative result because it says that if consumption remains sufficiently random, then θ converges to zero, thereby silencing Lemma 1.

LEMMA 2:

a) $\{\theta_t\}$ is a non-negative martingale. Therefore, $\theta_t \to \bar{\theta}$ a.s. for a random variable $\bar{\theta}$ that is finite with probability one.

b) for any given realization ω such that $\theta_t(\omega) \to \bar{\theta}(\omega) > 0$, $\frac{u_{c,t}(\omega)}{E_{t-1}(u_{c,t})(\omega)} \to 1$ as $t \to \infty$.

PROOF: To prove part a):

$$E_t(\theta_{t+1}) = \theta_t E_t\left(\frac{u_{c,t+1}}{E_t(u_{c,t+1})}\right) = \theta_t$$

To prove part b), if $\theta_t(\omega) \to \bar{\theta}(\omega) > 0$, then

$$\log \theta_t(\omega) = \sum_{\tau=1}^{t} [\log u_{c,\tau}(\omega) - \log E_{\tau-1}(u_{c,\tau}(\omega))] \to \log \bar{\theta}(\omega) > -\infty$$

as $t \to \infty$. Convergence of this sum implies $\log u_{c,t}(\omega) - \log E_{t-1}(u_{c,t}(\omega)) \to 0$ and $\frac{u_{c,t}(\omega)}{E_{t-1}(u_{c,t})(\omega)} \to 1$ as $t \to \infty$. ∎

Part b) of Lemma 2 implies that, if the marginal utility of consumption is sufficiently unpredictable as time passes, then θ_t converges to zero a.s. Unfortunately, we expect that c_t will generally remain random in many models of interest.

Lemma 1 becomes silent about both the limiting behavior of ψ_t and the Ramsey allocation under risk-free government debt. This is a negative conclusion, because it means that the martingale approach cannot be used in some important cases. For example, one possibility we would like to know about is whether (c_t, i_t) converges to a stationary distribution as time goes on. Another is whether, as in example 3, it is possible for the allocation to converge to one for a complete markets economy with some multiplier. The next corollary shows that Lemma 2 is silent about these possibilities.

COROLLARY 1: a) If the allocation converges to a stationary distribution with $u_{c,t} \neq E_{t-1}(u_{c,t})$, then $\theta_t \to 0$.

[21] Note that Chamberlain and Wilson do not have many results for the case where θ_t converges to zero.

b) If $u_{c,t}^{CM} \neq E_{t-1}(u_{c,t}^{CM})$ for the complete market Ramsey equilibria with any multiplier γ_0, then $\theta_t \to 0$.

PROOF:

a) In this case, $\frac{u_{c,t}}{E_{t-1}(u_{c,t})}$ does not converge to 1 a.s. and the statement is implied directly by the contrapositive of Lemma 2, b).

b) Consider a realization for which $\bar{\theta}(\omega) > 0$. Then Lemma 1 implies that $\psi_t(\omega)$ converges, and the model converges to a complete markets equilibrium. Hence marginal utility converges to some complete market Ramsey equilibrium, under the assumption stated in part b) $\frac{u_{c,t}}{E_{t-1}(u_{c,t})}$ does not converge to 1, and the statement is implied by the contrapositive of part b) of Lemma 2. ▮

Notice that the conditions of part b) Corollary 1 are satisfied if u has some curvature and g has persistent randomness. Example 2 is one where u has insufficient curvature. Example 3 has insufficient randomness in g.

This means that, in the general case, $\theta_t \to 0$, leaving us unable to use Lemma 1 to determine whether the allocation converges. Furthermore, this is also a case where the boundary conditions of the equivalent measure required in the literature on risk-adjusted martingales also fail, since the distribution of the equivalent measure is degenerate and equal to zero in the limit.[22]

5. Another non-convergence result

In section 4, we discovered that the martingale approach is often inconclusive about the asymptotic behavior of the equilibrium. However, in example 3 the incomplete markets Ramsey allocation and tax policy converge to their complete markets counterparts. In this section, we explore whether the convergence in example 3 can be extended to more general government expenditure processes. It cannot. By working directly with the government budget constraints, under general conditions on the government expenditure process, we rule out convergence to the Ramsey equilibrium under complete markets (to be called the Lucas-Stokey or LS equilibrium).

The budget constraint of the government without lump sum transfers and for any debt limits can be rewritten as

$$b_t^g - b_{t-1}^g = \left(\frac{1}{p_t^b} - 1\right)\left(\frac{g_t - \tau_t(1-x_t)}{1-p_t^b} + b_{t-1}^g\right). \tag{31}$$

Here $g_t - \tau_t(1-x_t)$ is the net-of-interest or 'primary deficit'. Let $D(f, g_t) \equiv \frac{g_t - \tau_t^f(1-x_t^f)}{1-p_t^{b,f}}$, where the f superscript denotes the LS equilibrium with a multiplier $\gamma_0 = f$.

[22] See Duffie (1996) for a precise description of the conditions that the equivalent measure approach requires.

DEFINITION: We say that $D(f, g_t)$ is *sufficiently random* if there exists an $\epsilon > 0$ such that for t large enough and any constant K either

$$P(\ D(f, g_j) > K + \epsilon \quad \text{for all } j = t, ...t + k \ \mid \ g_{t-1}, ...g_0) > 0 \tag{32}$$

or

$$P(\ D(f, g_j) < K - \epsilon \quad \text{for all } j = t, ...t + k \ \mid \ g_{t-1}, ...g_0) > 0 \tag{33}$$

for all $k > 0$ for almost all realizations.[23]

Clearly, g_t is *in*sufficiently random if $D(f, g_t)$ converges a.s.

Notice that examples of *in*sufficient randomness in the above sense occur in example 3, or cases where the complete markets solution implied a constant D. Also, it is easy to see that if g_t is stationary with positive conditional variance, D has sufficient randomness under most utility functions.

Notice that convergence of the incomplete markets allocation to the LS equilibrium requires that ψ_t converges to a non-zero value and that the multipliers ν of the debt limits become zero. The following lemma shows that if there is sufficient randomness in D, the incomplete markets allocation cannot converge to a LS allocation.

LEMMA 3: Assume that the interest rate is bounded away from zero with probability one. Also, assume that the first order conditions for optimality in the Ramsey problem (13) define a continuous function mapping $(\psi_t, \gamma_t, b^g_{t-1})$ to the endogenous variables (τ_t, x_t, p^b_t). Then

$$P\left(\omega : \psi_t(\omega) \to \varsigma(\omega)\right) \neq 0 \text{ as } t \to \infty$$
$$\text{and } D(\varsigma(\omega), g_t) \text{ sufficiently random}) = 0$$

Furthermore, for a particular realization where $\psi_t(\omega) \to \varsigma(\omega)$, we have $b^g_t(\omega) \to D(\varsigma(\omega), g_t)$.

PROOF: Consider a realization ω such that $\psi_t(\omega) \to \varsigma(\omega) \neq 0$. In this case, $(\psi_t - \psi_{t-1})(\omega) \to 0$ and (13) implies that (τ_t, x_t, p^b_t) converge to the LS equilibrium with Lagrange multiplier $\varsigma(\omega)$, and $\left|\frac{g_t - \tau_t(1-x_t)}{1-p^b_t}(\omega) - D(\varsigma(\omega), g_t)\right| \to 0$.

Now if $D(\varsigma(\omega), g_t)$ is sufficiently random, there is an $\epsilon > 0$ (possibly dependent on $\varsigma(\omega)$) as in the definition of sufficient randomness.

Since the endogenous variables converge to the LS equilibrium with Lagrange multiplier $\varsigma(\omega)$, there is a t such that for all $\bar{t} \geq t$ we have $\left|\frac{g_{\bar{t}} - \tau_{\bar{t}}(1-x_{\bar{t}})}{1-p^b_{\bar{t}}}(\omega) - D(\varsigma(\omega), g_{\bar{t}}(\omega))\right| < \frac{\epsilon}{2}$. Now if $D(\varsigma(\omega), g_t)$ is sufficiently random, either (32) or (33) is satisfied. Let's say that

[23] Notice that ϵ can depend on f, the t 'large enough' can depend on ϵ, f, but these have to be uniform on K and k.

for $K = -b^g_{t-1}$ it is (32) that occurs. Using equation (31) we have that with positive probability

$$b^g_{\bar{t}} - b^g_{\bar{t}-1} > \left(\frac{1}{p^b_{\bar{t}}} - 1\right)\left(D(\varsigma(\omega), g_{\bar{t}}(\omega)) - \epsilon/2 + b^g_{\bar{t}-1}\right)$$

$$> \left(\frac{1}{p^b_{\bar{t}}} - 1\right)\left(\epsilon - b^g_{t-1} - \epsilon/2 + b^g_{\bar{t}-1}\right)$$

for all $\bar{t} \geq t$, where the first inequality follows from convergence to the LS equilibrium and the second inequality from equation (32) for $K = -b^g_{t-1}$. This equation for $\bar{t} = t$ implies that $b^g_t - b^g_{t-1} > 0$ so that, by induction, $-b^g_{t-1} + b^g_{\bar{t}-1} > 0$ and

$$b^g_{\bar{t}} - b^g_{\bar{t}-1} > \left(\frac{1}{p^b_{\bar{t}}} - 1\right)\epsilon/2$$

for all $\bar{t} \geq t$. Since $\frac{1}{p^b_{\bar{t}}}$ is larger than, and bounded away from, 1 this equation implies that the debt grows without bound and that the upper bound of debt would be violated with positive conditional probability. Similarly, if we had (33) holding for $K = -b^g_{t-1}$ we would have the lower bound on debt being violated. Therefore, with sufficient randomness of D, it is impossible for the allocation to converge to a LS allocation. ∎

Summary

In general, with sufficient randomness we can rule out the example 3 outcome that the Ramsey allocation with only risk-free debt converges to a Ramsey allocation with state contingent debt. But at least two possibilities remain: ψ_t may have a non-degenerate distribution in the limit or it may converge to the first best, as in example 2 under the natural asset limit.

6. A numerical example

Sections 4 and 5 tell why it is generally difficult to characterize the Ramsey allocation for the incomplete markets economy for more general preferences than those for example 2. It is reasonable to emerge from sections 3, 4, and 5 with the prejudice that in the general case the allocation would exhibit behavior somehow between those of examples 2 and 3. The results in this section support that prejudice by presenting approximate Ramsey plans for both complete and incomplete markets economies with a serially independent government purchase process.

From the point of view of someone used to solving dynamic programming problems by discretizing the state space and iterating on the Bellman equation, obtaining numerical solutions of this model seems daunting. First of all, the solution is time-inconsistent, so that the policy function (as a function of the states g_t) changes every period. Second, there are several endogenous continuous state variables, so that discretization is very costly computationally, and second order approximations are likely to be inexact. We approach the first issue by using the framework of recursive contracts to characterize the (time inconsistent) optimal solution by a recursive dynamic Lagrangean problem with few state variables. As we argued in section 2, a sufficient set of state variables is $(g_t, b^g_{t-1}, \psi_{t-1})$. Then we can solve the first order conditions by numerically approximating the law of motion with some continuous flexible functional form.[24]

Parameters

We rescaled the feasibility constraint so that $c_t + x_t + g_t = 100$, and set government purchases to have mean 30. The stochastic process for g_t is

$$g_{t+1} = \overline{g} + \frac{\epsilon_{t+1}}{\alpha},$$

where ϵ_t is an independently and identically distributed sequence distributed $\mathcal{N}(0,1)$, and α is a scale factor. Our utility function is

$$u(c,x) = \frac{c^{1-\sigma_1} - 1}{1-\sigma_1} + \eta \left(\frac{x^{1-\sigma_2} - 1}{1-\sigma_2} \right). \tag{34}$$

We set $(\beta, \sigma_1, \sigma_2, \eta) = (.95, .5, 2, 1)$ and $(\overline{g}, \alpha, b^g_{-1}) = (30, .4, 0)$, and $(\underline{M}, \overline{M}) = (-1000, 1000)$.[25]

For the complete markets Ramsey plan, Figure 1 displays linear impulse response functions to the innovation in government expenditures. The impulse responses confirm that every variable of interest inherits the serial correlation pattern of government purchases. We can estimate the variance of each variable by squaring the coefficient at zero lag, then multiplying by the innovation variance of g_t. Notice that the tax rate τ_t has very low variance, as indicated by its low zero-lag coefficient of about 7×10^{-4}. These impulse response functions tell us how extensively the government relies on the proceeds of the 'insurance' it has purchased from the private sector. In particular, the net-of-interest deficit is about 93 percent of the innovation to government purchases. The deficit is covered by state-contingent payments from the private sector.

[24] See Marcet, Sargent, and Seppälä (1995) for a description of these and other calculations. To approximate a solution, we apply the parameterized expectations algorithm of Marcet (1988). This approach is convenient since it avoids discretization of the state variables, and in our problem we have at least two endogenous continuous state variables. A number of other approaches to solve this kind of first order conditions are also available in the literature.

[25] Various computational details are described in depth in an earlier draft of this paper available at `ftp://zia.stanford.edu/pub/sargent/webdocs/research/albert8.ps`.

Figure 2 displays linear impulse responses for the incomplete markets economy. The impulse response function for b_t^g shows what a good approximation it is to assert, as Barro did, that an innovation in government expenditures induces a permanent increase in debt. This contrasts sharply with the pattern under complete markets with serially independent g_t, for which an innovation in government expenditures has *no* effect on the present value of debt passed into future periods. Figure 2 shows that ψ_t is well approximated by a martingale. The impulse response functions for the tax rate τ_t and tax revenues deviate from the 'random walk' predicted by Barro mainly in their first-period responses. (A random walk would have perfectly flat impulse response function.) These impulse response functions resemble a weighted sum of the random walk response predicted by Barro and the white noise response predicted by Lucas and Stokey.[26]

Notice that the lag zero impulse coefficient for the tax rate is about 1/4 higher than for the complete markets case, so that the one-step ahead prediction error variance is correspondingly higher. Because of the near-unit root behavior of the tax rate under incomplete markets, the j-step ahead prediction error variance grows steadily with j, at least for a long while. The unconditional variance of tax rates under incomplete markets is therefore much higher than under complete markets.

Welfare comparison

Despite differences of behaviors for taxes, surpluses, and debts, the impulse response functions for consumption and leisure, respectively, in the complete and incomplete market economies are very close. The proximity of the impulse response functions for (c_t, x_t) implies proximity of the Ramsey allocations in the two economies. This is confirmed by some welfare calculations. We calculated the expected utility of the household to be 298.80 in the complete markets economy and 298.79 in the incomplete markets economy.[27] This comparison indicates the capacity of tax-smoothing over time to substitute for tax-smoothing across states.

7. Concluding remarks

Lucas and Stokey (1983, p. 77) drew three lessons: (1) Budget balance in a present value sense must be respected;[28] (2) No case can be made for budget balance on a continual basis; (3) State-contingent debt is an important feature of an optimal policy under complete

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[26] The impulse response functions for tax rates and for tax revenues reveal that these variables are well approximated as univariate processes whose first differences are first order moving averages.

[27] For pairs of similar economies except with first order autoregressive government expenditures with a.r. coefficient $\rho = .75$ and the same values of the other parameters, we calculated expected utilities 299.04 and 298.97.

[28] According to Keynes, 'What the government spends, the public pays for.'

markets.[29] Our results support 1, amplify 2, but may qualify 3. At least for our computed example, the welfare achieved by the incomplete markets Ramsey allocation is close to the complete markets Ramsey allocation, testimony to the efficacy of the incomplete market Ramsey policy's use of 'self-insurance.' Here, government uses debt as a buffer stock, just as savings allow smooth consumption in the 'savings problem'. For a general equilibrium version of a model whose residents all face versions of the savings problem, Krusell and Smith (1998) display incomplete markets allocations close to ones under complete markets.

The analogy to the literature on the savings problem leads us to suspect that for the optimal tax problem, the proximity of Ramsey allocations under complete and incomplete market structures will depend sensitively on (a) the persistence of the government expenditure process and the volatility of innovations to it, (b) the curvature of the household's utility function, and (c) the debt and asset limits set for the government.[30] More serially dependent government expenditure processes are more difficult for a government to self-insure, and will strengthen Lucas and Stokey's lesson 3. Our numerical example has modest curvature of the utility function, and no serial correlation of the government expenditure process, weakening that lesson.

In affirming Barro's characterization of tax-smoothing as imparting near-unit root components to tax rates and government debt, our incomplete markets model enlivens a view of 18th century British fiscal outcomes as Ramsey outcomes. The time series of debt service and government expenditure for 18th century Britain resemble a simulation of Barro's model or ours, not a complete markets model.[31]

Appendix A

PROOF OF PROPOSITION 1: First we show that the constraints (3), (4), and (6) imply (9) and (10). From (3) and the household's first-order conditions with respect to bonds we have

$$\omega_t + \beta\, E_t \left(\frac{u_{c,t+1}}{u_{c,t}}\; b^g_t \right) = b^g_{t-1}.$$

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[29] Lucas and Stokey write: "... even those most skeptical about the efficacy of actual government policy may be led to wonder why governments forego gains in everyone's welfare by issuing only debt that purports to be a *certain* claim on future goods." Our computed calculations do not diminish the relevance of this statement as a comment about the role of state-contingent debt in making possible a *debt structure* that renders their Ramsey tax policy time-consistent.

[30] Thus, for their settings of other parameters, Krusell and Smith's allocations under complete and incomplete markets would be brought even closer together if they replaced the no-borrowing constraint they impose with the natural debt limits.

[31] See Figure 2 of Sargent and Velde (1995).

Using forward substitution on b_t^g and also the law of iterated expectations, we have

$$E_t \sum_{j=0}^{T-1} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} + \beta^T \; E_t \left(\frac{u_{c,t+T}}{u_{c,t}} \; b_{t+T-1}^g \right) = b_{t-1}^g,$$

for all T, which implies

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} = b_{t-1}^g,$$

Since according to Definition I, b_{t-1}^g is known at $t-1$ and (4) is satisfied, the last equation implies that (9) and (10) are satisfied.

To prove the reverse implication, we have

$$\begin{aligned} B_t &\equiv \omega_t + E_t \sum_{j=1}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} \\ &= \omega_t + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} \frac{u_{c,t+1}}{u_{c,t}} \; \frac{u_{c,t+j}}{u_{c,t+1}} \omega_{t+j}. \end{aligned} \tag{35}$$

Applying the law of iterated expectations, we can condition the term inside E_t on information at $t+1$ to get

$$\begin{aligned} &\omega_t + \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} \; E_{t+1} \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+1+j}}{u_{c,t+1}} \omega_{t+j+1} \right] \\ &= \omega_t + \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} \; B_{t+1} \right] = \omega_t + \beta E_t \left(\frac{u_{c,t+1}}{u_{c,t}} \right) B_{t+1}, \end{aligned}$$

using (10) in the last line. With formula (6) for bond prices we have:

$$B_t = \omega_t + p_t^b \; B_{t+1},$$

which guarantees that (3) and (4) are satisfied precisely for $b_{t-1}^g = B_t$. ∎

Appendix B: Recursive saddle point formulation

According to Definition III, the Ramsey problem under incomplete markets is to maximize the utility of the household subject to measurability conditions expressed as

$$E_t \sum_{j=0}^{\infty} \beta^j \; u_{c,t+j}\omega_{t+j} = u_{c,t}\, b^g_{t-1} \,, \; \forall t \geq 0 \tag{36}$$

with b^g_{-1} given. The presence of future choice variables in constraints (36) makes the problem non-recursive in terms of the natural state variables and renders the solution time-inconsistent. Further, the solution is *not* of the form $c_t = f(g_t, b^g_{t-1})$; rather, the policy functions are time-dependent and may have the whole past history of g's as arguments. Finding a solution for taxes in terms of past g's is, therefore, demanding.

In this appendix, we re-formulate the problem with an eye to recovering a recursive structure and facilitating computation. We use the apparatus of Marcet and Marimon (1996) (M&M)[32], and keep our notation close to theirs.

Consider the following problem

Program 1

$$\sup_{\{X_t\}} \; E_0 \sum_{t=0}^{\infty} \beta^t \; U(X_t, X_{t+1}, S_t)$$

subject to

$$\mathcal{T}(X_t,\, X_{t+1},\, S_t) \geq 0, \; t \geq 0; \; X_0 = \overline{X} \tag{37}$$

$$E_t \sum_{j=0}^{\infty} \beta^j \; \mathcal{V}(S_{t+j}, X_{t+j+1}) \geq \Phi(X_t, X_{t+1}, S_t)\,, \quad \forall t \geq 0 \tag{38}$$

$$X_{t+1} \text{ measurable with respect to } \quad (S_0 \ldots, S_{t-1}, S_t). \tag{39}$$

Here, the initial values S_0, $\overline{X}$, the constant β, and the mappings $u, \mathcal{V}, \Phi$ and $\mathcal{T}$ are given, and $\{S_t\}$ is a stochastic Markov process.

The approach uses three steps:

Step 1:

Show that solving *Program 1* is equivalent with solving

Program 2

$$\inf_{\{\Gamma_t, \mu_t\}} \sup_{\{X_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \; \left(U(X_t, X_{t+1}, S_t) + \mu_{t+1} \; \mathcal{V}(S_t, X_t, X_{t+1}) \; - \; \Gamma_{t+1} \Phi(X_t, X_{t+1}, S_t)\right)$$

[32] There are several approaches in the literature that can be used to find a recursive structure in models where the Bellman equation is not satisfied. Nevertheless, they are not applicable without modifications to our problem: the approach of Kydland and Prescott (1980) does not incorporate uncertainty; the recursive formulation of Abreu, Pierce and Stachetti (1990) only provides necessary conditions for an optimum, but is not designed for solving Ramsey problems. The M&M approach is closely related to that used for the linear-quadratic case by Hansen, Epple, and Roberds (1985).

subject to

$$\mu_{t+1} = \mu_t + \Gamma_{t+1} \quad \text{for all} \quad t, \ \mu_0 = 0 \tag{40}$$

and constraints (37) and (39).

This step follows from algebra. The form of Program 2 offers hope for finding a recursive approach, because future variables appear neither in the constraints nor return functions. Nevertheless, we cannot formulate a Bellman equation immediately for Program 2, because it is a saddle point, not a maximization, problem. We require a theory for recursive saddle point problems.

Thus, define a

Recursive Saddle Point Problem (RSPP):

$$\inf_{\{\mu_t\}} \sup_{\{X_t\}} E_0 \sum_{t=0}^{\infty} \beta^t h(X_t, \, X_{t+1}, \, \mu_t, \, \mu_{t+1}, \, S_t)$$

subject to

$$\mathcal{T}(X_t, \, X_{t+1}, \, S_t) \geq 0 \tag{41}$$

$$\mathcal{Q}(\mu_t, \, \mu_{t+1}, \, S_t) \geq 0 \tag{42}$$

$$X_0 = \overline{X}, \ \ \mu_0 = \overline{\mu}, \ \ S_0 = \overline{S} \tag{43}$$

$$(X_{t+1}, \mu_{t+1}) \text{ measurable with respect to } \ (S_0 \ldots, S_{t-1}, S_t), \tag{44}$$

and let W be the value function for this problem; i.e., $W(\overline{X}, \overline{\mu}, \overline{S})$ is the value attained by the objective function at the saddle point for the given initial conditions.

Step 2:

Show that the value function W of a RSPP satisfies the following analog of a Bellman equation:

Saddle Point Functional Equation (SPFE)

$$W(X, \mu, S) = \inf_{\mu'} \sup_{X'} \left\{ h(X, \, X', \, \mu, \, \mu', \, S) + \beta E \left[W(X', \mu', S') | \, S \right] \right\}$$

$$\text{s.t.} \quad \mathcal{T}(X, \, X', \, S) \geq 0$$
$$\mathcal{Q}(\mu, \, \mu', \, S) \geq 0$$

Letting $f(X, \mu, S)$ be the optimal choice in the right side of the SPFE, it can be shown that the optimal solution to program 2 satisfies $X_t = f(X_{t-1}, \mu_{t-1}, S_{t-1})$ for all t and with $(X_{-1}, \mu_{-1}, S_{-1}) = (\overline{X}, 0, \overline{S})$.

Step 3:

Evidently, *Program 2* is a special case of a RSPP, because we can take

$$h(X_t, \, X_{t+1}, \, \mu_t, \, \mu_{t+1}, \, S_t) = U(X_t, X_{t+1}, S_t) + \mu_{t+1} \, \mathcal{V}(S_t, X_t, X_{t+1}) - \Gamma_{t+1} \, \Phi(X_t, X_{t+1})$$

$$\mathcal{Q}(\mu, \Gamma, \, \mu', \Gamma' \, S) \equiv \begin{cases} \mu' - \mu - \Gamma' \\ -\mu' + \mu + \Gamma' \end{cases}$$

and

$$\overline{\mu} = 0.$$

Recall that Step 1 implies that the solutions to Program 1 and 2 are equivalent; therefore, the solution to Program 1 satisfies $X_t = f(X_{t-1}, \mu_{t-1}, S_{t-1})$.

The Ramsey problem in the current paper is a special case of program 1 if we take

$$\begin{aligned} X_t \equiv (c_t, b_t^g), S_{t-1} &\equiv g_t, \mu_t \equiv \psi_t \\ U(X_t, X_{t+1}) &\equiv u(c_{t+1}, 1 - c_{t+1} - g_{t+1}), \\ \mathcal{V}(S_t, X_{t+j+1}) &\equiv u_{c,t+1}\omega_{t+1} \\ \Phi(X_t, X_{t+1}, S_t) &\equiv u_{c,t+1}\ b_t^g \\ T(X_t, X_{t+1}, S_t) &\equiv \left\{ \begin{array}{c} b_t^g - \overline{M} \\ -b_t^g + \underline{M} \end{array} \right\} \end{aligned}$$

The corresponding RSPP (or *Program 2*) is displayed in equation (11) in the main text. Therefore, the corresponding SPFE for the Ramsey problem is

$$W(b^g, \psi, g) \ = \inf_{\mu', \gamma', \nu_1', \nu_2'} \sup_{X'} \left\{ u(c', 1 - c' - g) + \psi' u_c' \omega - \right.$$

$$\left. u_{c'}\ (b^g \gamma + \nu_1 \overline{M} - \nu_2 \underline{M}) + \beta E \left[W(b^{g\prime}, \psi', g') | \, g \right] \right\}$$

$$\begin{aligned} \text{s.t.} \qquad \underline{M} \leq b^{g\prime} &\leq \overline{M} \\ \nu_1', \nu_2' &\geq 0 \\ \psi' &= \gamma' + \nu_1' + \nu_2' + \psi \end{aligned}$$

Using the framework in M&M, we conclude that the solution of the Ramsey problem satisfies:

$$c_t = h(g_t, \mu_{t-1}, b_{t-1}^g) \qquad \text{for all } t,$$

and $(\mu_{-1}, b_{-1}^g) = (0, \overline{b}^g)$, where h is the decision function for the above SPFE.

Figures

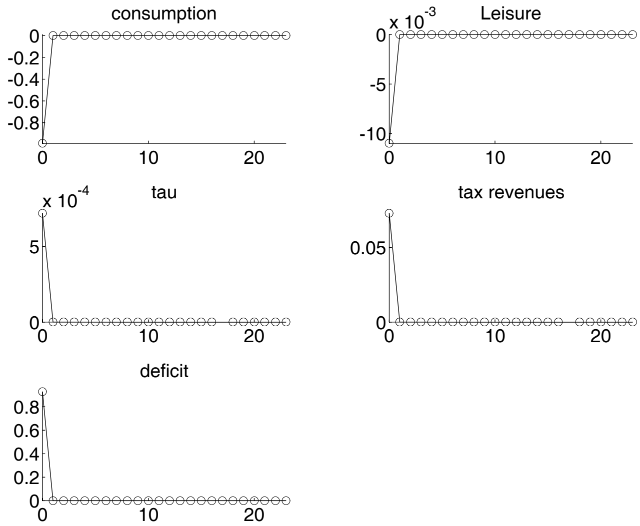

Figure 1. Impulse response functions for complete markets economy, serially independent government purchases. From left to right, top to bottom, are impulse response functions for consumption, leisure, tax rate, tax revenues, and the government deficit.

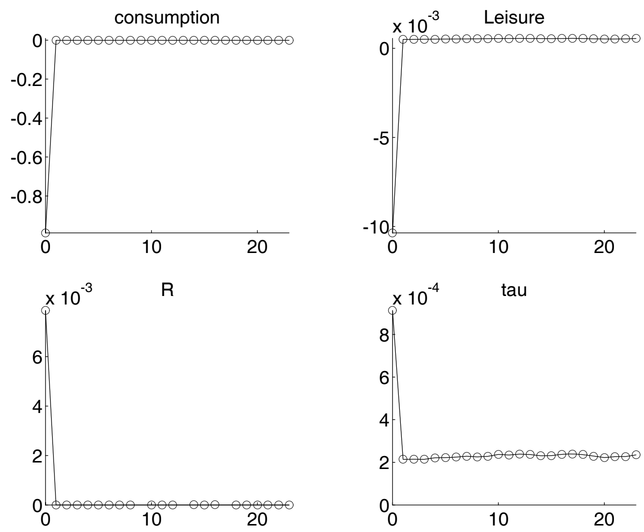

Figure 2a. Impulse response functions for incomplete markets economy, serially independent government purchases. From left to right, top to bottom, are impulse responses of consumption, leisure, the gross real interest rate, and the tax rate.

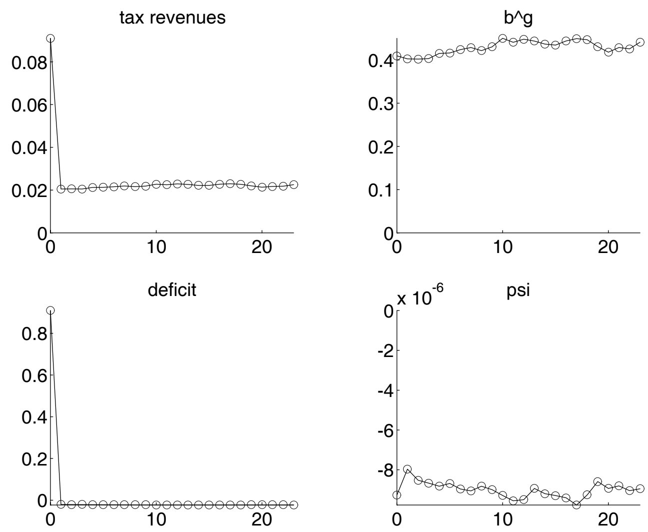

Figure 2b. Impulse response function for incomplete markets economy, serially independent government purchases. From left to right, top to bottom, are impulse responses of tax revenues, the debt level b^g, the deficit, and the multiplier ψ_t.

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