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6

The Convergence of Vector Autoregressions to Rational Expectations Equilibria

ALBERT MARCET and THOMAS J. SARGENT

1. Introduction

In economic affairs, the way that the future unfolds from the past depends partly on how people expect it to unfold from the past. Economic systems can thus be described as self-referential, because outcomes depend partly on what people expect those outcomes to be. This self-referential aspect of economic systems gives rise to enormous theoretical problems of indeterminacy (i.e. multiple equilibria) when people's expectations are left as 'free variables' that are not restricted by economic theory. To fight that threat of indeterminacy, economists have embraced the hypothesis of rational expectations. This hypothesis instructs us to focus only on outcomes and systems of beliefs that are consistent with one another, allowing for whatever differences between outcomes and beliefs can be attributed to uncertainty and limited information.

A rational expectations equilibrium is a fixed point of a particular mapping from beliefs to outcomes. When agents have an arbitrarily given set of beliefs about the laws of motion of the economy, their behaviour causes the actual laws of motion to be determined. We can think of people's behaviour collectively as inducing a mapping from their believed laws of motion to the actual laws of motion for the economy. A rational expectations equilibrium is a fixed point of that mapping, which is a set of beliefs about laws of motion that is consistent with realized outcomes.

Much work in economic theory and rational expectations econometrics simply assumes that the economy is described by a rational

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expectations equilibrium. This type of work is silent on the question of how the economy could have arrived at such a situation in which beliefs are consistent with outcomes. That is, the theory is silent about how agents might learn to have correct beliefs if they had started with beliefs that are wrong. Recently, a number of researchers have begun studying this 'learning' problem. One reason for studying the problem is that the notion of a rational expectations equilibrium would be a more attractive one if there were plausible and undemanding learning schemes which would drive the system towards a rational expectations equilibrium. A second reason for studying learning theories—one emphasized in the work of George Evans—is that, even though it restricts systems of beliefs, the concept of rational expectations is often not restrictive enough to prevent a multiplicity of equilibria from occurring. That is, for some economic environments, there can occur multiple systems of beliefs that are consistent with outcomes. One use of a learning theory that converges to a rational expectations equilibrium would be to select which of several equilibria might be expected to prevail in practice because learning schemes are attracted towards them and repelled from others.

Margaret Bray (1982, 1983), Bray and Savin (1986), and Fourgeaud *et al*. (1986) have studied the learning problem in environments that can be described by particular linear rational expectations equilibria. They relax the assumption of rational expectations and instead assume that agents must learn about some aspect of the environment through the sequential application of linear least squares. Each of these authors displays technical conditions under which their systems converge almost surely to a rational expectations equilibrium.

This paper summarizes and applies some of our research on least-squares learning mechanisms in the context of linear rational expectations models with private information.[1] We model the agents in the economy as forming beliefs by fitting vector autoregressions. Each period, the agents add the latest observations and update their vector autoregressions. They use these updated vector autoregressions to form forecasts that influence decisions that they make, which in turn influence the motion of variables in the economy. We study conditions under which such an economy converges to a rational expectations equilibrium.

Studying the convergence of least-squares estimators in setups like ours involves technical difficulties because these setups involve departures from the standard assumptions used in time-series econometrics to deliver convergence. Under standard conditions maintained in econometrics (e.g. covariance stationarity, and ergodicity of the stochastic process for which the vector autoregression is being estimated), least-squares estimators of vector autoregressions are known to converge strongly. Such convergence results fail to cover the cases that we want to study, because in our systems agents' learning behaviour causes the stochastic process being learned about to be non-stationary. In effect, agents are shooting at (learning about) a moving target, rather than the fixed target assumed in the standard econometric setting. This feature is what has made convergence results difficult to attain.

[1] This work is contained in four papers by Marcet and Sargent (1988, 1989*a*, 1989*b*, 1989*c*).

In self-referential linear models, the least-squares estimators of vector autoregressions follow a complicated stochastic difference equation whose limiting behaviour we want to analyse. By building on technical results of Lennart Ljung (1977), it can be shown that the limiting behaviour of this stochastic difference equation is described by a much simpler ordinary differential equation.[2] The differential equation involves the operator mapping perceived laws of motion into actual laws of motion, which appears in the work of DeCanio (1979) and Evans (1983, 1985). Using this approach, one immediately obtains Margaret Bray's (1982) result that the only possible limit points of least-squares learning schemes are rational expectations equilibria. Further, the local stability of least-squares learning schemes about a rational expectations equilibrium can be determined by studying the stability of the associated differential equation at the rational expectations equilibrium.

We apply our results to three models that have appeared in the literature. These examples illustrate the power of the differential equation approach to shorten and unify proofs of convergence that have appeared in the literature. The differential equation approach also permits a unified interpretation of apparently diverse results in terms of the character of the operator mapping perceived to actual laws of motion. We first study a model of Margaret Bray, and show how our methods can be used to represent and somewhat strengthen her results. Next we study a model that we have created by modifying a model contributed by Roman Frydman. Under our modifications, the model converges strongly to a rational expectations equilibrium. The third model is a private-information version of a hyperinflation or stock price model. The rational expectations version of this model has many equilibria, all but one of which exhibit a 'speculative bubble'. Our results suggest that the equilibria with bubbles will not be attractors under our learning scheme.

[2] The ordinary differential equations approach is described and applied by Ljung and Soderstrom (1983) and Goodwin and Sin (1983). Margaritis (1985) gave an early application of the approach in economics. Woodford (1986) applies some of Ljung's methods to a nonlinear dynamic model. Also see Kushner and Clark (1978) and Robbins and Monro (1951).

2. The Model and a Convergence Proposition[3]

This section describes the technical results obtained by Marcet and Sargent (1989*b*), where we study the convergence of a system driven by the behaviour of two types of differentially informed agents, each of whom is learning through the sequential application of linear least squares. That distinct agents are differentially informed creates the situation that for each class of agent there are hidden state variables.[4] Our setup is as follows.

There is an $(n \times 1)$ state vector z_t. We let z_{it} be an $n_i \times 1$ subvector, where $1 \leq n_i \leq n$, for $i = a, b, c, d$. There are two types of agents, types a and b, who observe $z_{at} = e_a z_t$ and $z_{bt} = e_b z_t$, respectively, possibly distinct subvectors of z_t. Agents of type j want to predict future value of possibly distinct subvectors $z_{k(j)} = e_{k(j)} z_t$, where $k(a) = c$ and $k(b) = d$, and to use the current observation on z_{jt} in order to form those predictions. The selection matrices e_a, e_b, e_c, e_d are constant through time. There is an economic model which maps beliefs of agents a and b into actual outcomes in the following way. If the beliefs of agents of type a and type b were given by the time-invariant rules,

$$E^*(z_{ct} | z_{at-1}) = \beta_a \, z_{at-1}$$

$$E^*(z_{dt} | z_{bt-1}) = \beta_b \, z_{bt-1} \qquad \text{for all } t, \tag{1}$$

then the actual law of motion of z_t would be given by

$$z_t = T(\beta) z_{t-1} + V(\beta) \epsilon_t \tag{2}$$

where ϵ_t is an $(m \times 1)$-vector white noise, $\beta = (\beta_a, \beta_b)$, and T and V are operators that map matrices comformable to the objects they operate upon. A particular economic model will determine the operators T and V. In subsequent sections, we describe several economic models and display the operators T and V that are associated with them.

We will work in regions of the parameter space β for which (2) implies that z_t is a covariance stationary stochastic process. For this purpose, we define the following set:

$$D_s = \{\beta | \text{the operators } T(\beta) \text{ and } V(\beta) \text{ are well defined, and the eigenvalues of } T(\beta) \text{ are less than unity in modulus}\}.$$

For $\beta \in D_s$, (2) generates a covariance stationary stochastic process, for which the second-moment matrix $E z_t z_t'$ is well defined. Letting $M_z(\beta) = E z_t z_t'$, this moment matrix can be computed as the solution of the discrete Lyapunov equation,

$$M_z(\beta) = T(\beta) M_z(\beta) T(\beta)' + V(\beta) \Omega V(\beta)',$$

where $\Omega = E \epsilon_t \epsilon_t'$. We use the following notation for some submatrices of $E z_t z_t'$:

$$M_{z_j}(\beta) = E z_{jt} z_{jt}', \qquad j = a, b$$

$$M_{z_j, z}(\beta) = E z_{jt} z_t', \qquad j = a, b. \tag{3}$$

In general, each of these moment matrices is a function of β.

If the actual law of motion for z_t is (2), then it can be calculated that the linear least-squares projection of $z_{k(j)t}$ on z_{jt-1} is given by

$$\hat{E}(z_{k(j)t} | z_{jt-1}) = S_j(\beta) z_{jt-1}, \tag{4}$$

where

$$S_j(\beta) = e_{k(j)} T(\beta) [M_{z_j}(\beta)^{-1} M_{z_j, z}(\beta)]', \qquad \text{for } j = a, b. \tag{5}$$

The operators $S_j(\beta)$ map the perceptions $\beta = (\beta_a, \beta_b)$ into the projection coefficients (in the linear least-squares sense), $S_a(\beta)$, $S_b(\beta)$. Let us define $S(\beta) = [S_a(\beta), S_b(\beta)]$.

We now advance the following.

DEFINITION. A rational expectations equilibrium with asymmetric private information occurs when perceptions in (1) are given by a matrix $\beta = (\beta_a, \beta_b)$ that satisfies

$$\beta_f \equiv [\beta_{af}, \beta_{bf}] = [S_a(\beta_f), S_b(\beta_f)] = S(\beta_f).$$

Thus, a rational expectations equilibrium is a fixed point of the mapping S. Notice that this concept of a rational expectations equilibrium is relative to the fixed information sets z_{at-1} and z_{bt-1} specified by the model-builder.

We now describe the model of learning. The learning scheme is a recursive version of least squares, modified to permit agents to disregard observations that threaten to drive the estimates outside of some pre-specified sets; D_{1j}, $j = a, b$. These pre-specified sets play an important role in governing the global convergence of the least-squares estimators. For $j = a, b$, we let $\{\alpha_{jt}\}$ be a positive, non-decreasing sequence with $\lim_{t \to \infty} \alpha_{jt} = 1$. Beliefs of agents of type $j (= a, b)$ evolve according to the following scheme. Define $\bar{\beta}_{jt}$ and $\bar{R}_{jt}$ by

$$\bar{\beta}_{jt}' = \beta_{jt-1}' + (\alpha_{jt-1}/t) R_{jt-1}^{-1} \{ z_{jt-2} [z_{k(j)t-1} - \beta_{jt-1} z_{jt-2}]' \}$$

$$\bar{R}_{jt} = R_{jt-1} + (\alpha_{jt-1}/t)(z_{jt-1} z_{jt-1}' - R_{jt-1}/\alpha_{jt-1}). \tag{6a}$$

Agent j specifies two sets D_{2j} and D_{1j}, $j = a, b$. Let $D_{2j} \subset D_{1j} \subset \mathbb{R}^{n_{k(j)} \times (n_j)^3}$, $j = a, b$. The algorithm generating beliefs at t is then

[3] This section parallels and elaborates on the first section of Marcet and Sargent (1989*b*).

[4] Note that the models of Bray and Savin (1986) and of Fourgeaud *et al.* (1986) do not have hidden state variables.

$$(\beta_{jt}, R_{jt}) = \begin{cases} (\bar\beta_{jt}, \bar R_{jt}) & \text{if} \quad (\bar\beta_{jt}, \bar R_{jt}) \in D_{1j} \\ \text{some value in } D_{2j} & \text{if} \quad (\bar\beta_{jt}, \bar R_{jt}) \notin D_{1j}. \end{cases} \tag{6b}$$

Equations (6*b*) define a 'projection facility' whose function is to keep beliefs (β_{jt}, R_{jt}) within the set D_{1j}. Two distinct sets, D_{1j} and D_{2j}, are used in defining the projection facility in order properly to invoke some technical arguments made by Ljung (1977). In practice, we shall be free to choose D_{2j} to be a set contained within, but arbitrarily close to, D_{1j}. In the applications below, we shall always think of D_{2j} as being arbitrarily close to D_{1j}, and thus will focus our attention on specification of the sets D_{1j}.[5]

If $D_{2j} = D_{1j} = \mathbb{R}^{n_{k(j)}\times(n_j)^2}$, then the 'projection facility' on the second branch of (6*b*) is never invoked, and with suitable initial conditions, (6*a*, *b*) simply becomes a recursive version of weighted least squares:

$$\beta_{jt} = \left(\sum_{i=1}^{t-1} \alpha_{ji} z_{ji-1} z'_{ji-1}\right)^{-1} \left(\sum_{i=1}^{t-1} \alpha_{ji} z_{ji-1} z'_{k(j)i}\right).$$

In the special case that $\{\alpha_{jt}\} = \{1\}$, the above formula is just ordinary least squares. In cases in which a nontrivial projection facility is specified by choosing D_{1j} to be a proper subset of $\mathbb{R}^{n_{k(j)}\times(n_j)^3}$, it is natural to set 'some point in D_{2j}' in (6*b*) equal to $(\beta_{jt'}, R_{jt'})$, where t' is the last time that $(\beta_{jt'}, R_{jt'}) \in D_{2j}$. With D_{2j} set arbitrarily close to D_{1j}, (6*a*, *b*) then amounts to least squares adjusted sequentially to ignore observations that threaten to drive (β_{jt}, R_{jt}) outside of the set D_{1j}. When the sequence $\{\alpha_{jt}\}$ is chosen to be strictly increasing, it leads to adjusting the least-squares algorithm to weight more recent observations more heavily. (The condition that $\lim_{t\to\infty} \alpha_{jt} = 1$ restricts the eventual rate of forgetting in a way sufficient to permit convergence of β_{jt} within the system to be studied below.)

The sets D_{1j} and D_{2j} will play important roles in one part of the proposition to be stated below. One role of the sets D_{1j} and D_{2j} can be to force the learning algorithm to remain the set D_s defined above.

We assume that, when agents are learning according to (6*a*, *b*), the actual law of motion is determined by substituting $\beta_t = (\beta_{at}, \beta_{bt})$ from (6*a*, *b*) for β on the right-hand side of (2):

$$z_t = T(\beta_{t-1}) z_{t-1} + V(\beta_{t-1})\epsilon_t. \tag{7}$$

The system that we want to study is (6*a*, *b*) and (7). Equation (7) indicates the sense in which agents' process of learning influences the

[5] Ljung and Soderstrom (1983) frequently proceed in this way, specifying a projection facility in terms of a single set.

actual law of motion of the system. Agents' learning behaviour evidently causes the z_t process to be non-stationary. This property of the z_t process implies that the recursive estimation scheme (6*a*, *b*), which could be rationalized as being optimal only in a constant coefficient environment, is in general a suboptimal way of learning for this class of environments. Our model is thus an irrational model of learning, a point emphasized by Bray and Savin (1986) and Bray and Kreps (1987).

We want to study (6*a*, *b*)–(7), which is a complicated system of stochastic difference equations. We shall use the method of Ljung (1977), whose approach is to find an ordinary differential equation that is associated with (6*a*, *b*)–(7) in the sense that the limiting behaviour of (6*a*, *b*)–(7) is described by that differential equation. It turns out that associated with the system of stochastic difference equations (6*a*, *b*) and (7) is the following ordinary differential equation:

$$\frac{d}{dt}\begin{bmatrix} \beta_a \\ \beta_b \\ R_a \\ R_b \end{bmatrix} = \begin{bmatrix} R_a^{-1} M_{z_a}(\beta)[S_a(\beta) - \beta_a]' \\ R_b^{-1} M_{z_b}(\beta)[S_b(\beta) - \beta_b]' \\ M_{z_a}(\beta) - R_a \\ M_{z_b}(\beta) - R_b \end{bmatrix}. \tag{8}$$

Defining $R = (R_a, R_b)$, we can represent (8) in the vector form:

$$\frac{d}{dt}\begin{pmatrix} \mathrm{col}(\beta) \\ \mathrm{col}(R) \end{pmatrix} = g(\beta, R),$$

where $\mathrm{col}(\beta)$ is a vector obtained by stacking columns of β on top of each other, and $\mathrm{col}(R)$ is a vector obtained by stacking columns of R on top of one another. In the interest of studying the linear approximations that govern the local behaviour of (8), we define

$$h(\beta, R) = \frac{\mathrm{d}}{\mathrm{d}(\mathrm{col}\,\beta, \mathrm{col}\,R)'} g(\beta, R).$$

Let $\{(\beta(t), R(t))\}_{t\in[0,\infty)}$ denote the trajectories of (8). We define the set D_A to be the domain of attraction of the fixed point (β_f, R_f) of (8), which we assume to be unique. That is, D_A consists of the set of $(\beta(0), R(0))$ such that, when $(\beta(0), R(0)) \in D_A$, (8) implies $\lim_{t\to\infty}(\beta(t), R(t)) = (\beta_f, R_f)$.

We use a set of six assumptions about system (6)–(7) which are described in the Appendix. Among these, the first five are in the nature of regularity conditions which are easy to verify and are typically satisfied for the kinds of applications we have encountered. Assumption (A1), which states that S has a unique fixed point, could be relaxed to permit multiple fixed point; then our propositions would transform in a readily seen way to statements about each fixed point of $S(\beta)$.

Assumption (A6) can be considerably more difficult to verify than (A1)–(A5), as we discuss below. This assumption is used in only the first part of our four-part proposition. For this first part, we also use the following additional assumption.

Assumption A7. For $j = a, b$, assume that D_{2j} is closed, that D_{1j} is open and bounded, and that $\beta \in D_s$ for all $(\beta_a, R_a, \beta_b, R_b) \in D_{1a} \times D_{1b}$. Assume that the trajectories of (8) with initial conditions $(\beta_a(0), R_a(0), \beta_b(0), R_b(0)) \in D_{2a} \times D_{2b}$ never leave a closed subset of $D_{1a} \times D_{1b}$.

We now state the following.

Proposition 1. Assume that (β_t, R_t, z_t) are determined by (6*a*, *b*), (7). Assume that (A1), (A2), (A3), (A4), and (A5) are satisfied.

(i) Assume also that (A6) and (A7) are satisfied and that $D_{1a} \times D_{1b} \subset D_A$, where D_A is the domain of attraction of (β_f, R_f) in (8). Then

$$P[\beta_t \to \beta_f] = 1.$$

(ii) Let $\hat{\beta} \neq \beta_f$, and assume that $M_{z_j}(\hat{\beta})$ is positive definite for $j = a, b$. Then

$$P[\beta_t \to \hat{\beta}] = 0.$$

(iii) If $h(R_f, \beta_f)$ has one or more eigenvalues with strictly positive real part, then

$$P[\beta_t \to \beta_f] = 0.$$

(iv) $h(\beta_f, R_f)$ has $(n_a)^2 + (n_b)^2$ repeated eigenvalues of -1. The remaining eigenvalues are the same as those of the following derivative matrix:

$$(\partial/\partial\beta)\begin{bmatrix}\mathrm{col}\,[S_a(\beta) - \beta_a] \\ \mathrm{col}\,[S_b(\beta) - \beta_b]\end{bmatrix}\Bigg|_{\beta=\beta_f}$$

This concludes the proposition.

Statement (i) asserts that sufficient conditions for $\beta_t \to \beta_f$ almost surely as $t \to \infty$ are that the set $D_{1a} \times D_{1b}$ generated in the projection facility be contained in D_A, and that, at (and close to) the boundary of $D_{1a} \times D_{1b}$, the differential equation (8) has trajectories that point towards the interior of $D_{1a} \times D_{1b}$. The situation described in part (i) is depicted in Fig. 6.1. Ljung (1977), Ljung and Soderstrom (1983), and Marcet and Sargent (1989*a*) describe what can happen when some of the trajectories of (8) point outside $D_{1a} \times D_{1b}$ at the boundary of $D_{1a} \times D_{1b}$. Statement (ii) asserts that the only candidate (β, R) as limit-points of the learning scheme are rational expectations equilibria. Statement (iii) asserts sufficient conditions for non-convergence of the learning scheme. Statement (iv) implies that everything can be learned

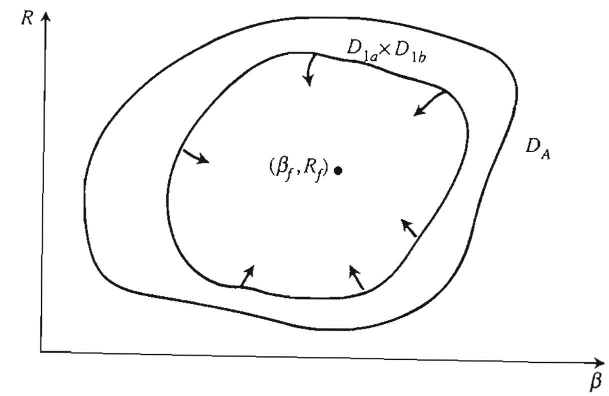

Fig. 6.1. Global convergence

The set D_A is the domain of attraction of the fixed point (β_f, R_f) of equation (8). The set $(D_{1a} \times D_{1b})$ determines the projection facility defined in (6*b*). When the trajectories of (8) point inward along the boundary of $D_{1a} \times D_{1b}$, and when $D_{1a} \times D_{1b}$ is inside D_A, then β_t converges almost surely to β_f.

about the local stability of the learning scheme by studying the differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\beta_a \\ \beta_b\end{pmatrix} = \begin{bmatrix}S_a(\beta) - \beta_a \\ S_b(\beta) - \beta_b\end{bmatrix} = S(\beta) - \beta. \tag{9}$$

Proposition 1 can be proved by retracing the steps used to prove propositions 1, 2, and 3 of Marcet and Sargent (1989*a*). Here we confine ourselves to relating an heuristic account of the mechanism underlying the proposition. This account is obtained simply by adapting the heuristic account of Ljung (1977) and Ljung and Soderstrom (1983) to our setting. In order to conserve notation, we describe only the special case of model (6*a*, *b*), (7) that emerges when we set $z_{at} = z_{bt}$, $z_{ct} = z_{dt}$, $\alpha_{at} = \alpha_{bt}$, $\beta_{at} = \beta_{bt}$, $R_{at} = R_{bt}$. This is an interesting special case, of which the model of Bray analysed in Section 3 and the model of hyperinflation analysed in Section 5 are further special cases. With the preceding special settings, the model is one in which there are homogeneous expectations but hidden state variables. For this special case, the trajectory of ordinary differential equation (8) is fully determined by the smaller ordinary differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\beta_a' \\ R_a\end{pmatrix} = \begin{matrix}R_a^{-1}M_{z_a}(\beta)[S_a(\beta) - \beta_a]' \\ M_{z_a}(\beta) - R_a,\end{matrix} \tag{10}$$

where it is understood that $\beta = (\beta_a, \beta_a)$. It will be obvious how, with some proliferation of notation, the following heuristic account would work for the model (6*a*, *b*), (7) without the restriction to homogeneous expectations.

Here is how the heuristic account of Ljung (1977) and Ljung and Soderstrom (1983) applies to our system. As t becomes large, the values of β_{at} and R_{at} determined by $(6a, b)$ and (7) change very little from $t-1$ to t. This is partly a result of the regularity conditions imposed in assumption (A4), which require that $\alpha_{at-1}/t \to 0$ as $t \to \infty$. However, while (β_{at}, R_{at}) is changing very little, z_t given by (7) continues to vary quite a bit, owing to the imposition of the random shocks ϵ_t. Since β_{at} is not changing much while the z_t are, for large t the movement over long stretches of time of $(6a, b)$ and (7) is well approximated by the system that emerges when we replace α_{jt} by 1 and the terms in brackets on the right-hand side of $(6a)$ and (7) with their expected values, evaluated at the (nearly) fixed values $\beta_{at-1} = \tilde{\beta}_a$, $R_{at-1} = \tilde{R}_a$:

$$\beta'_{at} = \beta'_{at-1} + \frac{1}{t}\tilde{R}_a^{-1}[M_{z_a,z}(\tilde{\beta}_a)T(\tilde{\beta}_a)'e'_c - M_{z_a}(\tilde{\beta}_a)\tilde{\beta}'_a]$$

$$R_{at} = R_{at-1} + \frac{1}{t}[M_{z_a}(\tilde{\beta}_a) - \tilde{R}_a].$$

Using the definition of $S_a(\beta)$ in (3) and the approximation $\tilde{\beta}_a = \beta_{at-1}$ and $\tilde{R}_a = R_{at-1}$, the above equations become

$$\beta'_{at} = \beta'_{at-1} + \frac{1}{t}R_{at-1}^{-1}M_{z_a}(\beta_{at-1})[S(\beta_{at-1}) - \beta_{at-1}]'$$

$$R_{at} = R_{at-1} + \frac{1}{t}[M_{z_a}(\beta_{at-1}) - R_{at-1}]. \tag{11}$$

Equation (11) can be recognized as Euler's method for solving the ordinary differential equation (10) where the step size is $1/t$. Thus, for large t, the algorithms $(6a, b)$ and (7) provide a way of solving the differential equation (10). This suggests that the limiting behaviour of $(6a, b)$–(7) is governed by the trajectories of the associated differential equation (10). This concludes our transcription of the heuristic account of Ljung (1977) and Ljung and Soderstrom (1983).

The proof of the key parts of proposition 1 in effect involves verifying formally the approximations used in the various steps of the preceding heuristic argument. We refer the reader to Ljung (1977) and to Marcet and Sargent (1989a) for the proof. Partly as an aid to extending the proofs in Marcet and Sargent (1989a) to the present environment, we now briefly describe precisely the sense in which the model of Marcet and Sargent (1989a) is a special case of the present model.

We again work with the special case of the model that emerges when we set $a = b$, $c = d$, $e_c = e_d$, $\beta_a = \beta_b$, $R_a = R_b$. In the context of this special case, we use the following partitions of z_t:

$$z_t = \begin{bmatrix} z_{ct} \\ z^c_{ct} \end{bmatrix} = \begin{bmatrix} z^c_{at} \\ z_{at} \end{bmatrix},$$

where z^c_{ct} includes variables in z_t not in z_{ct}, and z^c_{at} includes variables in z_t not in z_{at}. Partitioning $T(\beta)$ conformably with the above partitions permits us to represent (2) in the form

$$\begin{bmatrix} z_{ct} \\ z^c_{ct} \end{bmatrix} = \begin{bmatrix} T_{11}(\beta) & T_{12}(\beta) \\ T_{21}(\beta) & T_{22}(\beta) \end{bmatrix} \begin{bmatrix} z^c_{at-1} \\ z_{at-1} \end{bmatrix} + V(\beta)\epsilon_t.$$

Using these partitions, it is possible to represent (5) as

$$S_a(\beta) = \begin{bmatrix} I_{n_c} & 0 \end{bmatrix} \begin{bmatrix} T_{11}(\beta) & T_{12}(\beta) \\ T_{21}(\beta) & T_{22}(\beta) \end{bmatrix} \begin{bmatrix} M_{z_a z^c_a}(\beta)' M_{z_a}(\beta)^{-1} \\ I_{n_a} \end{bmatrix}, \tag{12}$$

where I_{n_j} is the $n_j \times n_j$ identity matrix. In (12), the matrix $M_{z_a z^c_a}(\beta)' M_{z_a}(\beta)^{-1} = \gamma'$, where $\hat{E} z^c_{at} | z_{at} = \gamma z_{at}$ is the linear least-squares projection of z^c_{at} on z_{at}. Representation (12) shows the way in which e_c, $T(\beta)$, and the regression coefficient γ of excluded on included variables interact to compose $S_a(\beta)$.

Marcet and Sargent (1989a) restricted themselves to analysing the further special case of the present model, which emerges when (12) can be written with

$$T_{11} \equiv 0, \tag{13}$$

so that the perceived law of motion $E^*(z_{ct}|z_{at-1})$ omits no variables from z_t which the actual law of motion would imply belonged in the regression of z_{ct} on z_t. In this special case, in which from agents' point of view there are no hidden state variables, (12) simplifies to

$$S_a(\beta) = T_{12}(\beta),$$

and the differential equations (8) and (9) simplify to those studied by Marcet and Sargent (with the notation $T_{12}(\beta)$ matching the notation $T(\beta)$ in Marcet and Sargent 1989a, and with $T_{11}(\beta)$ being 0, and $[T_{21}(\beta), T_{22}(\beta)] = A(\beta)$ in Marcet and Sargent 1989a).

Marcet and Sargent (1989a) describe a variety of models in the literature which satisfy the no-hidden-states assumption (13). The models described in subsequent sections of this paper all violate that assumption. It bears emphasizing that, when (13) is violated, the concept of a limited-information, rational expectations equilibrium used in this paper makes the equilibrium depend on the model-builder's specification of the information sets z_{at}, z_{bt}. In models with sufficiently rich dynamic structures, in specifying z_{at} and z_{bt} the model-builder will be inducing a concept of equilibrium which depends on his truncating lag distributions in specifying (1). For example, the equilibrium described in Section 5 and in Marcet and Sargent (1989b) are sensitive to such truncation. Truncations are unavoidable so long as one stays with a finite-dimensional z_t vector. Sargent (1989) suggests a modification of structures (1) and (2) which in effect permits agents to condition on an infinite number of past values of the variables in their information sets.

3. A Model of Bray

This section applies proposition 1 to the model of Margaret Bray (1982). Although there are differentially informed agents in Bray's model, the model fits into the framework of Section 2 with the special settings $\beta_a = \beta_b$, $z_{at} = z_{bt}$, $e_c = e_d$, $R_a = R_b$. We confirm a conjecture of Bray concerning the necessity of the stability condition that she discovered.

There are N_u uninformed traders and N_i informed traders. Let (r_t, s_t) be a sequence of independently and identically distributed random (2×1) vectors which obey a bivariate normal distribution.[6] Here r_t denotes the return on an asset, and s_t denotes the exogenous supply of the asset. Informed agents observe s_t at t, while uninformed agents only observe p_t, the price of the asset. Informed agents know all the moments of the model, while uninformed agents do not, and use a linear least-squares learning scheme to decide their demand. Each informed trader's demand is given by

$$\theta_i[E(r_t|s_t) - p_t], \qquad \theta_i > 0,$$

where $E(r_t|s_t)$ is the mathematical expectation of r_t conditional on s_t. Each uninformed agent's demand is given by

$$\theta_u[E^*_{t-1}(r_t|p_t) - p_t], \qquad \theta_u > 0,$$

where $E^*_{t-1}(r_t|p_t) = a_{t-1} + b_{t-1}p_t$, and (a_{t-1}, b_{t-1}) are obtained from a least-squares regression of r_s on p_s for $s = 1, \ldots, t-1$. The equilibrium price at t is determined by equating supply and demand for the asset:

$$N_i\theta_i[E(r_t|s_t) - p_t] + N_u\theta_u[a_{t-1} + b_{t-1}p_t - p_t] = s_t.$$

We map this model into the setting of Section 2. We define

$$z_t = \begin{bmatrix} r_t \\ r_{t-1} \\ r_{t-2} \\ s_t \\ p_{t-1} \\ 1 \end{bmatrix}; \qquad \epsilon_t = \begin{bmatrix} r_t - E(r_t) \\ s_t - E(s_t) \end{bmatrix}; \qquad \beta_t = (a_{t-1}, b_{t-1})$$

$$z_{ct} = r_{t-2}; \qquad z_{at} = \begin{bmatrix} 1 \\ p_{t-1} \end{bmatrix}; \qquad \beta = (a, b).$$

[6] In Bray's (1982) notation, we assume that the information set $I_t = s_t$. Expanding I_t to include additional variables would modify the analytics in the text in only trivial ways. The state vector z_t would simply have to be extended to include the additional components of I_t.

The actual law of motion of z_t is given by

$$z_t = \begin{bmatrix} r_t \\ r_{t-1} \\ r_{t-2} \\ s_t \\ p_{t-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & Er_t \\ 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & Es_t \\ 0, & 0, & 0, & T^{54}(\beta_t), & 0, & T^{56}(\beta_t) \\ 0, & 0, & 0, & 0, & 0, & 1 \end{bmatrix} \times \begin{bmatrix} r_{t-1} \\ r_{t-2} \\ r_{t-3} \\ s_{t-1} \\ p_{t-2} \\ 1 \end{bmatrix} + V(\beta_t)\epsilon_t, \tag{14}$$

where, letting $\rho = \text{cov}(r_t, s_t)/\text{var}(s_t)$,

$$T^{54}(\beta) = \frac{N_i\theta_i\rho - 1}{N_u\theta_u(1 - b) + N_i\theta_i}$$

$$T^{56}(\beta) = \frac{N_u\theta_u a + N_i\theta_i[E(r_t) - E(s_t)\rho]}{N_u\theta_u(1 - b) + N_i\theta_i},$$

where $V(\cdot)$ is a constant function such that $V^{11}(\cdot) \equiv V^{42}(\cdot) \equiv 1$ and the rest of the elements of V are equal to 0. Note that the operator $T(\beta)$ is not well defined at points (a, b) for which $b = 1 + (N_i\theta_i)/(N_u\theta_u)$. Thus, for this model, the set D_s defined in Section 2 excludes such points. To invoke part (i) of our proposition, a nontrivial projection facility (6*b*) must be used.

Following Bray, define the random variable $x_t = [N_i\theta_i E(r_t|s_t) - s_t]$. Then calculations on (14) show that[7]

$$S(a, b) = [E(r_t) - kE(x_t)/(N_u\theta_u) - ka,\ k(N_i\theta_i + N_u\theta_u)/N_u\theta_u - kb], \tag{15}$$

where, as in Bray, $k = N_u\theta_u/[N_i\theta_i - 1/\rho]$.

[7] The mapping S can be derived from formula (5). Alternatively, it can be found directly by computing the linear projection of z_{ct} on z_{at-1} for fixed beliefs β, and letting $S(\beta)$ be the coefficients multiplying z_{at-1} in this projection. For Bray's model this method is probably the most convenient: fix $\beta = (a, b)$; then, using $p_t = T^{56}(\beta) + T^{54}(\beta)s_t$, we have

$$\begin{aligned} E(r_t|p_t) &= E(r_t) - \rho\, E(s_t) - \rho\, T^{56}(\beta)/T^{54}(\beta) + \rho\, p_t/T^{54}(\beta) \\ &= \{E(r_t) - k[N_i\theta_i E(r_t) - E(s_t)]/N_u\theta_u - k\, a\} \\ &\quad + \{k(N_i\theta_i + N_u\theta_u)(1 - b)/N_u\theta_u\} pt \\ &\equiv S_1(\beta) + S_2(\beta)p_t. \end{aligned}$$

It is readily verified that the fixed point $(a^*, b^*) = S(a^*, b^*)$ is given by Bray's formulas (3.10), (3.11). Differentiating the right-hand side of (15) and evaluating at (a^*, b^*) gives

$$\left.\frac{\partial[S(a, b) - (a, b)]}{\partial(a, b)}\right|_{\substack{a=a^* \\ b=b^*}} = \begin{bmatrix} -k - 1, & 0 \\ 0, & -k - 1 \end{bmatrix}, \qquad (16)$$

which has a repeated eigenvalue of $(-k - 1)$. Using parts (i), (iii), and (iv) of proposition 1, we can establish that the learning model is locally stable if and only if $k > -1$.[8] Whether or not this inequality is satisfied will depend on the values of the parameters of the model. Using (15), when $Es_t = Er_t = 0$, the unstable region of the parameter space can be deduced to be the shaded region depicted in Fig. 6.2, namely:

- If $1/\rho < 0$, then the stability region includes all $(N_i\theta_i, N_u\theta_u) \in R^2_+$.
- If $1/\rho > 0$, then the stability region includes all $(N_i\theta_i, N_u\theta_u)$ satisfying $0 < N_i\theta_i < 1/\rho$, *and* $N_u\theta_u + N_i\theta_i > 1/\rho$.

We note that, given any pair (θ_i, θ_u), any $1/\rho > 0$, and any positive ratio N_i/N_u of informed to uninformed agents, there exists an absolute number of traders $N_u + N_i$ sufficiently small to guarantee local stability of the learning mechanism. Under the same conditions, there also exists a number of traders sufficiently large to guarantee local stability.

Notice that all of our results are local, being based on the parts of proposition 1 that depend on an analysis of the 'small' ordinary differential equation (9). Parts (i) and (iv) of the proposition assure us

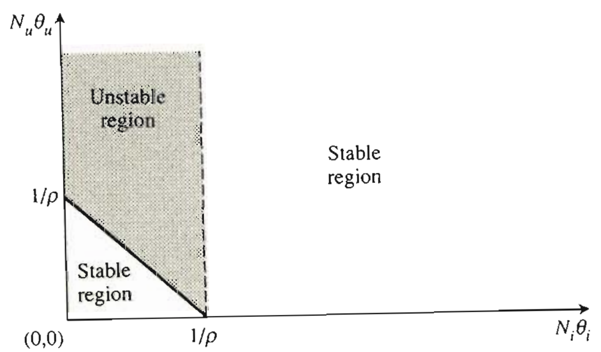

FIG. 6.2. Stability region of parameter space for Bray's model

[8] For Bray's model, as well as the models of Sections 4 and 5, it is straightforward to verify that assumptions (A1)–(A5) are satisfied. Assumption (A6) is satisfied since, if D_1 is bounded, $|p_t| < K_1 + K_2|s_t|$ for some constants K_1, K_2. Normality of s_t then implies (A6).

that there exists some nontrivial choice of set D_{1a} under which almost sure convergence obtains, but are silent about how large that set can be. To get a stronger result, the 'larger' ordinary differential equation (8) must be analysed, and in particular its trajectories must be verified to point towards the interior of D_{1a} at points on the boundary of D_{1a}. For a model as complicated as Bray's, in which z_{at} involves endogenous variables and the $S(\beta)$ operator involves a nontrivial operator $M_{z_a}(\beta)^{-1}M_{z_az_c}(\beta)'$, it is difficult to carry out this analysis analytically. Numerical techniques, like those illustrated for Townsend's model by Sargent (1991), could be used.

Note that assertion (iii) of proposition 1 confirms the conjecture about conditions for non-convergence made by Bray (1982).

4. Convergence in a Version of Frydman's Model

We briefly study least squares learning in a variant of a model due to Roman Frydman (1982). We alter the learning scheme relative to the one used by Frydman, but retain the economic structure contributed by him. The resulting model provides a useful technical contrast to Bray's, it being much easier to obtain global convergence results because agents are regressing on exogenous variables.

There are n competitive firms indexed by $i = 1, \ldots, n$. Demand for the firms' output is given by

$$p_t = a - b \sum_{i=1}^{n} y_{it} + u_t, \qquad a, b > 0,$$

where y_{it} is output of the ith firm, and u_t is an independently and identically distributed disturbance distributed $N(0, \sigma_u^2)$. Firm i's cost function at t is given by

$$C(y_{it}) = (1/2s)(y_{it})^2 + k_{it}y_{it} + C_f,$$

where C_f and s are positive constants and k_{it} is a random variable given by

$$k_{it} = \alpha_t + \epsilon_{it}, \qquad \text{for all } i, t,$$

where α and ϵ_i are independently and identically distributed with $\alpha_t \sim N(0, \sigma_\alpha^2)$ and $\epsilon_{it} \sim N(0, \sigma_\epsilon^2)$.

Firm i observes k_{it} at time t, but not p_t at t. The firm observes a history of $\{p_s, k_{is}\}$ for $s = 0, \ldots, t - 1$, and forms an expectation of p_t at time t according to

$$E^*(p_t|k_{it}) = \beta^i_{1t} + \beta^i_{2t}k_{it}, \qquad i = 1, \ldots, n$$

where $(\beta^i_{1t}, \beta^i_{2t})$ are least-squares regression coefficients based on

$\{p_s, k_{is}\}$, $s = 0, \ldots, t-1$. Firm i's output at t is given by $s[E^*(p_t|k_{it}) - k_{it}]$. The equilibrium price at t is determined from

$$p_t = a - bs \sum_{i=1}^{n} [E^*(p_t|k_{it}) - k_{it}] + u_t.$$

We show how the system behaves for the case $n = 2$. To map this model into the setup of Section 2, define

$$z_t = \begin{bmatrix} k_{1t} \\ k_{2t} \\ p_{t-1} \\ 1 \end{bmatrix}, \qquad \epsilon_t = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \alpha_t \\ u_{t-1} \end{bmatrix}$$

$$z_{ct} = z_{dt} = p_{t-1}; \qquad z_{at} = k_{1t}; \qquad z_{bt} = k_{2t}.$$

The transition law corresponding to (2) is readily verified to be

$$z_t = \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ T_{31}(\beta), & T_{32}(\beta), & 0, & T_{34}(\beta) \\ 0, & 0, & 0, & 1 \end{bmatrix} z_{t-1} + V(\beta)\epsilon_t,$$

where $V(\cdot)$ is a constant function, with zeroes everywhere except for elements V_{11}, V_{13}, V_{22}, V_{23}, V_{34}, which are unity; and where

$$T_{31}(\beta) = -bs(\beta_2^1 - 1)$$
$$T_{32}(\beta) = -bs(\beta_2^2 - 1)$$
$$T_{34}(\beta) = (a - bs)(\beta_1^1 + \beta_1^2).$$

It follows that

$$\begin{bmatrix} S_a(\beta) \\ S_b(\beta) \end{bmatrix} = \begin{bmatrix} a - bs(\beta_1^1 + \beta_1^2, & -bs[\beta_2^1 - 1 + (\beta_2^2 - 1)\gamma] \\ a - bs(\beta_1^1 + \beta_1^2), & -bs[(\beta_2^1 - 1)\gamma + \beta_2^2 - 1] \end{bmatrix},$$

where $\gamma = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\epsilon^2)$. Notice that $0 < \gamma < 1$. The rational expectations equilibrium is given by

$$\beta_{1f}^1 = \beta_{1f}^2 = a/(1 + 2bs)$$
$$\beta_{2f}^1 = \beta_{2f}^2 = bs(1 + \gamma)/[1 + bs(1 + \gamma)].$$

Evaluating the derivative of

$$\text{col}\begin{bmatrix} S_a(\beta) - \beta_a \\ S_b(\beta) - \beta_b \end{bmatrix}$$

with respect to β at $\beta = \beta_f$, we obtain

$$\begin{bmatrix} -(bs + 1), & 0, & -bs, & 0 \\ 0, & -(bs + 1), & 0, & -bs\gamma \\ -bs, & 0, & -(bs + 1), & 0 \\ 0, & -bs\gamma, & 0, & -(bs + 1) \end{bmatrix}.$$

The eigenvalues of this matrix are[9]

$$[-(1 + 2bs), -1, -bs(1 + \gamma) - 1, -bs(1 - \gamma) - 1].$$

Since $b > 0$, the eigenvalues are all less than 0, and, by proposition 1(iv), the associated differential equation (9) is locally stable. Furthermore, a version of corollary 2 of Marcet and Sargent (1989*a*) implies that proposition 1(i) applies with sets D_{1a}, D_{1b} chosen as

$$D_{1a} = \{(\beta_a, R_a) \| \beta_a - \beta_{af}| < K_a\}$$
$$D_{1b} = \{(\beta_b, R_b) \| \beta_b - \beta_{bf}| < K_b\},$$

for arbitrarily large positive constants K_a and K_b. This means that, for arbitrarily large sets D_{1a} and D_{1b} in the projection facilities (6*b*), proposition 1(i) holds. Thus, we get a strong global convergence result for this model, in contrast to Bray's model, in which a much more difficult analysis would have to be used to achieve such a result. The reason things are simpler in the current model depends on the fact that, since k_{jt} is exogenous for each j, $M_{z_j}(\beta)$ is independent of β for $j = a, b$, making S linear and corollary 2 of Marcet and Sargent (1989*a*) applicable.

It is evident how the analysis of this section could be extended to handle larger values of n.

5. Cagan's Hyperinflation Model and 'Bubbles'

This section applies proposition 1 to study a version of Cagan's hyperinflation model with a restricted information set. We analyse the ordinary differential equation (9) and display a set D_{1a} for which proposition 1(i) applies. We also describe some of the insights that (9) seems to yield about this model even for (β, R) pairs lying outside the set D_s defined in Section 2.

Consider the model

$$y_t = \lambda E^*(y_{t+1}|y_t) + x_t + v_t$$
$$x_t = \rho x_{t-1} + u_t, \tag{17}$$

where $|\rho|, |\lambda| < 1$; $Eu_t = Ev_t = Eu_s v_t = 0$ for all t, s, $Eu_t^2 = \sigma_u^2$, $Ev_t^2 =$

[9] The matrix of eigenvectors is

$$\begin{bmatrix} 1, & -1, & 0, & 0 \\ 0, & 0, & 1, & -1 \\ 1, & 1, & 0, & 0 \\ 0, & 0, & 1, & 1 \end{bmatrix}.$$

$\sigma_v^2 > 0$. The processes u_t and v_t are assumed serially uncorrelated.[10] Agents are restricted to forming expectations about future y on the basis of one lagged value of y. We study a learning version of the model in which

$$E^*(y_{t+1}|y_t) = \beta_t y_t, \tag{18}$$

where β_t is the least-squares estimator of the regression of y_s on y_{s-1} based on data for dates $s = 1, \ldots, t-1$.

This model falls within the setting of Section 2, with $\beta_a = \beta_b$, $R_a = R_b$, $z_{at} = z_{bt} = z_{ct} = z_{dt}$. Define

$$z_t = \begin{bmatrix} y_t \\ x_t \end{bmatrix}; \qquad z_{at} = z_{bt} = y_t; \qquad \epsilon_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix}.$$

The model can be represented by

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0, & \rho/(1-\lambda\beta_t) \\ 0, & \rho \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} 1/(1-\lambda\beta_t), & 1/(1-\lambda\beta_t) \\ 1, & 0 \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix}. \tag{19}$$

It can be verified from (19) that

$$M_{z_a,z}(\beta) = E\begin{bmatrix} y_t \\ x_t \end{bmatrix} y_t = \begin{bmatrix} \dfrac{\sigma_x^2 + \sigma_v^2}{(1-\lambda\beta)^2} \\ \dfrac{\sigma_x^2}{1-\lambda\beta} \end{bmatrix}$$

$$M_{z_a}(\beta) = E(y_t)^2 = \frac{\sigma_x^2 + \sigma_v^2}{(1-\lambda\beta)^2}, \tag{20}$$

where $\sigma_x^2 = \sigma_u^2/(1-\rho^2)$. Using the definition of $T(\beta)$ associated with (19) and formulas (20), we have that

$$S(\beta) = \rho\sigma_x^2/(\sigma_x^2 + \sigma_v^2).$$

A stationary rational expectations equilibrium is determined by

$$\beta_f = \frac{\rho\sigma_x^2}{\sigma_x^2 + \sigma_v^2} \quad \text{and} \quad R_f = \frac{\sigma_x^2 + \sigma_v^2}{(1-\lambda\beta_f)^2} \tag{21}$$

Corresponding to (8), we have the associated differential equation

[10] This model is a version of one used by Shiller (1981) and LeRoy and Porter (1982), where y_t is interpreted as the price of a stock and x_t is its dividend. It is also Sargent and Wallace's (1973) version of Cagan's (1956) model of hyperinflation, with y_t representing the log of the price level and x_t the log of the money supply.

$$\left(\frac{d}{dt}\right)\begin{bmatrix} \beta \\ R \end{bmatrix} = \begin{bmatrix} R^{-1}[K_1 - \beta K_2]/(1-\lambda\beta)^2 \\ K_2/(1-\lambda\beta)^2 - R \end{bmatrix}, \tag{22}$$

where $K_1 = \rho\sigma_x^2$ and $K_2 = \sigma_x^2 + \sigma_v^2$. Corresponding to (9), we have

$$\left(\frac{d}{dt}\right)\beta = \frac{\rho\sigma_x^2}{\sigma_v^2 + \sigma_x^2} - \beta. \tag{23}$$

Examination of 23 reveals that it is locally stable about the fixed point β_f. Therefore, by proposition 1(iv), (22) is also locally stable about (β_f, R_f). More can be inferred about (22) from Fig. 6.3, which displays the phase diagram in the (β, R) plane, and Fig. 6.4, which displays some trajectories of (22). Note the vertical line at λ^{-1}, a value of β at which there is a singularity of the right-hand side of (22). Trajectories originating on the right-hand side of this vertical line can never cross

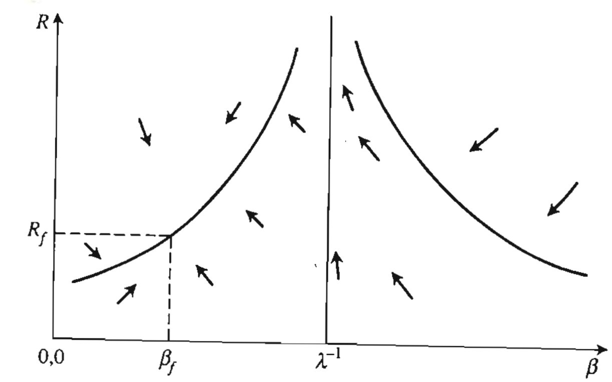

FIG. 6.3. Phase diagram of the ordinary differential equation associated with the hyperinflation model

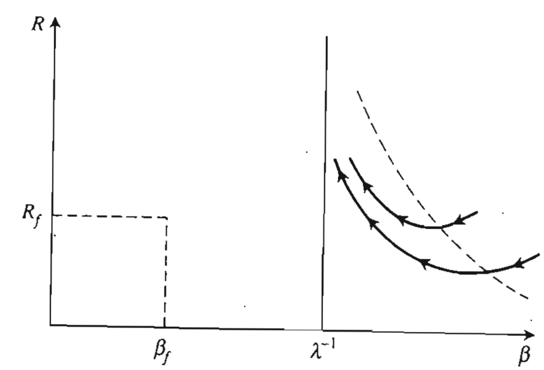

FIG. 6.4. Some trajectories of the ordinary differential equation associated with the hyperinflation model that start to the right of the singularity at λ^{-1}

this line, and so cannot converge to (β_f, R_f). The phase diagram informs us that trajectories starting from $\beta(0) > \lambda^{-1}$ and $R(0) > 0$ have $\lim_{t\to\infty}(\beta t), R(t)) = (\lambda^{-1}, +\infty)$.

The singularity on the right-hand side of (22) has an interpretation in terms of 'bubbles'. The solutions of the model (17) under rational expectations and no learning, where agents' information set is extended to include x_t at t, are given by

$$y_t = \frac{x_t}{1 - \lambda\rho} + c_t\lambda^{-t} + v_t,$$

where c_t is any martingale. The term $c_t\lambda^{-t}$ is called a 'bubble'. Note that one martingale is the process $c_t = c^{-t}x_t$ for any constant c. Thus, one rational expectations equilibrium with a bubble is given by

$$y_t = \frac{x_t}{1 - \lambda\rho} + c(\rho\lambda)^{-t}x_t + v_t,$$

for any constant $c \neq 0$. It can be verified that, along any such solution with $c \neq 0$,

$$\lim_{t\to\infty} \hat{E}(y_{t+1}|y_t) = \lambda^{-1}y_t$$

$$\lim_{t\to\infty} \frac{1}{t}\sum_{s=0}^{t} y_s^2 = +\infty.$$

Thus, the singularity of (22) at $\beta = \lambda^{-1}$ corresponds to a 'bubble' solution of the model. In specifying that agents form expectations by regressing y_{t+1} on y_t, it seems that we have left open the possibility that the model with learning converges to a non-stationary 'bubble'. We shall discuss this possibility below, and shall argue informally that the learning system is unlikely to converge to a bubble solution, even though the differential equation (22) does converge to values of (β, R) associated with a bubble solution.

To satisfy the hypotheses of proposition 1(i), we have to use a projection facility that keeps β_t from crossing the singular value λ^{-1}. A set that works is formed by selecting two real numbers $\bar{\epsilon} > 0$, $K > 0$ and defining

$$D_{1a} = \{(\beta, R)\big|\beta < \lambda^{-1} - \bar{\epsilon} \quad \text{and} \quad |\beta - \beta_f| < K\}.$$

For such a D_{1a} set, with K arbitrarily large and $\bar{\epsilon} > 0$ arbitrarily small, assumption (A7) is satisfied; in particular, the trajectories of (22) point towards the interior of D_{1a} at points on the boundary of D_{1a}. With such a D_{1a}, proposition 1(i) applies for this model.

With the preceding statement, we have exhausted what we can say about global convergence of our model on the basis of proposition 1. However, this is not the end of the matter because proposition 1(i) states sufficient, but not necessary, conditions for global convergence of the learning scheme. What would occur if there is no projection facility (i.e. if $D_{1a} = \mathbb{R}^2$), so that β can exceed λ^{-1}? We can gain some additional understanding about the behaviour of the system when β exceeds λ^{-1} by studying for large t how closely the recursive algorithm (6) keeps to the trajectories of (22). (Notice that the heuristic argument of Ljung cited above applies in the region for which $\beta_t > \lambda^{-1}$, breaking down only at the point $\beta = \lambda^{-1}$.) We have simulated (17)–(18) many times to address this issue, starting the simulations with $\beta > \lambda^{-1}$ and using normally distributed and orthogonal pseudo-random numbers for v_t and u_t in (17)–(18). We started the simulations at a large value for t, to make system (17)–(18) mimic (22) well from the beginning of our simulations. We used no projection facility. From every one of these simulations, the following pattern emerged. So long as β_t stayed far enough to the right of the singularity of (22) at λ^{-1}, the simulations traced out paths that mimicked the trajectories of (22). However, for β_t close to λ^{-1}, the stochastic version of 'Euler's method' that (17)–(18) constitutes turns out to be a bad one for solving differential equation (22). In particular, for every simulation that we generated, (β_t, R_t) stayed close to trajectories of (22) until β_t approached λ^{-1} so closely that eventually β_t dropped below λ^{-1}. After this event occurred, (β_t, R_t) in each case seemed to follow trajectories of (22) for $\beta < \lambda^{-1}$, with (β_t, R_t) approaching the fixed point (β_f, R_f).

Figure 6.5 reports one representative simulation in which we set initial conditions in order to start the system to the right of the singularity at $\beta = \lambda^{-1}$. We set $\lambda = 0.5$, $\rho = 0.5$ $Eu_t^2 = Ev_t^2 = 1$, $\beta_{t_0} = 2.4$ and $R_{t_0} = 58.3333$. This choice of (β_{t_0}, R_{t_0}) sets $\beta_{t_0} > \lambda^{-1}$ and sets R at a value satisfying $(d/dt)R = 0$ in (22). We set the initial time t_0 equal to 10,000 in order to make the differential equation system (22) well approximate the stochastic system (17)–(18) that we are simulating. (Recall the role that a large value of t plays in the heuristic argument reported in Section 2.) The results of a simulation of 20,000 periods are reported in Fig. 10.4. Except for very close to the singularity at λ^{-1} ($= 2$), the behaviour of the simulated system (17)–(18) clings close to the trajectories of (22).

Here is our interpretation of these simulations. With no operative projection facility, the learning mechanism has a recursive representation of the form

$$\theta_{t+1} = \theta_t + (1/t)Q(z_t, \theta_t), \qquad (24)$$

where $\theta_t = (\beta_t, R_t)$. This is a stochastic version of Euler's method for solving the differential equation $(d/dt)\theta = f(\theta) \equiv E\,Q(z_t, \theta)$, namely,

$$\bar{\theta}_{t+1} = \bar{\theta}_t + (1/t)f(\bar{\theta}_t). \qquad (25)$$

Now given the behaviour of the function f associated with (22) in a neighbourhood of $\beta = \lambda^{-1}$, the linear nature of the approximation involved in using (24) to mimic $(d/dt)\theta = f(\theta)$ is disastrous, because it involves taking a step of positive length along a straight line that crosses λ^{-1} (see Fig. 6.5). The small variation arising from randomness in $Q(z_t, \theta_t)$ makes it even more likely that eventually (β_t, R_t) will jump over the vertical line at λ^{-1} into the region $(-\infty, \lambda^{-1})$, which is in the domain of attraction of (β_f, R_f). Once inside this region, all of the simulations we computed again stuck close to the trajectories of (22).

We also computed trajectories of the ordinary differential equation (22) using both Euler's method and the Runge–Kutta method. In all cases, when the trajectory was started out to the right of λ^{-1}, the trajectory crossed λ^{-1} and converged to (β_f, R_f). This illustrates how the inability of the learning algorithm to converge to a bubble non-stationary equilibrium is associated with the numerical properties of methods like Euler's for recovering the non-stationary solutions of (22).

In summary, even though trajectories of the associated differential equation cannot cross the vertical line through λ^{-1}, the random difference equation (6) for (β_t, R_t) *can* cross λ^{-1} from above, and always did so in our simulations. Our analysis fails to prove that (β_t, R_t) from (17)–(18) cannot converge to the bubble solution $(\lambda^{-1}, +\infty)$, but it does indicate the existence of forces impeding convergence to such a solution.[11,12]

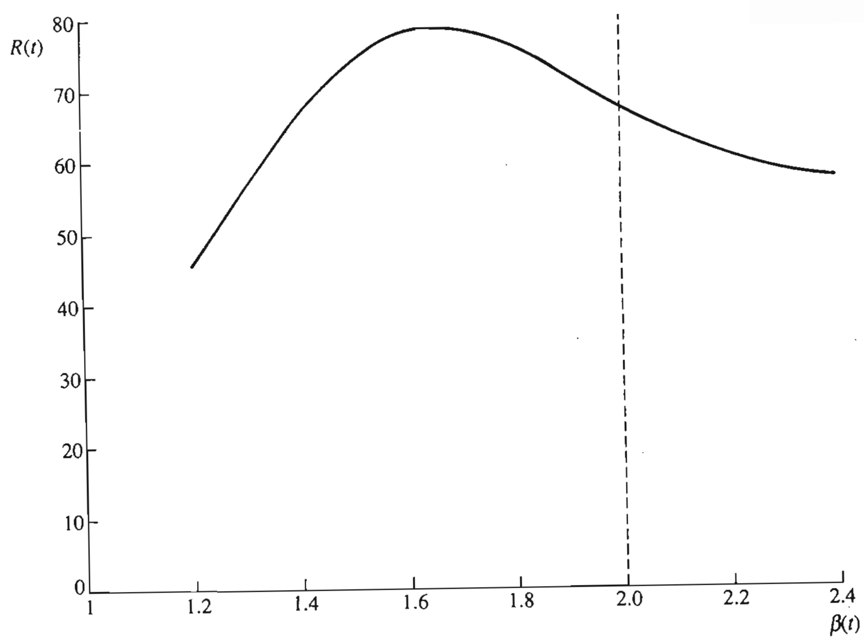

FIG. 6.5. Simulation of hyperinflation model with initial conditions to the right of the singularity at $\lambda^{-1} = 2$ (20,000 observations)

6. Conclusions

This paper has described the limiting behaviour of a class of self-referential systems in which differentially and imperfectly informed agents affect the motion of the system through their learning by least squares. We have described the senses in which the limiting behaviour of the system is governed by a particular ordinary differential equation. We have displayed some examples of rational expectations equilibria in which the learning schemes converge to a rational expectations equilibrium (Bray's model for some parameter settings, Frydman's model, and the hyperinflation model). We have also encountered cases in which the learning scheme does not produce convergences to a rational expectations equilibrium (Bray's model for some parameter settings) and some rational expectations equlibria from which the learning scheme is repelled (the bubbles equilibria of the hyperinflation model). These results all obtain in environments that satisfy Margaret Bray's theorem that, if the learning scheme converges, it must converge to a rational expectations equilibrium.

A remarkable feature of these results is how often a least-squares learning scheme performs well in eventually settling down to a rational expectations equilibrium. In fact, either the least-squares learning scheme or the ordinary differential equation associated with it can suggest effective algorithms for *computing* a rational expectations equilibrium for applied work.

When we encounter the strong sort of convergence to a rational expectations equilibrium embodied in much of the literature we have surveyed, one reaction might be that we analysts have chosen to make the learners 'too smart', and that the convergence results are the consequence of our choice. After all, although sequential application of linear least squares to vector autoregression is *not* rational for these

[11] The failure of our simulations to converge to $\beta = \lambda^{-1}$ is related to the numerical problems that arise in solving systems of Euler equations or Hamiltonian equations 'forward' in time on a computer. In these settings, once one gets close to the optimum, very small rounding errors propel the system away from the optimum solution.

[12] The presence of the random variable z_t in (25) means that in our system there can be a positive probability that β_t jumps back above λ^{-1} from below. For example, with normally distributed (u_t, v_t) in (17), even though remote, such an event has positive probability. This possibility would complicate the task of proving the conjecture that $P[\beta_t \rightarrow \lambda^{-1}] = 0$, which the behaviour of our simulations might suggest.

environments (again, see Bray and Kreps 1987), it is still a pretty sophisticated method. Further, in the setups that appear in this literature, the agents are typically supposed to be quite smart about all aspects of their decision making *except* their forecasting decisions. For example, agents typically are assumed to use the correct intertemporal marginal conditions. In effect, agents are supposed to be using state-of-the-art 'adaptive control' techniques (see e.g. Goodwin and Sin 1983).

The next frontier in the study of learning in self-referential systems is to study the consequences of withdrawing from agents some of the knowledge that has been attributed to them in the least-squares learning literature. The literature on genetic algorithms in artificial intelligence is a good source of ideas on how to proceed (see e.g. Holland 1975).

Appendix

We state six assumptions that we make about system (6*a*, *b*)–(7).

ASSUMPTION A1. The operator S has a unique fixed point $\beta_f = S(\beta_f)$ which satisfies $\beta_f \in D_s$.

ASSUMPTION A2. For $\beta \in D_s$, T is twice differentiable and V has one derivative.

ASSUMPTION A3. The covariance matrices M_{z_j} (β_f) are non-singular for $j = a, b$.

ASSUMPTION A4. For $j = a, b$ and for all t, $\alpha_{jt} > 0$; α_{jt} is increasing in t; $\alpha_{jt} \to 1$ as $t \to \infty$; and

$$\limsup_{t\to\infty} t|\alpha_{jt} - \alpha_{jt-1}| = K_j < \infty, \qquad j = a, b.$$

ASSUMPTION A5. The vector ϵ_t consists of m stationary random variables; ϵ_t is serially independent. Further, $E|\epsilon_{it}|^p < \infty$ for all $p > 1$, all $i = 1, \ldots, m$.

ASSUMPTION A6. There exists a subset Ω_0 of the sample space with $P(\Omega_0) = 1$, four random variables $C_a(\omega)$, $C_b(\omega)$, $G_a(\omega)$, $G_b(\omega)$, and a subsequence $\{t_h(\omega)\}$ for which

$$|z_{jt_h}(\omega)| < C_j(\omega) \qquad j = a, b$$

$$|R_{jt_h}(\omega)| < G_j(\omega) \qquad j = a, b$$

for all $\omega \in \Omega_0$ and all $h = 1, 2, \ldots$.

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PART III

Economic Fluctuations: Endogenous *v.* Exogenous Explanations