A, B, C'S (AND D)'S FOR UNDERSTANDING VARS

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ABSTRACT. The dynamics of a linear (or linearized) dynamic stochastic economic model can be expressed in terms of matrices (A, B, C, D) that define a state space system for a vector of observables. An associated state space system $(A, \hat{B}, C, \hat{D})$ determines a vector autoregression for those same observables. We present a simple condition for checking when these two state space systems match up and when they don't when there are equal numbers of economic and VAR shocks. We illustrates our condition with a permanent income example.

KEY WORDS: Vector autoregression, economic shocks, innovations, invertibility.

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I. Introduction

The last two years have witnessed a heated discussion questioning whether unrestricted VARs are informative about how particular economic models respond to preference, technology, and information shocks.[1] In the simplest possible setting, this paper provides a check for whether a theoretical model has the property in population that it is possible to infer economic shocks and impulse responses to them from the innovations and the impulse responses associated with a vector autoregression (VAR). We revisit an invertibility issue that is known to cause a potential problem for interpreting VARs and present a simple check for its presence.[2] We illustrate our check in the context of a permanent income model for which it can be applied by hand.

II. Two recursive representations of observables

II.1. **Recursive representation of an equilibrium.** Let an equilibrium of an economic model or an approximation to it have a representation for $\{y_{t+1}\}$ in the state space form

$$x_{t+1} = Ax_t + Bw_{t+1} \tag{1}$$

$$y_{t+1} = Cx_t + Dw_{t+1} \tag{2}$$

where x_t is an $n \times 1$ vector of possibly unobserved state variables, y_t is a $k \times 1$ vector of variables observed by an econometrician, and w_t is an $m \times 1$ vector of economic shocks impinging on the states and observables, i.e., shocks to preferences, technologies, agents' information sets, and the economist's measurements. The shocks w_t are Gaussian vector white noise satisfying $Ew_t = 0$, $Ew_t w_t' = I$, $Ew_t w_{t-j} = 0$ for $j \neq 0$, where the assumption of normality is for convenience and allows us to associate linear least squares predictions with conditional expectations. With m shocks in the economic model, n states, and k observables, A is $n \times n$, B is $n \times m$, C is $k \times n$, and D is $k \times m$. In general, $k \neq m$. The matrices A, B, C, and D are functions of parameters that define preferences, technology, and economics shocks. They incorporate the typical cross-equation restrictions embedded in modern macroeconomic models.

Equilibrium representations of the form (1)-(2) are obtained in one of two widely used procedures. The first is to compute a linear or loglinear approximation of a nonlinear model as exposited, for example, in Uhlig (1999). It is straightforward to collect the linear or log linear approximations to the equilibrium decision rules and to arrange them into the state space form (1)-(2). A second way is to derive (1)-(2) directly as a representation of a member of a class of dynamic stochastic general equilibrium models with

[1] See Chari, Kehoe, and McGrattan (2005) and Christiano, Eichenbaum, and Vigfusson (2006).

[2] Hansen and Sargent (1981), Hansen and Sargent (1991), Lippi and Reichlin (1994), Sims and Zha (2004), and Hansen and Sargent (2007) contain general treatments of this problem.

linear transition laws and quadratic preferences. For example, see Ryoo and Rosen (2003), Rosen, Murphy, and Scheinkman (1994), and Topel and Rosen (1988).[3]

II.2. **The question.** Our question is: under what conditions do the economic shocks in the state-space system (1)-(2) match up with the shocks associated with a vector autoregression? That is, under what conditions is

$$w_{t+1} = \Omega(y_{t+1} - E(y_{t+1}|y^t)) \tag{3}$$

where w_{t+1} are the economic shocks in (1)-(2), y^t denotes the semi-infinite history $y_t, y_{t-1}, \ldots$, $y_{t+1} - E(y_{t+1}|y^t)$ are the one-step-ahead forecast errors associated with an infinite order VAR, and Ω is a matrix of constants that can potentially be uncovered by 'Structural' VAR (SVAR) analysis? When (3) holds, impulse responses from the SVAR match the impulse responses from the economic model (1)-(2).

To begin to characterize conditions under which (3) holds, consider the prediction errors from (2) after conditioning on y^t, that is $y_{t+1} - E(y_{t+1}|y^t) = C(x_t - E(x_t|y^t)) + Dw_{t+1}$. Evidently, $C(x_t - E(x_t|y^t))$ drives a wedge between the VAR errors $y_{t+1} - E(y_{t+1}|y^t)$ and the structural errors w_{t+1}. What is required is a condition that eliminates this wedge. In Condition 1, we offer a simple condition that yields (3) in the interesting 'square case' in which $k = m$ and D has full rank.

II.3. **A poor man's invertibility condition.** When D is nonsingular, (2) implies $w_{t+1} = D^{-1}(y_{t+1} - Cx_t)$. Substituting this into (1) and rearranging gives:

$$[I - (A - BD^{-1}C)L]x_{t+1} = BD^{-1}y_{t+1} \tag{4}$$

where L is the lag operator. Consider

Condition 1. *The eigenvalues of $A - BD^{-1}C$ are strictly less than one in modulus.*

When Condition (1) is satisfied, we say that $A - BD^{-1}C$ is a stable matrix. The inverse of the operator on the left side of this equation gives a square summable polynomial in L if and only if Condition 1 is satisfied. In this case, x_{t+1} satisfies

$$x_{t+1} = \sum_{j=0}^{\infty} [A - BD^{-1}C]^j BD^{-1} y_{t+1-j}, \tag{5}$$

so that x_{t+1} is a square summable linear combination of the observations on the history of y at time $t+1$. This means that the complete state vector is in effect observed so that $var(x_t|y^t) = 0$. Shifting (5) back one period and substituting into (2), we obtain:

[3] Hansen and Sargent (2007) provide many other examples of this second approach.

$$y_{t+1} = C\sum_{j=0}^{\infty}[A-BD^{-1}C]^{j}BD^{-1}y_{t-j} + Dw_{t+1}. \tag{6}$$

If condition (1) is satisfied, equation (6) defines a vector autoregression for y_{t+1} because the infinite sum in (6) converges in mean square and Dw_{t+1} is orthogonal to y_{t-j} for all $j \geq 0$.

If one of the eigenvalues of $A-BD^{-1}C$ is strictly greater than unity in modulus, this argument fails because the infinite sum in (6) diverges. When $A-BD^{-1}C$ is an unstable matrix, the VAR is associated with another celebrated state space representation for $\{y_{t+1}\}$, to which we now turn.

II.4. **The innovations representation.** Associated with any state space system (A,B,C,D) for $\{y_{t+1}\}_{t=1}^{T}$ of the form (1)-(2) is another state space system: called the *innovations representation*:

$$\hat{x}_{t+1} = A\hat{x}_t + \hat{B}_{t+1}\varepsilon_{t+1} \tag{7}$$

$$y_{t+1} = C\hat{x}_t + \hat{D}_{t+1}\varepsilon_{t+1}, \tag{8}$$

where $x_0 \sim \mathscr{N}(\hat{x}_0,\Sigma_0)$, $\hat{x}_t = E(x_t|\{y_i\}_{i=1}^{t})$, $y_{t+1} - E(y_{t+1}|\{y_i\}_{i=1}^{t}) = \hat{D}_{t+1}\varepsilon_{t+1}$, ε_{t+1} is another iid Gaussian process with mean zero and identity covariance matrix, and the matrices $\hat{B}_{t+1}$ and $\hat{D}_{t+1}$ can be recursively computed by the Kalman filter. Under a general set of conditions, for any positive semi-definite Σ_0, as $t \to +\infty$, the matrices $\hat{B}_{t+1}$ and $\hat{D}_{t+1}$ converge to limits $\hat{B}$ and $\hat{D}$ that satisfy the equations:[4]

$$\begin{aligned}\Sigma &= A\Sigma A' + BB' - (A\Sigma C' + BD') \\ &\quad (C\Sigma C' + DD')^{-1}(A\Sigma C' + BD')'\end{aligned} \tag{9}$$

$$K = (A\Sigma C' + BD')(C\Sigma C' + DD')^{-1} \tag{10}$$

$$\hat{D}\hat{D}' = DD' + C\Sigma C' \tag{11}$$

$$\hat{B} = K\hat{D} \tag{12}$$

where $\Sigma = var(x_t|y^t)$.[5] When $A-BD^{-1}C$ is unstable, $\Sigma > 0$, meaning that at least some parts of the state x_t are hidden. This means the one-step-ahead forecast errors computed by the VAR, $y_{t+1} - E(y_{t+1}|y^t)$, contain the shocks Dw_{t+1} and the error from estimating the

[4] Alternative conditions for the existence of this time invariant innovations representation and for convergence of iterations on the Riccati equation are stated in Anderson and Moore (1979, ch. 4), Sargent (1980, chs. 5 and 6), Anderson, Hansen, McGrattan, and Sargent (1996), Hansen and Sargent (2007).

[5] With m shocks in the economic model, n states, and k observables, K, the steady-state Kalman gain, is $n \times k$, $\hat{D}$ is $k \times k$, and $\hat{B}$ is $n \times k$.

state $C(x_t - \hat{x}_t)$. Thus, (3) does not hold. These two components of $y_{t+1} - E(y_{t+1}|y^t)$ are uncorrelated, so that the variance of the VAR innovations $\hat{D}\varepsilon_{t+1}$ is larger than the variance of the economic model disturbances Dw_{t+1}. (Equivalently, from equation (11) $\hat{D}\hat{D}' > DD'$.)[6]

III. PERMANENT INCOME EXAMPLE

A state space representation for the surplus $y_{t+1} - c_{t+1}$ for the permanent income consumption model (e.g., see Sargent (1987), chapter XII) is

$$c_{t+1} = c_t + \sigma_w(1 - R^{-1})w_{t+1} \tag{13}$$

$$y_{t+1} - c_{t+1} = -c_t + \sigma_w R^{-1} w_{t+1} \tag{14}$$

where $y_{t+1} = \sigma_w w_{t+1}$ is an iid labor income process and $R > 1$ is a constant gross interest rate on financial assets. Equations (13) and (14) correspond to (1) and (2), where c_t is the unobserved state and $y_t - c_t$ is the variable observed by the econometrician. The impulse responses for the model are shown in panel (a) of figure 1 for the case that $R = 1.2$ and $\sigma_w = 1$. They show the familiar patterns: consumption increases permanently by the annuity value of the transitory increase in income; this leads to a large positive impact effect of w_t on $y_t - c_t$ and small negative values for all other periods.

For this example, it is easy to compute that $A - BD^{-1}C = R > 1$, so that Condition 1 does not hold. This failure of Condition 1 is part and parcel of the permanent income model because it is needed to verify that the present value of the coefficients describing the response of the surplus $y_{t+1} - c_{t+1}$ to an endowment innovation must be zero, an outcome that embodies the present value budget balance that is built into the permanent income model.

The innovations representation of the model is

$$\hat{c}_{t+1} = \hat{c}_t + \sigma_w(R^{-1} - 1)\varepsilon_{t+1} \tag{15}$$

$$y_{t+1} - c_{t+1} = -\hat{c}_t + \sigma_w \varepsilon_{t+1} \tag{16}$$

Equations (15) and (16) correspond to the steady state version of (7) and (8), where $\hat{c}_t = E(c_t|y^t - c^t)$ is the estimate of consumption constructed from the history of $y_t - c_t$. Because Condition 1 is not satisfied, c_t cannot be estimated perfectly from $y^t - c^t$, so that $\hat{c}_t \neq c_t$. Indeed, a simple calculation shows that $\Sigma = var(c_t|y^t - c^t) = \sigma_w^2(1 - R^{-2})$. Because $A - \hat{B}\hat{D}^{-1}C = R^{-1}$, which is stable, the errors computed by a VAR for $y_t - c_t$ are $\sigma_w \varepsilon_{t+1}$. (Of course, since $y_t - c_t$ is a scalar, the VAR is just a univariate autoregression.)

[6] Hansen and Sargent (2007), chapter 9, discusses the innovations representation, proves that $A - \hat{B}\hat{D}^{-1}C$ is a stable matrix and derives a general formula that describes the mapping from the economic shocks w_{t+1} to the VAR shocks ε_{t+1}.

Panel (b) of figure 1 shows the impulse responses of $\hat{y}_t$, $\hat{c}_t$, and $y_t - c_t$ with respect to the VAR shocks, ε_t. These are markedly different than the impulse responses shown in panel (a). Notably, while c_t responds *positively* and permanently to a shock in w_t in figure 1, $\hat{c}_t$ responds *negatively* and permanently to a VAR shock ε_t. In panel (a), the impulse response of the surplus $y_t - c_t$ has a present value of zero, implying present value budget balance; in panel (b) the the impulse response of the surplus has a positive present value so that the present value of the impulse response of consumption falls short of the present value of the impulse response of income.[7]

This example can be modified in instructive ways by altering what is observed. For example, if c_t, y_{t+1}, or the value of the consumer's accumulated assets were observed, then Condition 1 would be satisfied.[8]

IV. Concluding remarks

We hesitate to draw sweeping conclusions about VARs. Some applications of VARs are informative about the shapes of impulse-responses to some economic shocks that theories should attempt to match, while others are not.

It is easy to reiterate the recommendation to estimate the deep parameters of a complete and fully trusted model by likelihood-based methods. If you trust your model, you should accept that recommendation. However, the enterprise of identifying shocks and impulse-response functions to them by identifying economic shocks from VAR innovations aims to coax interesting patterns from the data that will prevail across a *set* of incompletely specified and not fully trusted models. Despite pitfalls, it is easy to sympathize with the enterprise of identifying economic shocks from VAR innovations if one is not dogmatic in favor of a particular fully specified model.

[7] For more discussion of this example, see Sargent (1987), Hansen, Roberds, and Sargent (1991), and Roberds (1991).

[8] See Watson (1994) for a more extensive discussion of how what is observed affects whether condition 1 is likely to be satisfied.

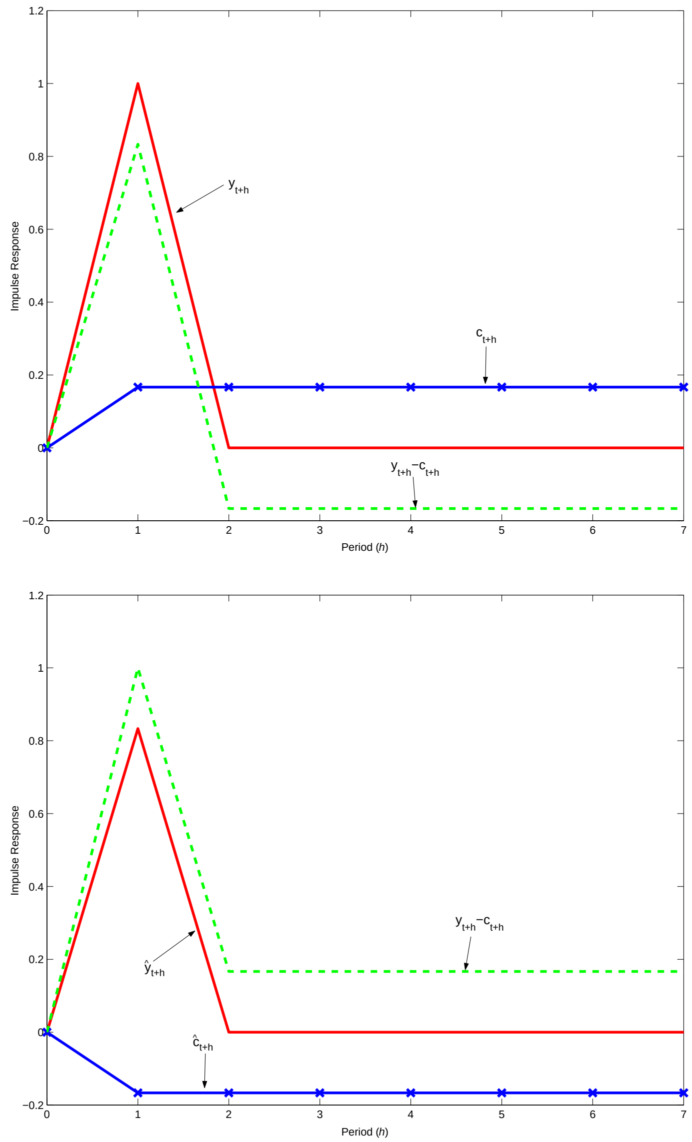

FIGURE 1. Panel (a): Impulse Responses of y_{t+h}, c_{t+h}, and $y_{t+h} - c_{t+h}$ to a shock in w_t. Panel (b): Impulse Responses of $\hat{y}_{t+h}, \hat{c}_{t+h}$, and $y_{t+h} - c_{t+h}$ to a shock in ε_t.

REFERENCES

ANDERSON, B., AND J. MOORE (1979): *Optimal Filtering*. Prentice Hall, Englewood Cliffs, New Jersey.

ANDERSON, E., L. P. HANSEN, E. R. MCGRATTAN, AND T. J. SARGENT (1996): "Mechanics of Forming and Estimating Dynamic Linear Economies," in *Handbook of Computational Economics, Volume 1*, ed. by H. M. Amman, D. A. Kendrick, and J. Rust, pp. 171–252. North-Holland.

CHARI, V. V., P. J. KEHOE, AND E. R. MCGRATTAN (2005): "A Critique of Structural VARs Using Real Business Cycle Theory," Federal Reserve Bank of Minneapolis Working Paper Number 631.

CHRISTIANO, L. J., M. EICHENBAUM, AND R. VIGFUSSON (2006): "Assessing Structural VARs," *NBER Macroeconomics Annual*, 21.

HANSEN, L. P., W. ROBERDS, AND T. J. SARGENT (1991): "Time Series Implications of Present Value Budget Balance and of Martingale Models of Consumption and Taxes," in *Rational expectations econometrics*, ed. by L. P. Hansen, and T. J. Sargent, pp. 121–162. Westview Press, Boulder.

HANSEN, L. P., AND T. J. SARGENT (1981): "Formulating and Estimating Dynamic Linear Rational Expectations Models," in *Rational Exectations and Econometric Practice*, ed. by J. Robert E. Lucas, and T. J. Sargent, pp. 127–158. University of Minnesota Press, Minneapolis, Minnesota.

——— (1991): "Two Difficulties in Interpreting Vector Autoregressions," in *Rational expectations econometrics*, ed. by L. P. Hansen, and T. J. Sargent, pp. 77–120. Westview Press, Boulder.

——— (2007): "Recursive Linear Models of Dynamic Economies," .

LIPPI, M., AND L. REICHLIN (1994): "VAR Analysis, Nonfundamental Representations, Blaschke Matrices," *Journal of Econometrics*, 63(1), 307–325.

ROBERDS, W. (1991): "Implications of Expected Present Value Budge Balance: Application to Postwar U.S. Data," in *Rational expectations econometrics*, ed. by L. P. Hansen, and T. J. Sargent, pp. 163–174. Westview Press, Boulder.

ROSEN, S., K. M. MURPHY, AND J. A. SCHEINKMAN (1994): "Cattle Cycles," *Journal of Political Economy*, 102(3), 468–492.

RYOO, J., AND S. ROSEN (2003): "The Engineering Labor Market," Manuscript: Hong Kong University of Science and Technology.

SARGENT, T. (1987): *Macroeconomic Theory, Second edition*. Academic Press, New York.

SARGENT, T. J. (1980): "Notes on Linear Control and Filtering Theory," University of Minnesota, Department of Economics.

SIMS, C. A., AND T. ZHA (2004): "Does Monetary Policy Generate Recessions?," Federal Reserve Bank of Minneapolis Working Paper Number 98-12.

TOPEL, R., AND S. ROSEN (1988): "Housing Investment in the United States," *Journal of Political Economy*, 96, 718–40.

UHLIG, H. (1999): "A toolkit for analysing nonlinear dynamic stochastic models easily," in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon, and A. Scott, pp. 30–61. Oxford.

WATSON, M. (1994): "Vector Autoregressions and Cointegration," in *Handbook of Econometrics, Vol. IV*, ed. by D. L. McFadden, and R. F. Engle, pp. 2843–2915. Elsevier Science.