# Implementing a Ramsey Plan*

Wei Jiang† Thomas J. Sargent‡ Neng Wang§

May 22, 2024

**Abstract**

We study how to induce future governments to continue a fiscal plan chosen by a benevolent government at time 0, an issue analyzed by Lucas and Stokey (1983). We implement a Ramsey plan by adding instantaneous debt to the contractible subspace and requiring each continuation government to preserve that debt's purchasing power over the next instant. After first using a Lagrangian to solve the Ramsey problem, we formulate the problem recursively and then apply that formulation to settings with various debt term structures and government spending processes. We extract implications about tax smoothing and effects of fiscal policies on bond markets.

**Keywords**: Ramsey planner, continuation Ramsey planner, implementability, instantaneous debt, dynamic programming.

JEL Classification:

*We thank Marco Bassetto, Pierre Yared for helpful comments.

†Department of Industrial Engineering and Decision Analytics, Hong Kong University of Science and Technology. E-mail: weijiang@ust.hk

‡New York University. E-mail: thomas.sargent@nyu.edu

§CKGSB and NBER. E-mail: neng.wang@columbia.edu

# 1 Introduction

This paper resolves a difficulty that Debortoli, Nunes, and Yared (2021) detected with the prescription for an optimal term structure of government debt proposed by Lucas and Stokey (1983, Sec. 3). In Lucas and Stokey's model, each of a sequence of governments is required to finance an exogenous and immutable joint stochastic process for government expenditures $\{G_t\}_{t=0}^{\infty}$ and debt service coupons $\{b_{0,t}\}_{t=0}^{\infty}$. At time 0, a Ramsey planner chooses a process of distorting flat rate taxes and possibly a restructured debt service coupon process $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$. At times $t > 0$, continuation Ramsey planners are free to redesign the flat rate tax process and to reschedule government debt from $t$ onward, but they must honor the continuation debt service coupon process that they inherited. Lucas and Stokey provided examples in which carefully restructured term structures of government debt induce continuation planners to choose to continue the original Ramsey tax rate process. However, Debortoli, Nunes, and Yared (2021) constructed examples in which Lucas and Stokey's way of restructuring government debt does not induce continuation planners to confirm the Ramsey plan.

To help set the stage for our expansion of the Ramsey planner's contractible subspace relative to Lucas and Stokey's, it is useful to read how Aguiar et al. (2019) summarized and contrasted Lucas and Stokey's (1983) model with theirs:

> *Lucas and Stokey (1983) studied optimal fiscal policy with complete markets and discussed at length how maturity choice is a useful tool to provide incentives to a government that lacks commitment to taxes and debt issuance, but cannot default. The government has an incentive to manipulate the risk-free real interest rate, by changing taxes which affects investors' marginal utility, to alter the value of outstanding long-term bonds, something ruled out by our small open-economy framework with risk-neutral investors. Their main result is that the maturity of debt should be spread out, resembling the issuance of consols. Our model instead emphasizes default risk, something absent from their work. Our main result is also the reverse, providing a force for the exclusive use of short-term debt.*

Short-term debt also plays an essential role in our model. But unlike Aguiar et al., we retain almost all other parts of Lucas and Stokey's and Debortoli et al.'s structure, including complete markets, a closed economy setting, the presence of incentives that continuation Ramsey planners have to use flat rate taxes to manipulate interest rates, and obligations of governments to honor all government debt that they inherit. To Lucas and Stokey's and Debortoli et al.'s contractable subspace for debt restructurings $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$, we add an instantaneous debt balance $\widehat{B}_t$ that time-$t$ continuation Ramsey planners must service, together with a local commitment condition. By managing $\widehat{B}_t$ appropriately, Lucas and Stokey's Ramsey plan can be implemented without restructuring $\{b_{0,t}\}_{t=0}^{\infty}$, i.e., by setting $\{\hat{b}_{0,t}\}_{t=0}^{\infty} = \{b_{0,t}\}_{t=0}^{\infty}$. (A Ramsey plan that can be implemented is typically described as being "time consistent.")[1] See Table 1 to see how the contractible subspace in our model compares with Lucas and Stokey's and Debortoli et al.'s.

To explain how we implement Lucas and Stokey's Ramsey plan, we note that bond price process $\{q^*_{0,u}; u \geqslant 0\}$ and a primary surplus process $\{S^*_{0,u}; u \geqslant 0\}$ are affiliated with a Ramsey plan. Together with the initial term debt structure $\{b_{0,u}; u \geqslant 0\}$, these two objects uniquely determine a process $\Pi^*_t = \int_0^t \frac{q^*_{0,u}}{q^*_{0,t}}\left(b_{0,u} - S^*_{0,u}\right)du$ that accumulates the government's unpaid liabilities from time 0 to $t$, conditional on the government not having rescheduled the initial term debt process $\{b_{0,t}\}_{t=0}^{\infty}$ before time $t$. Evidently. multiplication by the price $\frac{q^*_{0,u}}{q^*_{0,t}}$ converts the time $u$ flow liability $\left(b_{0,u} - S^*_{0,u}\right)$ into the time $t$ stock government liability $\Pi^*_t$. Our implementation of the Ramsey plan never restructures the initial debt term structure $\{b_{0,t}\}_{t=0}^{\infty}$ and instructs continuation governments to issue only instantaneous debt that has infinitesimal maturity. This arrangement takes into account all relevant quantity-and-equilibrium-price information about the initial term debt structure that concerns the Ramsey planner.

The Ramsey planner thus implements its plan by leaving the initial debt term structure untouched and paying for all government purchases and debt servicing by issuing instantaneous debt. We require that continuation Ramsey planners to preserve the "purchasing power" of the government's cumulative deficits, defined as the product of the government's accumulated primary deficits $\Pi^*_t$ and the representative household's marginal utility of consumption $U^*_{C,t}$. The Ramsey planner leaves each future government an instantaneous debt balance $\widehat{B}_t$ that equals $\Pi^*_t$ and insists that continuation Ramsey planners preserve its purchasing power.

Because instantaneous debt has the shortest possible maturity, among all government debts its value is least vulnerable to manipulation by future governments. Nevertheless, future governments in the guise of continuation planners have incentives to alter the con-

[1] Ljungqvist and Sargent (2025, ch. 21) assume that the government has access to only one-period debt in a discrete time version of Lucas and Stokey's model. It plays a counterpart to instantaneous debt in our continuous time model. It also leads to Bellman equations that can be used to express a discrete-time counterpart to the local commitment constraint that we impose in this paper.

tinuation Ramsey tax plan in order to manipulate government debt's purchasing power. Therefore, implementing the Ramsey plan requires that we supplement Lucas and Stokey's assumptions by adding our local commitment condition that requires that continuation Ramsey planners preserve the "purchasing power" of instantaneous government debt. This additional commitment is sufficient to induce them to confirm the Ramsey plan.

Table 1: Comparison with existing methods

| | Contractible space | Implementable? |
|---|---|---|
| Lucas and Stokey (1983) | $\{\hat{b}_{0,t}\}_{t=0}^{\infty} \neq \{b_{0,t}\}_{t=0}^{\infty}$ | Not always |
| Debortoli, Nunes, and Yared (2021) | $\{\hat{b}_{0,t}\}_{t=0}^{\infty} \neq \{b_{0,t}\}_{t=0}^{\infty}$ | Not always |
| This paper | $\widehat{B}_t$, $\{\hat{b}_{0,t}\}_{t=0}^{\infty} = \{b_{0,t}\}_{t=0}^{\infty}$ Local commitment | Always |

Section 2 sets out the environment for our continuous time version of Debortoli et al.'s model and poses the Ramsey problem. Section 3 formulates the Ramsey planner's Lagrangian and uses it to construct a Ramsey plan. Section 4 synthesizes a version of Debortoli et al.'s counterexample that prevails in our continuous time model. Section 5 shows how the Ramsey plan can be implemented in Debortoli et al.'s economy once we expand the contractible subspace for government debt by adding instantaneous debt and an additional local commitment not to dilute that debt. Section 6 uses the product of instantaneous debt and the representative consumer's marginal utility of consumption to define a state variable that allows us to formulate the Ramsey problem recursively. Sections 7 and 8 explain how instantaneous debt helps to implement the Ramsey plan. Subsection 8.2 argues that our local commitment condition is a mild requirement. Section 9 concludes. Appendix A provides some technical details.

# 2 Environment

Time $t \in [0, \infty)$ is continuous. A representative household and benevolent government participate in a complete set of perfectly competitive markets. The government finances an exogenous stream of expenditures with a stream of distorting flat tax rates. There is no uncertainty. At time 0, a Ramsey planner selects the best competitive equilibrium with distorting taxes.

**Assumption 1.** *Exogenous flows of government expenditures* $\overrightarrow{G_0} = \{G_t; t \geqslant 0\}$ *and debt-service payouts* $\overrightarrow{b_0} = \{b_{0,t}; t \geqslant 0\}$ *are Càdlàg processes, i.e.,* $\lim_{s\downarrow t} b_{0,s} = b_{0,t}$ *and* $\lim_{s\downarrow t} G_s = G_t$ *for all* $t \geqslant 0$*; and* $\lim_{s\uparrow t} b_{0,s}$ *and* $\lim_{s\uparrow t} G_s$ *exist for all* $t > 0$*.*

The representative household supplies a labor process $\overrightarrow{N_0} = \{N_{0,t}; t \geqslant 0\}$ that produces a flow of a single nonstorable good that can be divided between a consumption flow $\overrightarrow{C_0} = \{C_{0,t}; t \geqslant 0\}$ and a government expenditure flow $\overrightarrow{G_0}$:

$$C_{0,t} + G_t = N_{0,t}\,, \text{ for all } t \geqslant 0\,. \tag{1}$$

At time 0, the representative household orders consumption and labor supply streams $\{C_{0,t}; t \geqslant 0\}$ and $\{N_{0,t}; t \geqslant 0\}$ according to

$$\int_0^\infty e^{-\rho t} U(C_{0,t}, N_{0,t}) dt, \tag{2}$$

where $\rho > 0$ and $U(\cdot,\cdot)$ is strictly increasing in consumption $C$, strictly decreasing in labor supply $N$, globally concave, and continuously differentiable. Since consumption and labor decisions are convex controls and the household derives utility $U(C_{0,t}, N_{0,t})dt$ over a small time interval $dt$, we focus on policies that we call admissible in which $\overrightarrow{C_0}$ and $\overrightarrow{N_0}$ are Càdlàg processes. Along with Debortoli, Nunes, and Yared (2021), we assume that labor supply $N_{0,t}$ has no upper bound.

Let $\tau_{0,t}$ denote a tax rate at $t$ chosen at time 0 and $q_{0,t}$ be the time-0 value of a zero-coupon bond with a unit payoff at $t$. A first subscript denotes the time that a variable is chosen and that a second subscript denotes a time that the variable is realized.

The representative household faces a single intertemporal budget constraint:

$$\int_0^\infty C_{0,t} q_{0,t}\, dt \leqslant \int_0^\infty b_{0,t} q_{0,t} dt + \int_0^\infty (1 - \tau_{0,t}) N_{0,t} q_{0,t} dt\,. \tag{3}$$

The left side of (3) is the present value of the household's consumption stream $\{C_{0,t}; t \geqslant 0\}$ and the right side is the sum of the household's financial wealth and its human wealth in the form of the present value of after-tax labor income. Given tax rate process $\{\tau_{0,t}; t \geqslant 0\}$ and bond price process $\{q_{0,t}; t \geqslant 0\}$, the household chooses $\{C_{0,t}, N_{0,t}; t \geqslant 0\}$ to maximize (2) subject to constraint (3). That optimum problem brings the following first-order necessary conditions:

$$
\begin{aligned}
1-\tau_{0,t} &= -\frac{U_{N,t}}{U_{C,t}}\,, && (4)\\
q_{0,t} &= e^{-\rho t}\frac{U_{C,t}}{U_{C,0}}\,. && (5)
\end{aligned}
$$

The time$-0$ government finances an initial debt and its exogenous spending stream by taxing labor income:

$$
\int_0^\infty (\tau_{0,t}N_{0,t}-G_t)q_{0,t}\,dt \geqslant \int_0^\infty b_{0,t}q_{0,t}dt\,. \tag{6}
$$

Let $S_{0,t} = \tau_{0,t}N_{0,t} - G_t$ denote the primary surplus. The left side is time 0 value of $\{S_{0,t}; t \geqslant 0\}$ and the right side is time 0 value of payouts on government debt: $\{b_{0,t}; t \geqslant 0\}$.

**Definition 1.** *Given the government's initial debt term structure $\{b_{0,t}; t \geqslant 0\}$ and spending flow process $\{G_t; t \geqslant 0\}$, a competitive equilibrium is a feasible allocation $\{C_{0,t}, N_{0,t}; t \geqslant 0\}$, a flat rate tax $\{\tau_{0,t}; t \geqslant 0\}$ process, and a bond price process $\{q_{0,t}; t \geqslant 0\}$ for which*

- *the government's budget constraint* (6) *is satisfied, and*
- *given the government spending, tax, and bond price process, the allocation solves the household's optimization problem*

We follow Lucas and Stokey (1983) and use first-order conditions (4) and (5) together with feasibility constraint (1) to eliminate tax rates and bond prices from the government budget constraint. We thereby obtain the following implementability constraint on competitive equilbrium allocations $\{C_{0,t}, N_{0,t}\}_{t=0}^\infty$:

$$
\int_0^\infty e^{-\rho t}(C_{0,t}U_{C,t} + (C_{0,t}+G_t)U_{N,t})dt \geqslant \int_0^\infty e^{-\rho t}b_{0,t}U_{C,t}dt\,. \tag{7}
$$

**Proposition 1.** *Given an initial term-debt structure $\{b_{0,t}; t \geqslant 0\}$ and government spending path $\{G_t; t \geqslant 0\}$, an allocation $\{C_{0,t}, N_{0,t}; t \geqslant 0\}$ is a competitive equilibrium allocation if and only if it satisfies (1) for all $t \geqslant 0$ and (7) at $t = 0$.*

**Ramsey problem.** A Ramsey planner chooses a Càdlàg process $\overrightarrow{C_0} = \{C_{0,t}; t \geqslant 0\}$ that maximizes

$$
\int_0^\infty e^{-\rho t}U(C_{0,t}, C_{0,t}+G_t)dt \tag{8}
$$

subject to implementability constraint (7). By using first-order conditions (4) and (5), we can compute an associated tax rate process and bond price system. After the Ramsey planner chooses a tax plan and associated price system, the representative agent's best responses confirm the associated competitive equilibrium allocation.

We next turn to how Lucas and Stokey's Ramsey planner reschedules an initial debt term structure $\{b_{0,t}\}_{t=0}^{\infty}$ to another debt term structure $\{\hat{b}_{0,t}\}_{t=0}^{\infty} \neq \{b_{0,t}\}_{t=0}^{\infty}$, hoping to induce future planners to confirm the Ramsey plan it has designed.

**Continuation debt term structure.**

To begin, we note that at a Ramsey allocation $\{C_{0,t}^*, N_{0,t}^*; t \geqslant 0\}$, flat tax rate process $\{\tau_{0,t}^*; t \geqslant 0\}$, and price system $\{q_{0,t}^*; t \geqslant 0\}$, many possible debt term structures $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$ satisfy time 0 and time $t > 0$ continuation government budget constraints:

$$\int_0^{\infty} \hat{b}_{0,s} q_{0,s}^* ds \leqslant \int_0^{\infty} (\tau_{0,s}^* N_{0,s}^* - G_s) q_{0,s}^* \, ds \tag{9}$$

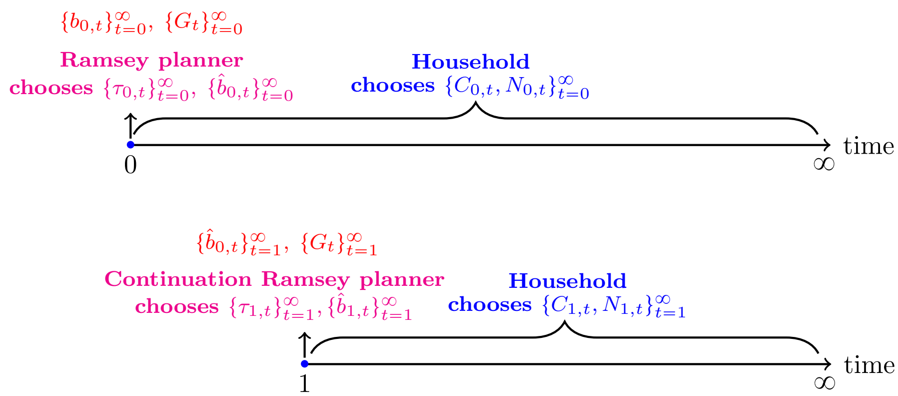
$$\int_t^{\infty} \hat{b}_{0,s} q_{0,s}^* ds \leqslant \int_t^{\infty} (\tau_{0,s}^* N_{0,s}^* - G_s) q_{0,s}^* \, ds \, . \tag{10}$$

**Definition 2.** *A continuation of a Ramsey plan at $t > 0$ is the tail $\{C_{0,s}^*, N_{0,s}^*, \tau_{0,s}^*, q_{0,s}^*; s \geqslant t\}$ of a Ramsey plan for $s \geqslant t$. A continuation Ramsey planner at time $t > 0$ must honor the (inherited) continuation debt term structure $\{\hat{b}_{0,s}\}_{s=t}^{\infty}$ and chooses a continuation plan $\{\tau_{t,s}\}_{s=t}^{\infty}$ for tax rates that need not equal the continuation of the original Ramsey plan, $\{\tau_{0,s}^*\}_{s=t}^{\infty}$, and a new debt term structure $\{\hat{b}_{t,s}\}_{s=t}^{\infty}$. A Ramsey plan is said to be implemented (or "time consistent") when all continuation Ramsey planners (i.e., for all $t > 0$) choose to confirm it.*

Figure 1 illustrates the timing protocol used by Debortoli, Nunes, and Yared (2021). At time 0, a Ramsey planner confronts a government purchase stream $\{G_t\}_{t=0}^{\infty}$ and an initial debt-service term structure $\{b_{0,t}\}_{t=0}^{\infty}$ that it must finance with a sequence of flat rate taxes $\{\tau_{0,t}\}_{t=0}^{\infty}$ and possibly restructured debt-service term structure $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$. Knowing the tax and debt structure sequences, at time 0 the representative agent chooses $\{C_{0,t}, N_{0,t}\}_{t=0}^{\infty}$. In Debortoli et al.'s laboratory, a single continuation Ramsey planner at time $t = 1$ confronts $\{G_t\}_{t=1}^{\infty}$ and $\{\hat{b}_{0,t}\}_{t=1}^{\infty}$ that it must finance with a time $t = 1$ continuation tax plan $\{\tau_{1,t}\}_{t=1}^{\infty}$.

Lucas and Stokey (1983) constructed examples in which budget-feasible debt term structures $\{\hat{b}_{0,t}\}_{t=1}^{\infty}$ induce a continuation Ramsey planner to confirm a continuation of a Ramsey tax plan $\{\tau_{0,t}^*; t \geqslant 1\}$ and its associated allocation and price system: $\{C_{0,t}^*, N_{0,t}^*, q_{0,t}^*; t \geqslant 1\}$.

Figure 1: Timing protocol for Ramsey planner and time 1 continuation Ramsey planner.




However, Debortoli, Nunes, and Yared (2021) constructed examples for which debt term structure $\{\hat{b}_{0,t}\}_{t=1}^{\infty}$ along lines recommended by Lucas and Stokey (1983) fails to induce a continuation Ramsey planner to confirm a continuation of the Ramsey plan. To set the stage for a continuous time version of Debortoli et al.'s counterexample, the next section presents the Ramsey planner's Lagrangian.

# 3 Ramsey Planner's Lagrangian

Attach a multiplier $\Lambda_0$ to implementability constraint (7) and form a Lagrangian:

$$\mathcal{L}_0 = \int_0^{\infty} e^{-\rho t} \left[ U(C_{0,t}, C_{0,t} + G_t) + \Lambda_0 \left( C_{0,t} U_{C,t} + (C_{0,t} + G_t) U_{N,t} - b_{0,t} U_{C,t} \right) \right] dt \,. \tag{11}$$

The Ramsey planner maximizes the right side of (11) over consumption plans $\{C_{0,t}\}_{t=0}^{\infty}$ and minimizes over the nonnegative multiplier $\Lambda_0$. Call the extremizing values: $\{C_{0,t}^*\}_{t=0}^{\infty}$ and $\Lambda_0^*$. For a given $\Lambda_0$, the optimal consumption plan satisfies[2]

$$C(\Lambda_0; b_{0,t}, G_t) := \arg\max_{C_{0,t}} \left[ U(C_{0,t}, C_{0,t} + G_t) + \Lambda_0 \left( C_{0,t} U_C + (C_{0,t} + G_t) U_N - b_{0,t} U_C \right) \right]. \tag{12}$$

Substituting (12) into implementability condition (7), we deduce that $\Lambda_0$ must satisfy:

$$\int_0^{\infty} e^{-\rho t} \big[ C(\Lambda_0; b_{0,t}, G_t) U_C + (C(\Lambda_0; b_{0,t}, G_t) + G_t) U_N \big] dt = \int_0^{\infty} e^{-\rho t} b_{0,t} U_C dt \,. \tag{13}$$

[2]We assume a regularity condition under which the optimal consumption plan for the problem defined in (12) is unique for a given $\Lambda_0$.

Substituting $C(\Lambda_0; b_{0,t}, G_t)$ into the right side of (11) lets us write $\mathcal{L}_0$ as a function $\mathcal{L}_0(\Lambda_0)$. An optimal $\Lambda_0^*$ is the non-negative root of equation (13) that maximizes $\mathcal{L}_0(\Lambda_0)$. In sum, the Ramsey allocation is given by: $C_{0,t}^* = C(\Lambda_0^*; b_{0,t}, G_t)$ and $N_{0,t}^* = C_{0,t}^* + G_t = N(\Lambda_0^*; b_{0,t}, G_t)$.

**Proposition 2.** *Under regularity conditions provided in Appendix A, there exists a non-negative root $\Lambda_0^*$ of equation (13) that solves*

$$\Lambda_0^* = \arg\min_{\Lambda_0} \ \mathcal{L}_0(\Lambda_0)\,,$$

*where the Ramsey allocation, tax rate, and bond price process satisfy*

$$C_{0,t}^* = C(\Lambda_0^*; b_{0,t}, G_t); \qquad N_{0,t}^* = C_{0,t}^* + G_t;$$
$$\tau_{0,t}^* = 1 + \tfrac{U_{N,t}(C_{0,t}^*, N_{0,t}^*)}{U_{C,t}(C_{0,t}^*, N_{0,t}^*)}; \qquad q_{0,t}^* = e^{-\rho t}\tfrac{U_{C,t}(C_{0,t}^*, N_{0,t}^*)}{U_{C,0}(C_{0,0}^*, N_{0,0}^*)}\,.$$

*The value $\mathcal{L}_0^*$ of a Ramsey plan is $\int_0^\infty e^{-\rho t} U(C_{0,t}^*, C_{0,t}^* + G_t)dt$.*

The primary surplus under the Ramsey plan is

$$S_{0,t}^* = S(C_{0,t}^*) = \tau_{0,t}^* N_{0,t}^* - G_t\,. \tag{14}$$

The time-$t$ value of cumulative deficits in the absent of any rescheduling the initial term debt debt process $\{b_{0,t}\}$ is

$$\Pi_t^* = \frac{1}{q_{0,t}^*}\int_0^t q_{0,u}^*\left(b_{0,u} - S(C_{0,u}^*)\right)du\,, \tag{15}$$

where the increment to the government's IOU's over a small $du$ interval is $\left(b_{0,u} - S(C_{0,u}^*)\right)du$ and $q_{0,u}^*/q_{0,t}^*$ converts a time $u$ deficit to its time $t$ value. The stock $\Pi_t^*$ plays a crucial role in our way of implementing the Ramsey plan.

# 4 The Counterexample

To construct a continuous-time version of Debortoli et al.'s counterexample to Lucas and Stokey's implementation of a Ramsey plan, we adopt the special instantaneous utility:

$$U(C, N) = \log C - \eta\frac{N^\gamma}{\gamma}\,, \tag{16}$$

where $\eta > 0$ and $\gamma \geqslant 1$. The Ramsey planner must finance the time-invariant government expenditure process $G_t = G$ for all $t \geqslant 0$ and its initial debt term structure:

$$b_{0,t} = \begin{cases} b_0 > 0 & t \in [0, T), \\ 0 & t \geqslant T, \end{cases} \tag{17}$$

for a given $T > 0$. We'll also assume $\gamma = 1$. For this specification, $U_C = 1/C$ and $U_N = -1$, which imply that the tax rate satisfies $\tau_{0,t} = 1 - \eta C_{0,t}$ for all $t \geqslant 0$, and the primary surplus $S_{0,t}$ given in (14) takes the form of:

$$S_{0,t} = S(C_{0,t}) = C_{0,t}\left(1 - \eta(C_{0,t} + G_t)\right) . \tag{18}$$

Next, we state a continuous-time version of Lemma 3 in Debortoli, Nunes, and Yared (2021) for the $\gamma = 1$ case.

**Lemma 1.** *When $b_0$ is not too high, a Ramsey plan exists and is given by*

$$C^*_{0,t} = C^*_0 1_{\{t\in[0,T)\}} + C^*_1 1_{\{t\in[T,\infty)\}}; \quad N^*_{0,t} = N^*_0 1_{\{t\in[0,T)\}} + N^*_1 1_{\{t\in[T,\infty)\}}, \tag{19}$$

$$\tau^*_{0,t} = \tau^*_0 1_{\{t\in[0,T)\}} + \tau^*_1 1_{\{t\in[T,\infty)\}}; \quad q^*_{0,t} = e^{-\rho t}\left(1_{\{t\in[0,T)\}} + C^*_0/C^*_1 1_{\{t\in[T,\infty)\}}\right), \tag{20}$$

*where $1_{\{A\}}$ is an indicator function that equals one if the event $A$ occurs and zero otherwise,*

$$C^*_0 = C(\Lambda^*_0; b_0, G) = \frac{2b_0\Lambda^*_0}{\sqrt{1 + 4\eta b_0\Lambda^*_0(1 + \Lambda^*_0)} - 1}, \quad C^*_1 = C(\Lambda^*_0; 0, G) = \frac{1}{\eta(1 + \Lambda^*_0)},$$

*$N^*_0 = C^*_0 + G$, and $N^*_1 = C^*_1 + G$. The Ramsey tax plan is given by $\tau^*_0 = 1 - \eta C^*_0$ and $\tau^*_1 = 1 - \eta C^*_1$. Finally, $\Lambda^*_0$ is the unique non-negative root of*

$$\frac{1 - e^{-\rho T}}{\rho}\left(1 - \eta(C(\Lambda_0; b_0, G) + G) - \frac{b_0}{C(\Lambda_0; b_0, G)}\right) + \frac{e^{-\rho T}}{\rho}\left(1 - \eta(C(\Lambda_0; 0, G) + G)\right) = 0\,. \tag{21}$$

As $C^*_{0,t} = C^*_0$ for all $t < T$ and $C^*_{0,t} = C^*_1$ for all $t \geqslant T$, the bond price satisfies

$$q^*_{0,t} = e^{-\rho t}, \text{ for } t < T; \quad q^*_{0,t} = e^{-\rho t}\frac{C^*_0}{C^*_1}, \text{ for } \ t \geqslant T. \tag{22}$$

Consequently, the competitive equilibrium interest rate is $r^*_{0,t} = \rho, t < T;\ r^*_{0,t} = \rho,\ t \geqslant T$.

Immediately before $t = T$, the interest rate approaches $\infty$, so $\{r^*_{0,t} - \rho\}_{t=0}^{\infty}$ is a Dirac delta function. We shall soon see how the Dirac delta interest rate at $t = T$ plays a key role.

In the context of this example, we now describe how Debortoli et al. use Lucas and Stokey way of structuring a continuation debt term structure in order to motivate the continuation Ramsey planner to implement the continuation of the Ramsey plan. The Ramsey allocation (19) is piece-wise linear, so the Ramsey planner repurchases $\{b_{0,t}\}_{t=0}^{\infty}$ at time 0 and sells debt with the following coupon payment schedule: $\hat{b}_{0,t} = \hat{b}_0$ for all $t < T$ and $\hat{b}_{0,t} = \hat{b}_1$ for all $t \geqslant T$, making sure that $\hat{b}_0$ satisfies both the budget constraint (9) at time 0 and the budget constraint (10) at time $t = T$. We compute $\hat{b}_0$ and $\hat{b}_1$ as follows.

- To compute $\hat{b}_1$, we use the bond price given in (22) and the property that the primary surplus under the Ramsey plan is constant for $t \geqslant 1$: $S(C_1^*) = \tau_1^* N_1^* - G = C_1^* \left(1 - \eta(C_1^* + G)\right)$ to simplify the time-$T$ budget equation (10) as follows:

$$\int_T^{\infty} \hat{b}_1 e^{-\rho t} \frac{C_0^*}{C_1^*} dt = \int_T^{\infty} S(C_1^*) e^{-\rho t} \frac{C_0^*}{C_1^*} dt \,.$$

  This yields: $\hat{b}_1 = S(C_1^*)$. Simplifying the time 0 budget constraint (6) for the initial debt term structure $\{b_{0,t}\}$ we obtain

$$\frac{1 - e^{-\rho}}{\rho} \left(S(C_0^*) - b_0\right) + \frac{e^{-\rho}}{\rho} S(C_1^*) \frac{C_0^*}{C_1^*} = 0 \,.$$

  Combining the above two results gives $\hat{b}_1 = \left(e^{\rho T} - 1\right) \left(\frac{b_0}{C_0^*} - 1 + \eta(C_0^* + G)\right) C_1^*$.

- Using the preceding expression for $\hat{b}_1$ to rewrite the time-0 budget constraint (9) as follows:

$$\int_0^T e^{-\rho t} \hat{b}_0 dt + \int_T^{\infty} e^{-\rho t} \hat{b}_1 \frac{C_0^*}{C_1^*} dt = \int_0^T e^{-\rho t} S(C_0^*) dt + \int_T^{\infty} e^{-\rho t} S(C_1^*) \frac{C_0^*}{C_1^*} dt \,,$$

  we obtain $\hat{b}_0 = C_0^* \left(1 - \eta(C_0^* + G)\right)$.

In summary, we obtain the following Lucas-Stokey debt restructuring policy:

$$\hat{b}_{0,t} = \begin{cases} \hat{b}_0 = C_0^* \left(1 - \eta(C_0^* + G)\right), & t \in [0, T)\,, \\ \hat{b}_1 = \left(e^{\rho T} - 1\right) \left(\frac{b_0}{C_0^*} - 1 + \eta(C_0^* + G)\right) C_1^*, & t \geqslant T\,. \end{cases} \tag{23}$$

Next, we state key findings of Debortoli, Nunes, and Yared (2021) in the context of our continuous-time version of their model.

**Lemma 2.** *When $b_0$ is not too high, debt term structure* (23) *induces a time $T$ continuation Ramsey planner to confirm the Ramsey plan. But for $b_0$ above a threshold $b^*$, debt term structure* (23) *is unable to induce the continuation planner to confirm the Ramsey plan.*

**Numerical illustrations.** To stay close to Debortoli et al.'s discrete time example, we set $T = 1$ and $(\gamma, \eta) = (1, 1)$, which makes $U(C, N) = \log C - N$. We use the Debortoli et al.'s one-period discount factor of 0.5 by settomg $e^{-\rho} = 0.5$, so that $T = 1$ corresponds to 13.9 years with an annual discount rate of 5%. For these parameter values, it turns out that the threshold in Lemma 2 is $b^* = 0.35$. By setting $b_0 = .3$, we recover an example in which committing the government to finance Lucas and Stokey's recommended restructured debt term structure works. By setting $b_0 = .5$, we'll construct a version of Debortoli et al.'s counterexample in which it doesn't work.

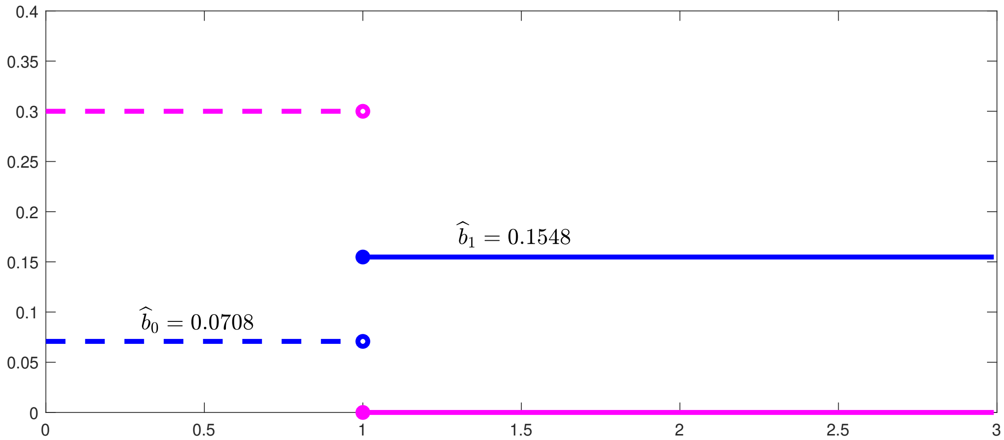
**Lucas-Stokey Implementation works when** $b_0 = 0.3$. When $b_{0,t} = b_0 = 0.3$ for $t \in [0, 1)$ and $b_{0,t} = 0$ for $t \geqslant 1$, the Ramsey planner sets $C^*_{0,t} = C^*_0 = 0.6987$ for $t \in [0, 1)$, $C^*_{0,t} = C^*_1 = 0.4721$ for $t \geqslant 1$, and associated tax rates: $\tau^*_0 = 0.3013$ and $\tau^*_1 = 0.5273$. To induce the continuation Ramsey planner to confirm the Ramsey plan, the Ramsey planner repurchases $\{b_{0,t}\}_{t=0}^{\infty}$ and sells $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$: $\hat{b}_{0,t} = \hat{b}_0 = 0.0708$ for $t \in [0, 1)$ and $\hat{b}_{0,t} = \hat{b}_1 = 0.1548$ for $t \geqslant 1$. This induces the continuation Ramsey planner at $t = 1$ to confirm the Ramsey plan: $C^*_{1,t} = C^*_1 = 0.4721$ for all $t \geqslant 1$.

**Lucas-Stokey Implementation doesn't work when** $b_0 = 0.5$. When $b_{0,t} = b_0 = 0.5$ for $t \in [0, 1)$ and $b_{0,t} = 0$ for $t \geqslant 1$, the Ramsey planner sets $C^*_{0,t} = C^*_0 = 0.7374$ for $t \in [0, 1)$ and $C^*_{0,t} = C^*_1 = 0.1846$ for $t \geqslant 1$. To induce the continuation Ramsey planner to confirm the continuation of the Ramsey plan, Lucas and Stokey advise the Ramsey planner to repurchase $\{b_{0,t}\}_{t=0}^{\infty}$ and to construct a restructured debt term structure $\{\hat{b}_{0,t}\}_{t=0}^{\infty}$ in which $\hat{b}_{0,t} = \hat{b}_0 = 0.0462$ for $t \in [0, 1)$ and $\hat{b}_{0,t} = \hat{b}_1 = 0.1136$ for $t \geqslant 1$. To construct the associated continuation Ramsey plan, first substitute the first order necessary condition $1 - \tau_{1,t} = C_{1,t}$ for labor into the primary surplus formula to get $S(C_{1,t}) = C_{1,t}(1 - (C_{1,t} + G))$ .The implementation constraint for the continuation Ramsey planner is:

$$\int_1^{\infty} e^{-\rho(t-1)} \frac{\hat{b}_1}{C_{1,t}} dt \leqslant \int_1^{\infty} e^{-\rho(t-1)} \frac{S(C_{1,t})}{C_{1,t}} dt \, .$$

Form a Lagrangian for the continuation Ramsey planner to obtain $C_{1,t} = C_{1,1}$ and $S(C_{1,t}) =$

(a) Initial debt term structure $b_{0,t}$ and restructured debt $\widehat{b}_{0,t}$




(b) Ramsey plan and continuation Ramsey plan.

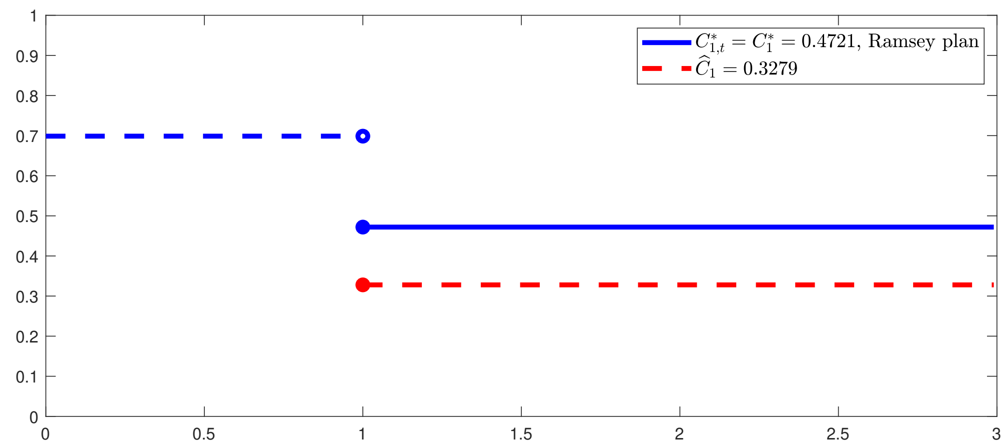



Figure 2: Implementation works when $b_0 = 0.3$. Ramsey plan: $C_0^* = 0.6987, C_1^* = 0.4721$, $\tau_0^* = 0.3013$, and $\tau_1^* = 0.5273$.

$\hat{b}_1$ for all $t \geqslant 1$. Solve $S(C_{1,1}) = \hat{b}_1$ for $C_{1,1}$ to obtain two roots: $C_{1,1} = C_1^* = 0.1846$, which describes the continuation of the Ramsey plan, and $C_{1,1} = \widehat{C}_1 = 0.6154$, which describes the continuation Ramsey plan. To verify that the continuation Ramsey planner sets $C_{1,1} = \widehat{C}_1 = 0.6154$ in order to maximize its objective function, form the Lagrangian:

$$\mathcal{L}_1 = \int_1^\infty e^{-\rho(t-1)}\left[\log C_{1,t} - (C_{1,t} + G) + \Lambda_1\left(1 - (C_{1,t} + G) - \frac{\hat{b}_1}{C_{1,t}}\right)\right]dt.$$

We can show that

$$U(\widehat{C}_1, \widehat{C}_1 + G) > U(C_1^*, C_1^* + G)$$

by using $\widehat{C}_1 = 0.6154$, $C_1^* = 0.1846$, and

$$-1.8769 = \int_1^\infty e^{-\rho(t-1)} U(\widehat{C}_1, \widehat{C}_1 + G) dt > \int_1^\infty e^{-\rho(t-1)} U(C_1^*, C_1^* + G) dt = -2.9926\,.$$

This completes our verification of Debortoli et al.'s counterexample to implementing a Ramsey plan by restructuring a Càdlàg $\{b_{0,t}; t \geqslant 0\}$ debt term structure.

It is not possible to find a counterexample to a counterexample, so we won't try. Instead, we'll seek an arrangement with minimal additional comments that continuation Ramsey planners are bound to honor, one that, in our context, is faithful to Lucas and Stokey's intention to give the government access to complete markets.

# 5 Expanding the Contractible Subspace

We implement a Ramsey plan by expanding the *contractible subspace* in the following way. In addition to confronting continuation Ramsey planners with a rescheduled debt term structure $\{\hat{b}_{0,t}, t \geqslant T\}$ that they must service, we now require that continuation Ramsey planners are also obligated to protect the purchasing power of a stock $\widehat{B}_t$ of instantaneous debt. We dub this additional obligation a "local commitment". We show how this enlargement of the contractible subspace allows the Ramsey planner to implement the Ramsey plan with a simple debt-management policy. In section 6, we show how our expansion of the contractible subspace relative to Debortoli et al.'s makes it possible to represent an implementable Ramsey plan with a dynamic program.

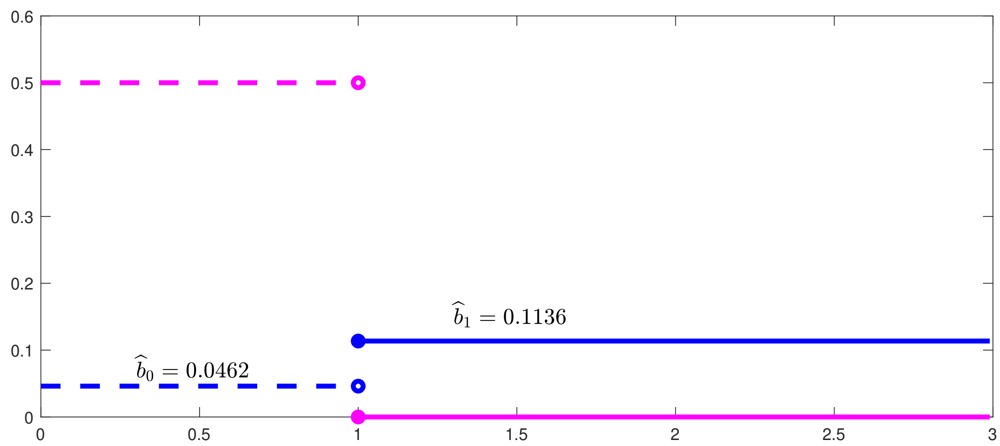
We focus on the setting of Debortoli et al.'s counterexample in which $T = 1$. Using $q_{0,t}^* = e^{-\rho t}$ and the formula for $S(C_0^*)$ given in (18) for $t < 1$, and then applying (15) to $t = 1-$, we obtain:

$$\Pi_{1-}^* = \frac{1}{e^{-\rho}} \int_0^{1-} e^{-\rho s} \left(b_0 - S(C_0^*)\right) ds = \frac{(e^\rho - 1)}{\rho} \left( \frac{b_0}{C_0^*} - 1 + (C_0^* + G) \right) C_0^*. \tag{24}$$

Next, using $q_{0,1-}^*/q_{0,1}^* = C_1^*/C_0^*$, (15), and (24), we obtain:

$$\Pi_1^* = \frac{1}{q_{0,1}^*} \int_0^1 q_{0,s}^* \left(b_{0,s} - S(C_{0,s}^*)\right) ds = \frac{q_{0,1-}^*}{q_{0,1}^*} \frac{1}{q_{0,1-}^*} \int_0^1 q_{0,s}^* \left(b_{0,s} - S(C_{0,s}^*)\right) ds = \frac{C_1^*}{C_0^*} \Pi_{1-}^*. \tag{25}$$

(a) Initial debt term structure $b_{0,t}$ and restructured debt $\widehat{b}_{0,t}$




(b) Ramsey plan and continuation Ramsey plan.

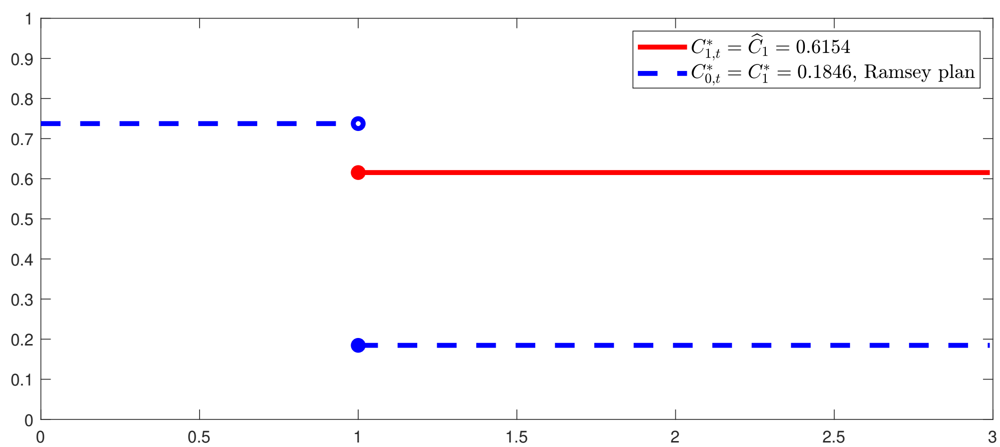



Figure 3: Implementation doesn't work when $b_0 = 0.5$. Ramsey plan: $C_0^* = 0.7374, C_1^* = 0.1846$, $\tau_0^* = 0.2626$, and $\tau_1^* = 0.8154$. Continuation Ramsey plan: $\widehat{C}_1 = 0.6154$ and $\hat{\tau}_1 = 0.3846$.

Combining (24) and (25) we deduce

$$\frac{\Pi^*_{1-}}{C^*_0} = \frac{\Pi^*_1}{C^*_1} = \frac{(e^\rho - 1)}{\rho}\left(\frac{b_0}{C^*_0} - 1 + (C^*_0 + G)\right), \tag{26}$$

which requires that a continuation Ramsey planner choose to smooth the representative household's "purchasing power" over time, including at $t = 1$.

We can restrict debt restructuring policies to a set in which the Ramsey planner leaves the initial debt term structure $\{b_{0,t}\}_{t=0}^{\infty}$ untouched and instead uses instantaneous debt to finance accumulated deficits $\Pi_t^*$. We propose the following debt restructuring policy.

**Debt Restructure 1.** *Refinance $\{G_t, b_{0,t}\}_{t=0}^{\infty}$ as follows:*

- *Set $\{\hat{b}_{0,t}\}_{t=0}^{\infty} = \{b_{0,t}\}_{t=0}^{\infty}$.*
- *Start with $B_0 = 0$, accumulate instantaneous debt balance $B_t$ according to $dB_t = d\Pi_t^*$ for $t \in [0, 1)$; restructure instantaneous debt from $B_{1-}$ to $\widehat{B}_1 = \frac{C_1^*}{C_0^*} B_{1-}$ and leave this new balance to time-1 government.*

Why do we require the Ramsey planner to restructure $B_{1-}$ to $\widehat{B}_1 = \frac{C_1^*}{C_0^*} B_{1-}$ at $t = 1-$? First note that under this restructuring, $B_{1-} = \Pi_{1-}^*$ and $\widehat{B}_1 = \frac{C_1^*}{C_0^*} B_{1-} = \frac{C_1^*}{C_0^*} \Pi_{1-}^* = \Pi_1^*$, where the last equality follows from (26). Therefore, this policy tracks the evolution of $\{\Pi_t^*; t \geqslant 0\}$ perfectly. This suggests that it is possible to implement Ramsey plan solely using instantaneous debt without adjusting the initial term debt structure at all. Intuitively, this restructuring policy preserves the "purchasing power" of instantaneous debt as $U_C(C_{0,1}^*, C_{0,1}^* + G)\widehat{B}_1 = U_C(C_{0,1-}^*, C_{0,1-}^* + G)B_{1-}$ under Ramsey plan.

How do we induce the continuation Ramsey planner to confirm the Ramsey plan? The key is to require that the "purchasing power" of instantaneous debt is preserved:

$$U_C(C_{1,1}, C_{1,1} + G)\widehat{B}_1 = U_C(C_{0,1-}, C_{0,1-} + G)B_{1-}\,. \tag{27}$$

where $C_{1,1}$ is chosen by the continuation Ramsey planner at $t = 1$. So we must verify that the continuation Ramsey planner confronts $\widehat{B}_1$ wants to set $C_{1,1} = C_1^*$. Substituting $C_{0,1-} = C_0^*$ into (27), we obtain

$$\frac{\widehat{B}_1}{C_{1,1}} = \frac{B_{1-}}{C_0^*}\,. \tag{28}$$

Substituting $B_{1-} = \Pi_{1-}^*$ and $\widehat{B}_1 = \Pi_1^*$ into (28) and using (26), we obtain $C_{1,1} = C_1^*$, which is the continuation of the Ramsey plan.

Facing $\widehat{B}_1$, the continuation Ramsey planner chooses $\{C_{1,t}; t > 1\}$ to maximize the household's utility subject to the following continuation implementability constraint:

$$\frac{\widehat{B}_1}{C_{1,1}} \leqslant \int_1^{\infty} e^{-\rho(t-1)} \frac{S(C_{1,t}) - 0}{C_{1,t}} dt\,.$$

By forming and extremizing the Lagrangian for the continuation Ramsey planner, we conclude that: 1) $C_{1,t}$ is constant for all $t > 1$; 2) Continuation IC and local commitment constraint (28) imply

$$\frac{1}{\rho}\frac{S(C_{1,t})-0}{C_{1,t}} = \frac{\widehat{B}_1}{C_{1,1}} = \frac{B_{1-}}{C_0^*} = \frac{1}{\rho}\frac{S(C_1^*)-0}{C_1^*}, \ \forall t > 1;$$

and 3) the continuation Ramsey planner chooses $C_{1,t} = C_1^*$ for all $t \geqslant 1$, thus confirming the tail of the Ramsey plan.

So for the special $b_0 = .5$ setting that leads to Debortoli et al.'s counterexample, $\widehat{B}_1 = \Pi_1^* = 0.1639$ and $C_{1,t} = C_1^* = 0.1846$ for $t \geqslant 1$, which confirms the Ramsey plan. Why in Debortoli et al.'s counterexample wasn't restructuring the initial term debt structure to a new term debt structure able to implement the Ramsey plan? It is because the restructured term debt stream $\{\hat{b}_{0,t}; t \geqslant 1\}$ in the implementation proposed by Lucas and Stokey has long duration, so in the absence of instantaneous debt and local commitment, the continuation Ramsey planner would choose to dilute its "purchasing power" by setting $C_{1,t} = \widehat{C}_1 = 0.6154$ (and $\tau_{1,t} = \hat{\tau}_1 = 0.3846$) for $t \geqslant 1$, instead of $C_{1,t} = C_1^* = 0.1846$ and $\tau_1^* = 0.8154$, as called for in the Ramsey plan.

The presence of instantaneous debt and the local commitment is what allows our restructuring policy 1 to implement the Ramsey plan. Note how the instantaneous debt balance $\widehat{B}_1$ includes information about both the price (via $r_{0,t} = \rho$) and the quantity ($\hat{b}_1$) of the government's liability: $\widehat{B}_1 = \int_1^\infty e^{-\rho(t-1)}\hat{b}_1 dt = \hat{b}_1/\rho$. Second, the (local) commitment condition (28) prohibits time-1 continuation planner from diluting the household's "purchasing power" of $\widehat{B}_1$, which is something that the continuation Ramsey planner wants to do when it confronts the Càdlàg $\{\hat{b}_{0,t}; t \geqslant 0\}$ debt term structure without instantaneous debt in Debortoli et al.'s counterexample.

In the next section, we develop a recursive method to solve the Ramsey problem and then reconfirm how the implementation method with instantaneous debt and the local commitment condition implements the Ramsey plan.

# 6 Ramsey problem as dynamic program

First, we introduce a state variable $\{X_t; t \geqslant 0\}$. Second, we formulate a dynamic programming problem for a given $X_0$. Third we characterize the Ramsey plan.

## 6.1 Introducing State Variable $X_t$

For debt service flow $\{b_{0,s}, s \geqslant 0\}$, we define:

$$\Pi_t = \int_0^t \frac{q_{0,s}}{q_{0,t}} \left(b_{0,s} + G_s - \tau_{0,s} N_{0,s}\right) ds\,, \quad \Pi_0 = 0\,. \tag{29}$$

Here $(b_{0,s} + G_s - \tau_{0,s} N_{0,s})\, ds$ is the (flow) increase of the government's liability over a small time interval $ds$ without touching term debt at all and the ratio $q_{0,s}/q_{0,t}$ compounds this flow's contribution to the government's time-$t$ liability stock.[3] Since we assume that $\{G_t\}$ and $\{b_{0,t}\}$ are Càdlàg processes, optimal allocations $\{C_{0,t}, N_{0,t}; t \geqslant 0\}$ may not be continuous. Let $\mathcal{T} = \{0 < t_1 < t_2 < \cdots\}$ denote a set of countable (possibly infinite) points where the $\{C_{0,t}; t \geqslant 0\}$ process jumps.

For $t \notin \mathcal{T}$ where $\{\Pi_t\}$ does not jump, $q_{0,t}$ and $\Pi_t$ evolve continuously: $dq_{0,t} = -r_{0,t} q_{0,t} dt$ and

$$d\Pi_t = r_{0,t} \Pi_t dt + \left(b_{0,t} + G_t - \tau_{0,t} N_{0,t}\right) dt\,. \tag{30}$$

Where $\{\Pi_t\}$ jumps at $t \in \mathcal{T}$, we have

$$\Pi_t = \Pi_{t-} \frac{q_{0,t-}}{q_{0,t}}\,, \quad t \in \mathcal{T}\,. \tag{31}$$

Thus, the time-0 value of the government's time $t$ cumulative liability $q_{0,t}\Pi_t$ is the same before and after a jump.

Define

$$X_t = \Pi_t U_{C,t}\,, \tag{32}$$

which will serve as a state variable for a recursive formulation of the Ramsey problem. Evidently, $X_t$ incorporates both information about the household's wealth $\Pi_t$ and its marginal utility of consumption $U_{C,t}$. For $t \notin \mathcal{T}$ where $\{\Pi_t\}$ does not jump, $X_t$ is continuous because $U_{C,t}$ is also continuous. Differentiating $X_t$ both sides of (32) with respect to $t$ and using (30) together with (4) and (5), we obtain:

$$\begin{aligned} dX_t &= U_{C,t} d\Pi_t + \Pi_t d\left(e^{\rho t} q_{0,t} U_{C,0}\right) \\ &= U_{C,t}\left[-\Pi_t \frac{dq_{0,t}}{q_{0,t}} + \left(b_{0,t} + G_t - \tau_{0,t} N_{0,t}\right) dt\right] + \Pi_t \left(q_{0,t} U_{C,0} \rho e^{\rho t} dt + e^{\rho t} U_{C,0} q_{0,t} \frac{dq_{0,t}}{q_{0,t}}\right) \end{aligned}$$

[3] When the interest rate is a constant $r$ over $(s,t)$, then $\frac{q_{0,s}}{q_{0,t}} = e^{r(t-s)} > 1$.

$$
\begin{aligned}
&= -X_t \frac{dq_{0,t}}{q_{0,t}} + (b_{0,t}U_{C,t} - C_{0,t}U_{C,t} - N_{0,t}U_{N,t})\, dt + \rho X_t dt + X_t \frac{dq_{0,t}}{q_{0,t}} \\
&= (\rho X_t - C_{0,t}U_{C,t} - N_{0,t}U_{N,t} + b_{0,t}U_{C,t})\, dt\,. \qquad (33)
\end{aligned}
$$

What happens where $\{\Pi_t\}$ jumps at $t \in \mathcal{T}$? Using (31) and first-order necessary condition (5), we obtain $X_{t-} = \Pi_{t-}U_{C,t-} = \Pi_{t-}q_{0,t-}e^{\rho t-}U_{C,0} = \Pi_t q_{0,t}e^{\rho t}U_{C,0} = \Pi_t U_{C,t} = X_t$. We have established:

**Lemma 3.** *$X_t$ defined in (32) is continuous in time. It evolves according to (33).*

Because welfare maximization calls for continuity of the household's net worth $\Pi_t$ multiplied by the marginal utility $U_{C,t}$, the Ramsey planner wants to make $X_t$ continuous, even where the initial debt structure $\{b_{0,t}\}$ or government spending process $\{G_t\}$ jumps.

Since the initial debt structure $b_{0,t}$ and government spending $G_t$ are generally time dependent, the Ramsey problem is time inhomogeneous, so $t$ will appear in Ramsey planner's value function. Consequently, we reformulate the Ramsey problem in the following may two steps. First, for a given $X_0$, we define and solve a dynamic programming (DP) problem with $X_t$ and time $t$ being the state variables in subsection 6.2. In subsection 6.3 we use the DP problem to characterize a Ramsey plan.

## 6.2 A Dynamic Program (DP)

**Definition 3 (DP Problem).** *Confronting $X_0$ at $t = 0$, a decision maker solves*

$$
\max_{\{C_{0,s};s\geqslant 0\}} \int_0^\infty e^{-\rho s} U(C_{0,s}, C_{0,s} + G_s) ds \qquad (34)
$$

*subject to the evolution (33) of state variable $X_t$.*

This can be formulated as a dynamic programming problem (see e.g., Fleming and Soner, 2006). Let $V(X_0, 0)$ denote the (optimal) value function for this problem. To solve $V(X_0, 0)$, we first construct the following time$-t$ optimization problem:

$$
\max_{\{C_{0,s};s\geqslant t\}} \int_t^\infty e^{-\rho(s-t)} U(C_{0,s}, C_{0,s} + G_s) ds. \qquad (35)
$$

subject to (33). Let $V(X_t, t)$ denote the (optimal) value function for (35). For all $t \geqslant 0$, the value function $V(X_t,t)$ defined in (35) satisfies the following HJB equation:

$$\rho V = \frac{\partial V}{\partial t} + \max_{C_{0,t}} \; U(C_{0,t}, C_{0,t}+G_t) + \frac{\partial V}{\partial X_t}\left(\rho X_t - C_{0,t}U_{C,t} - (C_{0,t}+G_t)U_{N,t} + b_{0,t}U_{C,t}\right) . \tag{36}$$

We guess and verify that $V(X_t,t)$ is additively separable in $X_t$ and $t$:

$$V(X_t,t) = -\Phi_0 X_t + H(t;\Phi_0); \;\; t \geqslant 0\,, \tag{37}$$

where $\Phi_0$ is an object to be chosen by the Ramsey planner. Substituting (37) into (36), we obtain the following differential equation for all $t \geqslant 0$:

$$\rho H(t;\Phi_0) = H'(t;\Phi_0) + \max_{C_{0,t}} \; U(C_{0,t}, C_{0,t}+G_t) + \Phi_0\left(C_{0,t}U_{C,t} + (C_{0,t}+G_t)U_{N,t} - b_{0,t}U_{C,t}\right). \tag{38}$$

Let $C(\Phi_0;G_t,b_{0,t})$ denote the function for $C_{0,t}$ that maximizes (38):

$$C(\Phi_0;G_t,b_{0,t}) = \arg\max_{C_{0,t}} \; U(C_{0,t}, C_{0,t}+G_t) + \Phi_0\left(C_{0,t}U_{C,t} + (C_{0,t}+G_t)U_{N,t} - b_{0,t}U_{C,t}\right). \tag{39}$$

**Proposition 3.** *Choices of $\{C_{0,t}\}$ for a* **DP Problem** *indexed by $\Phi_0 \geqslant 0$ satisfy (39) and attain the value function*

$$V(X_t,t;\Phi_0) = -\Phi_0 X_t + H(t;\Phi_0)\,, \tag{40}$$

*where $H(t;\Phi_0)$ is given by*

$$\begin{aligned} H(t;\Phi_0) = \int_t^\infty e^{-\rho(s-t)} \Big[ & U(C(\Phi_0;b_{0,s},G_s), C(\Phi_0;b_{0,s},G_s)+G_s) \\ & + \Phi_0\left(C(\Phi_0;b_{0,s},G_s)U_{C,s} + (C(\Phi_0;b_{0,s},G_s)+G_s)U_{N,s} - b_{0,s}U_{C,s}\right)\Big] ds. \end{aligned} \tag{41}$$

*and $N(\Phi_0,G_t,b_{0,t}) = C(\Phi_0,G_t,b_{0,t}) + G_t$.*

We next turn to how the Ramsey planner chooses the scalar $\Phi_0$ that pins down the value function for the **DP Problem**.

## 6.3 Characterizing the Ramsey Plan

The consumption stream $\{C_{0,t}\}$ is a Càdlàg process: $C_{0,0} = \lim_{s\downarrow 0} C_{0,s} = C(\Phi_0; b_{0,0}, G_0)$, which implies that $C_{0,0}$ is a function of $\Phi_0$. Subject to implementability constraint (7), the Ramsey planner chooses $\Phi_0$, or equivalently consumption $C_{0,0}$ according to

$$W = \max_{\Phi_0} V(X_0, 0; \Phi_0), \tag{42}$$

where $V(X_0, 0; \Phi_0)$ is given in (37). When solving the problem on the right side of (42), the Ramsey planner takes as given the initial debt term structure $\{b_{0,t}; t \geqslant 0\}$ and $X_0 = 0$ (implied by $\Pi_0 = 0$). The implementability constraint requires that $\Phi_0$ satisfies $I_0(\Phi) = 0$, where

$$I_0(\Phi_0) = \int_0^\infty e^{-\rho t} \left(C(\Phi_0, b_{0,t}, G_t)U_{C,t} + (C(\Phi_0, b_{0,t}, G_t) + G_t)U_{N,t} - b_{0,t}U_{C,t}\right) dt\,. \tag{43}$$

Let $\mathcal{A} = \{\Phi_0 \geqslant 0 : I_0(\Phi_0) = 0\}$ denote the admissible set for $\Phi_0$. Using (40), the Ramsey s problem is

$$W = \max_{\Phi_0 \in \mathcal{A}} V(0,0) = \max_{\Phi_0 \in \mathcal{A}} \int_0^\infty e^{-\rho t} U(C(\Phi_0, b_{0,t}, G_t), C(\Phi_0, b_{0,t}, G_t) + G_t) dt\,.$$

**Proposition 4.** *When it exists, a Ramsey plan satisfies*

$$C^*_{0,t} = C(\Phi^*_0, b_{0,t}, G_t); \quad N^*_{0,t} = C^*_{0,t} + G_t;$$
$$\tau^*_{0,t} = 1 + \frac{U_{N,t}(C^*_{0,t}, N^*_{0,t})}{U_{C,t}(C^*_{0,t}, N^*_{0,t})}; \quad q^*_{0,t} = e^{-\rho t} \frac{U_{C,t}(C^*_{0,t}, N^*_{0,t})}{U_{C,0}(C^*_{0,0}, N^*_{0,0})}$$

*where* $\Phi^*_0 = \arg\max_{\Phi_0 \in \mathcal{A}} V(0,0)$*. Under a Ramsey plan,* $X_t = X^*_t$ *where*

$$X^*_t = \int_t^\infty e^{-\rho(s-t)} \left[C^*_{0,s} U_{C,t}(C^*_{0,t}, N^*_{0,t}) + N^*_{0,s} U_{N,t}(C^*_{0,t}, N^*_{0,t}) - b_{0,s} U_{C,t}(C^*_{0,t}, N^*_{0,t})\right] ds \tag{44}$$

*and the household's value is*

$$W = \int_0^\infty e^{-\rho t} U(C^*_{0,t}, C^*_{0,t} + G_t) dt. \tag{45}$$

**Proposition 5.** *A Ramsey plan obtained via the section 2 Lagrangian method equals the plan described in Proposition 4.*

We next return to governments' management of deficits under a Ramsey plan.

# 7 Financing Deficits

Under a Ramsey plan, the equilibrium bond price $\{q^*_{0,t}; t \geqslant 0\}$ satisfies

$$q^*_{s,t} = e^{-\int_s^t r^*_{0,u} du} \tag{46}$$

and the associated "force of interest" (the instantaneous interest rate) $r_{0,t}$ is

$$r^*_{0,t} = \begin{cases} -\dfrac{d \ln q^*_{0,t}}{dt}, & t \notin \mathcal{T}, \\ -\ln\left(\dfrac{q^*_{0,t}}{q^*_{0,t-}}\right)\delta(t-t_i), & t = t_i \in \mathcal{T}, \end{cases} \tag{47}$$

where $\delta(\,\cdot\,)$ is the Dirac function.[4]

**General debt management equation.**

Let $B_t$ denote the instantaneous debt balance at $t$. The interest payment over an infinitesimal time interval $(t, t+dt)$ is $r^*_{0,t}B_t dt$. The government uses both instantaneous debt and term debt to finance its primary deficits, so the following dynamic budget constraint holds:[5]

$$d\Pi^*_t = dB_t + \underbrace{\left(\int_{s\geqslant t} q^*_{t,s}\partial_t b_{t,s} ds\right) dt}_{\text{term debt issued at } t} - \underbrace{\left(\int_0^t \partial_u b_{u,t} du\right) dt}_{\text{term debt due at } t}, \tag{48}$$

where $\Pi^*_t$ defined in (15) is the cumulative liabilities under the Ramsey plan absent term debt adjustments. Here $\partial_u b_{u,t}$ denotes a rate of increase the (incremental) debt coupon issued at time $u$ and due at $t$. The third term $\left(\int_0^t \partial_u b_{u,t} du\right) dt$ is the (incremental) term debt due at $t$, issued cumulatively from 0 to $t$. The second term is the net financing from new debt issuance over $dt$. the change of instantaneous debt balance $dB_t$ finances shortfalls

[4] A Dirac function can be considered as the differential of the Heaviside step function (Kanwal, 2012): $h'(t-t_i) \equiv \delta(t-t_i)$, where $h(\,\cdot\,)$ is the Heaviside step function: $h(t) = 0, \forall t < 0$ and $h(t) = 1, \forall t \geqslant 0$.

[5] When $t \in \mathcal{T}$, all instantaneous debt and term debt must be re-evaluated under the Ramsey plan from time $t-$ to time $t$ in order to preserve the purchasing power of $\Pi^*_t$ under the Ramsey plan. That is, the instantaneous debt balance shall change to $B_t = B_{t-}q^*_{0,t-}/q^*_{0,t}$ from $B_{t-}$ and the term debt service flows change to $b_{t,s} = b_{t-,s}q^*_{0,t-}/q^*_{0,t}$ from $b_{t-,s}$ for all $s \geqslant t$.

not financed by term debt changes.

**Managing $\Pi_t^*$ with instantaneous debt only.**

The structure of the contractible subspace allows us to restrict debt restructuring policies to a set in which $\partial_t b_{t,s} = 0$ so that $b_{t,s} = b_{0,s}$ for all $s \geqslant t, t \geqslant 0$, so that no term debt is issued at any $t \geqslant 0$. That in turn means that the instantaneous debt balance $B_t$ equals $\Pi_t^*$ for all $t \geqslant 0$. Thus, it is sufficient to only use instantaneous debt to finance the cumulative liabilities that are governed by $d\Pi_t^* = r_{0,t}^* \Pi_t^* dt + \left(b_{0,t} + G_t - \tau_{0,t}^* N_{0,t}^*\right) dt$ as given in (30). With $B_0 = \Pi_0^*$, we obtain the following instantaneous debt balance dynamics:

$$dB_t = r_{0,t}^* B_t dt + \left(b_{0,t} + G_t - \tau_{0,t}^* N_{0,t}^*\right) dt \,. \tag{49}$$

**Remark 4.** *The debt rescheduling policies of Lucas and Stokey (1983) use only term debt. Those policies use a coupon flow management policy of $\{\partial_t b_{t,s}; s \geqslant t, t \geqslant 0\}$ and $\{B_t = 0; t \geqslant 0\}$ to satisfy the dynamic budget constraint:*

$$d\Pi_t^* = \left( \int_{s \geqslant t} q_{t,s}^* \partial_t b_{t,s} ds \right) dt - \left( \int_0^t \partial_u b_{u,t} du \right) dt \,. \tag{50}$$

*This policy requires continuous adjustments to the term debt adjustment and makes it challenging to formulate the Ramsey problem recursively.*

When the Ramsey planner has access to instantaneous debt, the Ramsey planner gains nothing by rescheduling term debt because instantaneous debt balance $B_t$ perfectly tracks the cumulative deficits $\Pi_t^*$ and hence summarizes both price and quantity information about the Ramsey plan. Furthermore, because instantaneous debt has the shortest maturity, it is least subject to future governments' manipulation. In the next section, we describe how a (local) commitment condition that is sufficient to convince continuation Ramsey planners to confirm a Ramsey plan.

# 8 Implementing the Ramsey Plan

In subsection 8.1, we implement the Ramsey plan with instantaneous debt under a (local) commitment condition that we add to the commitments that Lucas and Stokey (1983) and

Debortoli, Nunes, and Yared (2021) impose on continuation planners. In subsection 8.2 we assess the strength of the (local) commitment condition that constrains our implementation.

## 8.1 Commitment Condition and Implementation of Ramsey Plan

In our implementation, the government at time $t-$ leaves a debt structure $\{\widehat{B}_t, \widehat{b}_{0,s}; s \geqslant t\}$ to the time $t$ government, where

$$\widehat{B}_t = \Pi_t^* \text{ and } \widehat{b}_{t-,\,s} = b_{0,\,s}; \;\; s \geqslant t. \tag{51}$$

**Assumption 2 (local commitment condition).** *To preserve the purchasing power of the instantaneous debt inherited by a time t government, we impose:*

$$\widehat{B}_t U_{C,t}(C_{t,t}, N_{t,t}) = B_{t-} U_{C,t-}(C_{t-,t-}, N_{t-,t-}), \tag{52}$$

*where $C_{t,t}$ and $N_{t,t}$ are the time t government's choices.*

Note that if the Ramsey plan has been implemented until time $t-$, then $C_{s,s} = C_{0,s}^*$ and $N_{s,s} = N_{0,s}^*$ for $0 < s < t$. We now state:

**Proposition 6.** *The Ramsey plan can be implemented by leaving $\{b_{0,t}\}$ untouched and adjusting instantaneous debt according to (51) under the local commitment condition stated in Assumption 2.*

*Proof.* We use a recursion. Under the assumption the Ramsey plan is implemented up to time $t-$, it is sufficient if we prove that the Ramsey plan is also implemented at time $t$.[6] We can show that at time $t$ the continuation Ramsey planner, confronting $\widehat{B}_t = \Pi_t^*$ and $\widehat{b}_{t-,s} = b_{0,s}$ for all $s \geqslant t$, will confirm the continuation of the Ramsey plan by choosing $C_{t,s} = C_{0,s}^*$ and $N_{t,s} = N_{0,s}^*$ for all $s \geqslant t$. We can prove the preceding claim by subdividing time-$t$ continuation Ramsey planner's problem into two subproblems.

First, the time-$t$ continuation Ramsey planner chooses $\{C_{t,s}, s \geqslant t\}$ to attain

$$\max_{\{C_{t,s}; s \geqslant t\}} \int_t^\infty e^{-\rho(s-t)} U(C_{t,s}, C_{t,s} + G_s) ds, \tag{53}$$

[6] At $t = 0$, the Ramsey planner is also a time 0 continuation Ramsey planner.

where maximization subject to $\widehat{X}_t = \widehat{B}_t U_{C,t}(C_{t,t}, N_{t,t})$ and the following dynamics for $s \geqslant t$:

$$d\widehat{X}_s = \big(\rho\widehat{X}_s - C_{t,s}U_{C,s} - N_{t,s}U_{N,s} + b_{0,s}U_{C,s}\big)ds\,. \tag{54}$$

Similar to our analysis in Section 6, we can show that the value function for (53), denoted as $\widehat{V}(\widehat{X}_t, t; \Phi_t)$, is additively separable: $\widehat{V}(\widehat{X}_t, t; \Phi_t) = \Phi_t \widehat{X}_t + \widehat{H}(t; \Phi_t)$, where $\widehat{H}(t; \Phi_t)$ solves a differential equation (A-8) in Appendix A, and that $C_{t,s} = C(\Phi_t; b_{0,s}, G_s)$ where $C(\Phi_t; b_{0,s}, G_s)$ is the same as the one given in (39). This is because the initial debt term structure $\{b_{0,t}; t \geqslant 0\}$ is unchanged in our implementation with instantaneous debt only.

Under the local commitment condition given in Assumption 2, the time $t$ continuation Ramsey planner optimally chooses $C_{t,t} = C^*_{0,t}$ to confirm the continuation of the Ramsey plan because 1.) $\widehat{B}_t = \Pi^*_t$ and $B_{t-} = \Pi^*_{t-}$ under our implementation; 2.) $\Pi^*_{t-}U_{C,t-}(C^*_{0,t-}, N^*_{0,t-}) = \Pi^*_t U_{C,t}(C^*_{0,t}, N^*_{0,t})$ under the Ramsey plan; and 3.) the local commitment condition: $\widehat{B}_t U_{C,t}(C_{t,t}, N_{t,t}) = B_{t-}U_{C,t-}(C^*_{0,t-}, N^*_{0,t-})$. So we have proved $\widehat{X}_t = X^*_t = \Pi^*_t U_{C,t}(C^*_{0,t}, N^*_{0,t})$ and $C_{t,t} = C^*_{0,t}$.

Next, we prove that the time-$t$ continuation Ramsey planner will choose $C_{t,s} = C^*_{0,s}$ for all $s > t$. Because $\{C_{t,s}, b_{0,s}, G_s; s \geqslant t\}$ are Càdlàg processes and $C_{t,s} = C(\Phi_t; b_{0,s}, G_s)$, we have $\lim_{s\to t} C_{t,s} = \lim_{s\to t} C(\Phi_t; b_{0,s}, G_s) = C^*_{0,t} = C(\Phi^*_0; b_{0,t}, G_t)$. Under regularity conditions, we obtain $\Phi_t = \Phi^*_0$ and $C_{t,s} = C(\Phi^*_t; b_{0,s}, G_s) = C^*_{0,s}$ for all $s \geqslant t$.

Second, the time-$t$ continuation Ramsey planner has to set $\Phi_t$ to $\Phi^*_0$ (for the Ramsey plan) given in Proposition 4 to satisfy the time $t$ implementability condition:

$$\int_t^\infty e^{-\rho(s-t)}(C_{t,s}U_{C,s} + (C_{t,s} + G_s)U_{N,s})ds = \int_t^\infty e^{-\rho(s-t)} b_{0,s}U_{C,s}dt + \widehat{B}_t U_{C,t} \tag{55}$$

when maximizing $\widehat{V}(\widehat{X}_t, t; \Phi_t)$. This follows because (55) holds when $\Phi_t = \Phi^*_0$. □

## 8.2 Strength of Local Commitment Condition

To understand the strength of our local commitment condition, we begin by studying special settings in which the initial debt term structure and government expenditure processes $\{b_{0,t}, G_t; t \geqslant 0\}$ are continuous in time. In these settings, the local-commitment condition boils down to requiring that the consumption rate is continuous in $t$. The class of $\{b_{0,t}, G_t; t \geqslant 0\}$ processes that are continuous in time unleash all of the key economic forces that also shape Ramsey plans in more general settings in which $\{b_{0,t}, G_t; t \geqslant 0\}$ are Càdlàg processes, such as the process assumed in Debortoli, Nunes, and Yared (2021)'s counterexample presented in section 4. We verify this claim by noting how Ramsey plans in Càdlàg settings can be approximated arbitrarily well by Ramsey plans in an approximating continuous $\{b_{0,t}, G_t; t \geqslant 0\}$ process setting.

### 8.2.1 Continuous $\{b_{0,t}, G_t; t \geqslant 0\}$ settings

**Assumption 3.** *Exogenous flows of government expenditures* $\overrightarrow{G_0} = \{G_t; t \geqslant 0\}$ *and debt-service payouts* $\overrightarrow{b_0} = \{b_{0,t}; t \geqslant 0\}$ *are continuous functions of time.*

Under Assumption 3, the debt price process $\{q^*_{0,t}; t \geqslant 0\}$ and cumulative liability process $\{\Pi^*_t; t \geqslant 0\}$ are both continuous in time. The following lemma describes a Ramsey plan.

**Lemma 5.** *Under Assumption 3, the Ramsey plan characterized in Proposition 4 is continuous in time, i.e., the consumption and labor supply processes* $\{C^*_{0,t} = C(\Phi^*_0; b_{0,t}, G_t), N^*_{0,t} = N(\Phi^*_0; b_{0,t}, G_t)\}$*, tax rate process* $\{\tau^*_{0,t}; t \geqslant 0\}$*, instantaneous interest rate process* $\{r^*_{0,t}; t \geqslant 0\}$*, and cumulative liabilities* $\Pi^*_t$ *process are all continuous in time.*

Lemma 5 follows immediately by adding Assumption 3 to the hypotheses of Proposition 4 . Using Lemma 5, we show that in settings satisfying Assumption 3, the local commitment condition can be expressed in another way.

**Lemma 6.** *Under Assumption 3, the local commitment condition (52) under our instantaneous-debt-based implementation is equivalent to requiring that consumption is continuous in time* $t$*:*

$$C_{t,t} = C_{t-,t-} \,. \tag{56}$$

*Proof.* We use a recursion. When (56) holds up to time $t-$, it is enough to show that our local commitment condition is satisfied at time $t$. Under our implementation, choosing $\widehat{B}_t = \Pi^*_t$ at time $t$ implies that the instantaneous debt balance is continuous at $t$:

$$\widehat{B}_t = \Pi^*_t = \Pi^*_{t-} = B_{t-} \,, \tag{57}$$

where the second equality follows from the result that $\Pi^*_t$ is continuous in $t$ under Ramsey plan (Lemma 5) and the third equality uses the result that the instantaneous debt balance is given by $B_{t-} = \Pi^*_{t-}$ under our implementation. Substituting (57) into the local commitment condition (52), we obtain (56). □

Thus, when $b_{0,t}$ and $G_t$ are continuous in time $t$, our local commitment condition is equivalent to requiring that consumption is continuous in time $t$. Recall that in the Ramsey plan $\tau^*_{0,t} = 1 + U_N(C^*_{0,t}, N^*_{0,t})/U_C(C^*_{0,t}, N^*_{0,t})$. Since the tax rate is continuous in $t$ instantaneously, using instantaneous debt as we have described implements the Ramsey plan.[7]

### 8.2.2 Setting with Càdlàg $\{b_{0,t}, G_t; t \geqslant 0\}$ processes and discontinuities

For most of this paper we chose to stay as close as possible to discrete-time settings like Debortoli, Nunes, and Yared (2021)'s by focusing on the settings where the $b_{0,t}$ has discontinuities. In such discontinuous $\{b_{0,t}\}$ settings, the interest rate process is a Dirac delta function and is $\infty$ at discontinuity points.

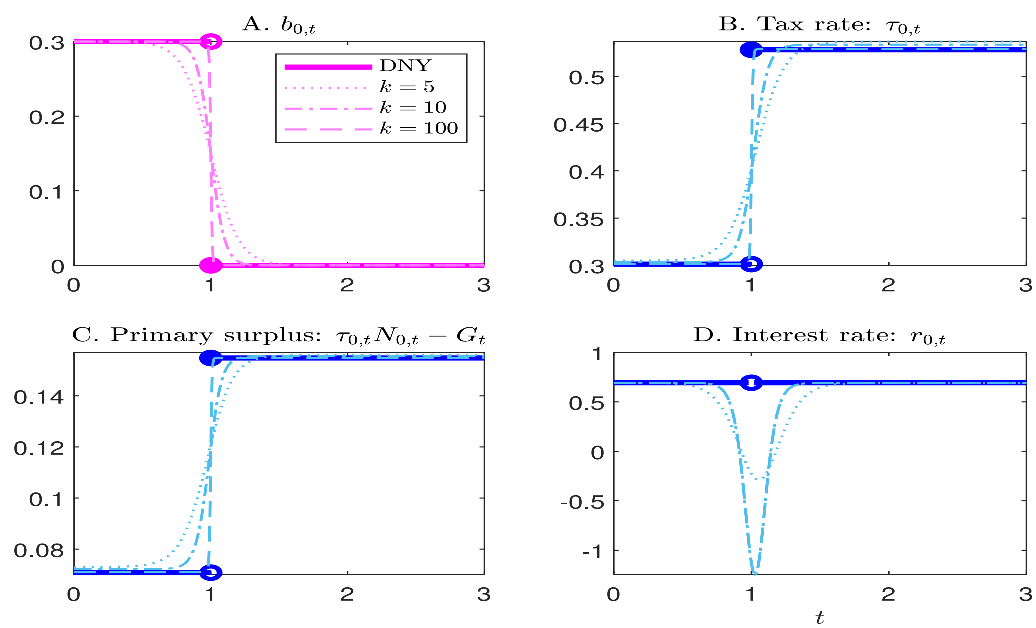



Figure 4: Ramsey plan for DNY economy ($\{b_{0,t} = 0.3, t < 1$ and $b_{0,t} = 0, t \geqslant 1\}$ with $G_t = 0.2, e^{-\rho} = 0.5, \eta = 1$) is obtained in the limit by a continuous function: $b^{(k)}_{0,t} = b_0(1 + e^{2k(t-1)})^{-1}$.

Nevertheless, because a discontinuous $\{b_{0,t}\}$ can be well approximated by the limit of

[7] In continuous $\{b_{0,t}, G_t; t \geqslant 0\}$ settings, our local commitment condition requires consumption rate to be continuous, it allows consumption to varying over every finite time interval. A way to interpret discrete-time models is that consumption flows in an underlying continuous-time models are restricted to be constant within each time period.

a sequence of continuous $\{b_{0,t}\}$ processes, confining ourselves to continuous $\{b_{0,t}\}$ and $\{G_t\}$ processes lets us isolate essential forces that shape a Ramsey plan. Figure 4 illustrates.[8]

# 9 Concluding Remarks

Along with Aguiar et al. (2019), we assign short-term government debt a central role as part of a Ramsey plan, but we do so in a closed economy setting in which endogenous interest rates are central. Disparate economic forces make short-term debt central in the two models. In our model, the Ramsey planner is a Stackelberg leader who manipulates bond prices to its advantage. Instantaneous debt is a powerful tool for the Ramsey planner to constrain continuation Ramsey planners in ways that facilitate implementing the Ramsey plan. Because the maturity of instantaneous debt is infinitesimal, its value is least vulnerable to manipulation by future governments. A local commitment condition is sufficient to prevent continuation Ramsey planners from diluting the instantaneous debt's purchasing power, inducing them to confirm a plan chosen by the time 0 government.

Debortoli et al.'s counterexample reveals that Lucas and Stokey's way of restructuring the term structure of government debt is sometimes unable to prevent future governments from wanting to manipulate bond prices, and so does not motivate them to follow the original Ramsey plan. This situation is related to a force in the durable goods monopoly problem of Coase (1972) and Stokey (1981) in which monopolists overproduce as time passes in the absence of commitment at time 0 to an intertemporal production plan. In our formulation with instantaneous debt and a (local) commitment condition, the Ramsey planner constrains its successors enough to induce them to confirm its plan.

Our continuous time version of Debortoli et al.'s model sets the stage for adding shocks and complete markets in the spirit of Lucas and Stokey. Doing that will allow us to deepen our understanding of how instantaneous debt and our local commitment condition connects to the contractible space assumed in Black and Scholes (1973) and Harrison and Kreps (1979).

We close by noting how the US Treasury and Congress Budget Office have gone partway towards the arrangement that we have used to implement the Ramsey plan. Although

[8]Our analysis of Ramsey plans in settings with continuous $\{b_{0,t}\}$ and $\{G_t\}$ processes can help us understand Ramsey plans and their implementations from discrete-time models. Thus, start with a Ramsey plan in a setting with continuous $\{b_{0,t}\}$ and $\{G_t\}$ processes. For a finite time interval $\Delta$, construct an associated discrete-time model by forming a discrete time consumption process $\{C_{0,t}\Delta\}_{t=0}^{\infty}$ and so on. The discrete time consumption process is "discontinuous in $t$" even though the consumption rate $C_{0,t}$ is continuous.

it doesn't now issue a counterpart to instantaneous debt $B_t$ as the government in our model do, it books and reports accumulated past primary deficits $\Pi_t^*$.

# References


Aguiar, M., M. Amador, H. Hopenhayn, and I. Werning (2019). Take the short route: Equilibrium default and debt maturity. *Econometrica 87*(2), 423–462.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 637–654.

Coase, R. H. (1972). Durability and monopoly. *The Journal of Law and Economics 15*(1), 143–149.

Debortoli, D., R. Nunes, and P. Yared (2021). Optimal fiscal policy without commitment: Revisiting lucas-stokey. *Journal of Political Economy 129*(5), 1640–1665.

Harrison, J. M. and D. M. Kreps (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic theory 20*(3), 381–408.

Kanwal, R. P. (2012). *Generalized functions: theory and applications.* Springer Science & Business Media.

Ljungqvist, L. and T. J. Sargent (2025). *Recursive Macroeconomic Theory, Fifth Edition.* Cambridge, Massachusetts: MIT Press.

Lucas, R. E. and N. L. Stokey (1983). Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics 12*(1), 55–93.

Stokey, N. L. (1981). Rational expectations and durable goods pricing. *The Bell Journal of Economics*, 112–128.

# Appendices

# A Details

*Proof of Lemma 2.* We want to prove Lemma 2 for the utility function given by (16) for $\eta > 0, \gamma \geqslant 1$, special initial debt structure of (17), and constant government spending process, as used in Debortoli, Nunes, and Yared (2021).

We first derive the unique debt restructuring policy that may induce future government to confirm the Ramsey plan. Follow the same procedure to derive (23) in Section 4, we obtain the debt restructuring policy for the case with $\eta > 0, \gamma \geqslant 1$ below

$$\hat{b}_{0,t} = \begin{cases} \hat{b}_0 = C_0^* \left(1 - \eta (C_0^* + G)^\gamma\right), & t \in [0, T) \\ \hat{b}_1 = \left(e^{\rho T} - 1\right) \left(\frac{b_0}{C_0^*} - 1 + \eta (C_0^* + G)^\gamma\right) C_1^*, & t \geqslant T \end{cases} \tag{A-1}$$

Next, we show that, under Ramsey plan, $C_1^*$ is a decreasing function of $b_0$, and $\widehat{b}_1 / C_1^*$ is an increasing function of $b_0$. This is because 1) $C(\Lambda_0; 0, G)$ is an decreasing function of $\Lambda_0$; 2) $\eta (C(\Lambda_0; b_0, G) + G)^\gamma + \frac{b_0}{C(\Lambda_0; b_0, G)}$ is an increasing function of $b_0$ when $b_0 \geqslant 0$; and 3) the following equation for $\Lambda_0$

$$\frac{1 - e^{-\rho T}}{\rho} \left(1 - \eta (C(\Lambda_0; b_0, G) + G)^\gamma - \frac{b_0}{C(\Lambda_0; b_0, G)}\right) + \frac{e^{-\rho T}}{\rho} \left(1 - \eta (C(\Lambda_0; 0, G) + G)^\gamma\right) = 0\,, \tag{A-2}$$

holds. Note that, for fixed $\Lambda_0$, the first part of (A-2) decreases in $b_0$, and $C(\Lambda_0; 0, G)$ decreases in $\Lambda_0$. To let (A-2) hold, it requires a higher $\Lambda_0$ for larger $b_0$. Thus, $\Lambda_0^*$ increases in $b_0$. As a result, $C_1^* = C(\Lambda_0^*; 0, G)$ decreases in $b_0$ and $\frac{\hat{b}_1}{C_1^*} = \left(e^{\rho T} - 1\right) \left(\frac{b_0}{C_0^*} - 1 + \eta (C_0^* + G)^\gamma\right)$ increases in $b_0$ because of (A-2).

Then, we formulate the continuation Lagrangian problem for the continuation Ramsey planner at time $t = T$. The Lagrangian multiplier $\Lambda_T$, implied by the first order condition under the continuation of the Ramsey plan, satisfies

$$\begin{aligned} \Lambda_T &= -\frac{1 - \eta C_1^* \left(C_1^* + G\right)^{\gamma - 1}}{\hat{b}_1 / C_1^* - \eta \gamma C_1^* \left(C_1^* + G\right)^{\gamma - 1}} \\ &= -\frac{1 - \eta C_1^* \left(C_1^* + G\right)^{\gamma - 1}}{1 - \eta \left(C_1^* + G\right)^\gamma - \eta \gamma C_1^* \left(C_1^* + G\right)^{\gamma - 1}} \end{aligned} \tag{A-3}$$

where the second equality follows from $\hat{b}_1/C_1^* = 1 - \eta\,(C_1^* + G)^\gamma$ because of the budget constraint for $\hat{b}_1$ under the Ramsey plan.

Lastly, when $b_0$ is too high, the implied $C_1^*$ is too low, $\Lambda_T$ given in (A-3) under the Ramsey plan could be negative, making the adoption of the continuation of the Ramsey plan for the continuation Ramsey planner suboptimal. Thus, there exists a $b^*$ such that only when $b_0 \leqslant b^*$, the Ramsey plan is implementable using the term debt only restructuring policy.

□

*Technical Details for Proposition 2.* Here we provide two regularity conditions such that the Ramsey plan for the Ramsey problem exists. First, we assume that, for the given initial debt structure, admissible allocation set is not empty, i.e., there exists allocations $\{C_{0,t}; t \geqslant 0\}$ that satisfy the implementability condition (7). Second, we assume that, for the given utility function $U(C, C+G)$, the function $(CU_C + (C+G)U_N - bU_C)$ is concave for $G > 0, b \geqslant 0$.

For instance, it is easy to verify that, with utility (16) and initial debt structure (17) proposed in Debortoli, Nunes, and Yared (2021) and given the condition that $b_0$ is not too high, i.e., lower than a given threshold, the two regularity conditions are satisfied. □

*Proof of Proposition 3.* Given $\Phi_0$, consumption at $t$, $C(\Phi_0; b_{0,t}, G_t)$, solves

$$\max_{C_{0,t}} \; U(C_{0,t}, C_{0,t} + G_t) + \Phi_0\left(C_{0,t}U_{C,t} + (C_{0,t} + G_t)U_{N,t} - b_{0,t}U_{C,t}\right). \tag{A-4}$$

As in Lucas and Stokey (1983) and Debortoli, Nunes, and Yared (2021), under regularity conditions (i.e., As provided in the technical details for Proposition 2, we may impose the following regularity condition: for the given utility function $U(C, C+G)$, the function $(CU_C + (C+G)U_N - bU_C)$ is concave in $C$ for $G > 0, b \geqslant 0$.) such that inner solution exits, $C(\Phi_0; b_{0,t}, G_t)$ satisfies the following first-order necessary condition for problem (A-4):

$$(1+\Phi_0)(U_{C,t} + U_{N,t}) + \Phi_0\left[(C_{0,t} - b_{0,t})(U_{CC,t} + U_{CN,t}) + (C_{0,t} + G_t)(U_{NC,t} + U_{NN,t})\right] = 0, \tag{A-5}$$

The associated labor supply at time $t$ is given by $N(\Phi_0; b_{0,t}, G_t) = C(\Phi_0; b_{0,t}, G_t) + G_t$.

Now we solve ODE (38) for $H(t; \Phi_0)$. Using $C(\Phi_0; b_{0,t}, G_t), N(\Phi_0; b_{0,t}, G_t)$, we can rewrite (38) as

$$d(-e^{-\rho t}H(t;\Phi_0)) = e^{-\rho t}\Big[U(C(\Phi_0;b_{0,t},G_t),N(\Phi_0;b_{0,t},G_t)) \\ + \Phi_0\left(N(\Phi_0;b_{0,t},G_t)U_{C,t} + N(\Phi_0;b_{0,t},G_t)U_{N,t} - b_{0,t}U_{C,t}\right)\Big]dt.$$

Integrating both sides from $t$ to $\infty$ and using the transversality condition $\lim_{t\to\infty} e^{-\rho t}H(t;\Phi_0) = 0$, we obtain

$$H(t;\Phi_0) = \int_t^\infty e^{-\rho(s-t)}\Big[U(C(\Phi_0;b_{0,s},G_s),N(\Phi_0;b_{0,s},G_s)) \\ + \Phi_0\left(N(\Phi_0;b_{0,s},G_s)U_{C,s} + N(\Phi_0;b_{0,s},G_s)U_{N,s} - b_{0,s}U_{C,s}\right)\Big]dt.$$

Consequently, value function $V(X_t,t)$ for the DP problem defined in (3) satisfies

$$\begin{aligned} V(X_t,t) &= -\Phi_0 X_t + H(t;\Phi_0) \\ &= \int_t^\infty e^{-\rho(s-t)}U(C(\Phi_0;b_{0,s},G_s),N(\Phi_0;b_{0,s},G_s))ds + \Phi_0\Pi_t U_{C,t} \\ &\quad -\Phi_0\int_t^\infty e^{-\rho(s-t)}\left(N(\Phi_0;b_{0,t},G_t)U_{C,t} + N(\Phi_0;b_{0,t},G_t)U_{N,t} - b_{0,t}U_{C,t}\right)ds \\ &= \int_t^\infty e^{-\rho(s-t)}U(C(\Phi_0;b_{0,s},G_s),N(\Phi_0;b_{0,s},G_s))ds \end{aligned}$$

where the last equality holds because that, given the definition of $\Pi_t$ in (29), the time$-0$ budget constraint is equivalent to

$$-\Pi_t U_{C,t} + \int_t^\infty e^{-\rho(s-t)}U(C(\Phi_0;b_{0,s},G_s),N(\Phi_0;b_{0,s},G_s))ds = 0\,.$$

□

*Technical Details for Proposition 6.* Here we provide technical details when solve the time $t$ continuation Ramsey planner's dynamic problem. First, the value function $\widehat{V}(\widehat{X}_s,s;\Phi_t)$ for $s \geqslant t$ satisfies the following partial differential equation

$$\rho\widehat{V} = \frac{\partial\widehat{V}}{\partial s} + \max_{C_{t,s}}\ U(C_{t,s},C_{t,s}+G_s) + \frac{\partial\widehat{V}}{\partial X_s}\left(\rho X_s - C_{t,s}U_{C,s} - (C_{t,s}+G_t)U_{N,s} + b_{0,s}U_{C,s}\right)\,. \tag{A-6}$$

We guess and verify that $\widehat{V}(\widehat{X}_s, s)$ is additively separable in $\widehat{X}_t$ and $s$:

$$\widehat{V}(\widehat{X}_s, s) = -\Phi_t \widehat{X}_s + \widehat{H}(s; \Phi_t); \;\; s \geqslant t \,, \tag{A-7}$$

where $\widehat{H}(s; \Phi_t)$ satisfies the following ordinary differential equation (ODE) for $s \geqslant t$:

$$\rho \widehat{H}(s; \Phi_t) = \widehat{H}'(s; \Phi_t) + \max_{C_{t,s}} \;\; U(C_{t,s}, C_{t,s} + G_s) + \Phi_t \left( C_{t,s} U_{C,s} + (C_{t,s} + G_s) U_{N,s} - b_{0,s} U_{C,s} \right). \tag{A-8}$$

Solving this ODE yields

$$\begin{aligned} \widehat{H}(s; \Phi_t) = \int_s^\infty e^{-\rho(u-s)} \left[ U(C(\Phi_t; b_{0,u}, G_u), C(\Phi_t; b_{0,u}, G_u) + G_u) \right. \\ \left. + \Phi_t \left( C(\Phi_t; b_{0,u}, G_u) U_{C,u} + (C(\Phi_t; b_{0,u}, G_u) + G_u) U_{N,u} - b_{0,u} U_{C,u} \right) \right] du. \end{aligned} \tag{A-9}$$

□