Implementing an Optimal Tax Plan with Short Term Debt*

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Abstract

Debortoli, Nunes, and Yared (2021) showed that when initial government debt is too high, Lucas and Stokey's (1983) debt management policy fails to implement a Ramsey plan for flat-rate taxes on labor. We show that preventing continuation Ramsey planners from resetting current-period tax rates and using short-term debt to finance accumulated primary deficits implements all Ramsey plans, including those with the high debt levels that trouble Lucas and Stokey's implementation. Our implementation equalizes Lagrange multipliers on distinct implementability constraints that face the Ramsey planner and the continuation planners, an essential indication of a successful implementation. We provide examples of our implementation under different initial debt structures and government spending patterns.

Keywords: Ramsey planner, continuation Ramsey planner, implementability, short term debt

JEL Classification: E3, E6, C7

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1 Introduction

Lucas and Stokey (1983) implemented an optimal plan for flat-rate taxes on labor by managing the term structure of government debt under a timing protocol that represents an "intermediate situation" of "partial commitment".[1] In their model, a government finances exogenous sequences $\{G_i\}_{i=0}^{\infty}$ of government expenditures and initial debt service coupons $\{b_{-1,i}\}_{i=0}^{\infty}$ with a sequence of flat-rate taxes on labor, the only factor of production. At time 0, a planner chooses a sequence of flat-rate taxes $\{\tau_{0,i}\}_{i=0}^{\infty}$ and restructures an initial debt coupon process $\{b_{-1,i}\}_{i=0}^{\infty}$ to become $\{b_{0,i}\}_{i=0}^{\infty}$. In each period $i > 0$, a continuation planner must honor the debt service sequence that it inherits from a period $i-1$ planner,[2] but is free to redesign the continuation of the flat-rate tax sequence and to reschedule government debt service from period i onward.[3] Lucas and Stokey said that an optimal plan for setting tax rates is "time consistent"[4] if all continuation planners choose to continue it. Lucas and Stokey (1983, Sec. 3) provided eight examples in which restructuring government debts and issuing consols implements a Ramsey plan for flat-rate taxes on labor.

Debortoli et al. (2021) constructed examples in which initial government debts are so high that in Lucas and Stokey's way restructuring government debt fails to motivate continuation planners to continue the Ramsey tax plan; coincidentally, the Ramsey plan sets a tax rate above the peak of the Laffer curve; the continuation Ramsey planner instead sets that tax rate below the peak of the Laffer curve.

We modify Lucas and Stokey's timing protocol by withdrawing from each continuation planner the authority to reset today's tax rate and adding the authority to set tomorrow's tax rate. Under our timing protocol, there exist debt management policies, different from Lucas and Stokey's, that implement a Ramsey plan whenever one exists. We focus on one such policy that each period involves minimally rescheduling the initial debt sequence. This policy uses short-term debt to finance primary government deficits accumulated under the Ramsey tax plan and payments due each period under the initial debt structure $\{b_{-1,i}\}_{i=0}^{\infty}$.[5]

To appreciate this debt management policy, it is useful to recall how Aguiar, Amador, Hopenhayn, and Werning (2019) contrasted Lucas and Stokey's (1983) model with theirs:

[1] Because they are not implemented under sequential timing protocols that actually characterize government decision-making processes, Kydland and Prescott (1977) and Calvo (1978) argued that optimal plans are implausible.

[2] This is the "commitment" part of Lucas and Stokey's "partial commitment."

[3] This is the "partial" part of Lucas and Stokey's "partial commitment."

[4] We say that it is "implemented."

[5] Subsection 5.3 describes other possible policies.

Lucas and Stokey (1983) studied optimal fiscal policy with complete markets and discussed at length how maturity choice is a useful tool to provide incentives to a government that lacks commitment to taxes and debt issuance, but cannot default. The government has an incentive to manipulate the risk-free real interest rate, by changing taxes which affects investors' marginal utility, to alter the value of outstanding long-term bonds, something ruled out by our small open-economy framework with risk-neutral investors. Their main result is that the maturity of debt should be spread out, resembling the issuance of consols. Our model instead emphasizes default risk, something absent from their work. Our main result is also the reverse, providing a force for the exclusive use of short-term debt.

Short-term debt takes center stage in our model too, but for different reasons because our economic environment differs from Aguiar et al.'s. Unlike Aguiar et al. (2019), we retain almost all parts of Lucas and Stokey's and Debortoli et al.'s structure including complete markets, a closed economy with endogenous interest rates, the presence of incentives that continuation Ramsey planners have to use tax rates to manipulate interest rates, obligations of governments to pay their debts, and Lucas and Stokey's "partial commitment" assumption:

We focus on a situation in which there are no binding commitments on future taxes but in which debt commitments are fully binding. . . . (this) seems to come closest to the institutional arrangement we observe in any stable, democratically governed countries. . . . Our main finding, in this intermediate situation, is that being unable to make commitments about future tax rates is not a constraint. In the absence of any ability to bind choices about tax rates directly, each government restructures the debt in a way that induces its successors to continue with the optimal tax policy. For this to be possible, a rich enough mix of debt instruments must be available, where a rich enough means, roughly, one security for each dated, state-contingent good being traded.

Lucas and Stokey (1983, p. 69)

Lucas and Stokey concluded that commitment to servicing debt suffices to induce successor governments to implement the Ramsey tax plan, but Debortoli et al.'s counterexample shows that actually it may not be enough. Provided that we also substantially modify their debt management policy, by marginally altering Lucas and Stokey's timing protocol we can make it be enough.

Related literature

Barro (1979), Lucas and Stokey (1983), and Debortoli et al. (2021) are the most pertinent references for us, but many other papers are related to aspects of this paper. Angeletos (2002) shows that by choosing the maturity structure of non-state-contingent bonds, a government can smooth tax rates in response to shocks to its expenditures as effectively as if it had access to complete markets in state-contingent government debt. He provides an example in which the government does that by trading a risk-free one-period bond and a consol. Buera and Nicolini (2004) showed quantitatively that such a government portfolio policy entails large trades, hundreds of times GDP), that would be very sensitive to a model's parameter values. Aiyagari et al. (2002) study optimal taxation in a setting in which a government can issue only one-period risk-free debt. They show that an optimal policy combines features of Barro's (1979) model (a near-unit root component in taxes and debt) and Lucas and Stokey's (1983) (strong dependence of taxes and deficits on current shocks). Like Angeletos (2002) and Buera and Nicolini (2004), Shin (2007) shows that a government can implement complete-market Ramsey allocations by actively managing just a few bonds. Shin uses his model to explain why in the 18th century Britain issued long-term bonds during peacetime and short-term bonds during wars. Bassetto (2014) studies how heterogeneity among economic agents affects optimal government responses to wartime surges in expenditures. If a government favors taxpayers, it runs larger deficits during wars, while if a government favors rentiers who earn most of their income from assets, it raises taxes immediately. He uses his model to understand differences between British and French fiscal policies during the 17th and 18th centuries.

Bhandari et al. (2017) provide conditions under which government debt under a Ramsey plan converges to an invariant distribution. They construct an approximation to that asymptotically invariant distribution as well as an approximation to the rate of convergence to it. They show how binding implementability constraints push government debt in a direction that eventually lets the government use fluctuations in equilibrium interest rates to insure against shocks to government expenditures, rather than using fluctuations in the par value of debt. Aparisi de Lannoy et al. (2025) study optimal taxation and government debts of different maturities, specifying preferences and shock process to match macro and asset pricing facts, something that previous papers had not done. They provide formulas that express an optimal government portfolio policy as a function of statistics formed from financial and macroeconomic data. For US data, they find that it is optimal for the government to issue bonds in amounts that decline approximately exponentially with increases in their times to maturity.

Zhu (1992) constructs a complete markets model with capital. He finds that while optimal capital income tax rates are indeterminate, their one-period ahead conditional expectation equals zero. Chari et al. (1994) construct a quantitative version of a similar model. They find that in a stationary equilibrium the optimal *ex ante* tax rate on capital income is approximately zero, that optimal labor tax rates exhibit the same persistence as exogenous economic shocks and have low volatility, and that most welfare gains come from setting the initial tax rate at an arbitrary exogenous upper bound. Chari et al. (2020) also study optimal capital taxation. They find that when a government has access to enough instruments, capital should not be taxed. Barro and Chari (2024) show that zero taxation of capital income is optimal when a government can't impose a capital levy.

An essential aspect of Lucas and Stokey (1983) is that the initial debt structure $\{b_{-1,i}\}_{i=0}^{\infty}$ can be any admissible function of calendar time, which shapes and substantially complicates their analysis and our analysis here. By way of contrast, some related papers about debt dynamics have for tractability assumed exponentially maturing term debt with a constant amortization rate.[6] Using such exponential term debt structures, DeMarzo and He (2021) extended the classic capital-structure model of (Leland, 1994) in a corporate finance context, while DeMarzo, He, and Tourre (2023) studied a sovereign finance environment. In their settings, a borrower continuously ratchets its term debt upward until it eventually defaults. A borrower's incentive to dilute the value of existing debt claims shape dynamics in these papers.[7]

Organization

Section 2 describes a setting with an infinite horizon and a discrete time increment $\Delta > 0$. A representative household and benevolent government participate in competitive markets with distorting flat-rate taxes on labor. The household supplies labor to produce goods that

[6]See Arellano and Ramanarayanan (2012) and Chatterjee and Eyigungor (2012) for examples in sovereign debt settings.

[7]This debt ratcheting mechanism resembles a mechanism that drives the price of a durable good to zero in a durable goods monopoly setting without commitment studied by Coase (1972), Stokey (1981).

are allocated between consumption and government expenditures. The government finances an exogenous sequence of government expenditures and an exogenous initial sequence of debt payments by collecting a flat-rate tax on labor. A manifold of competitive equilibria with distorting taxes is indexed by sequences of budget-feasible flat-rate taxes on labor.

Subsection 3.1 describes how a Ramsey planner chooses tax rates to maximize household utility subject to resource and implementability constraints implied by the household's first-order conditions. Subsection 3.2 introduces continuation Ramsey planners who must honor debt structures inherited from previous planners when they decide whether to confirm the Ramsey planner's intertemporal plan for setting tax rates. Understanding that, the Ramsey planner chooses how to restructure the exogenous initial government debt structure. This subsection also introduces a different timing protocol that lets continuation planners set taxes one period in advance and prevents them from resetting current-period tax rates.

Section 4 presents a debt management policy that under the new timing protocol implements the optimal plan. Our debt management policy "takes a short route" by leaving longer-term debts unchanged and using short-term debt to finance the resulting accumulated unpaid government liabilities under the optimal plan, defined as $\Pi_j^* = \sum_0^{j-1} \frac{q_{0,i}^*}{q_{0,j}^*}\left(b_{-1,i} - S_{0,i}^*\right)$, where $\{q_{0,i}^*\}_{i=0}^{\infty}$ and $\{S_{0,i}^*\}_{i=0}^{\infty}$ are bond price and primary surplus processes associated with the Ramsey plan. Note how a period i flow liability $\left(b_{-1,i} - S_{0,i}^*\right)$ times $\frac{q_{0,i}^*}{q_{0,j}^*}$ contributes to the period j value of government liabilities. The $\{\Pi_j^*\}_{j=0}^{\infty}$ process consolidates all equilibrium price and quantity information about the exogenous initial term debt structure that concerns the Ramsey planner. We use forward induction to prove that this debt management policy implements the optimal plan under our timing protocol.

Subsection 4.2 connects conditions for implementing the optimal plan with the invariance and equality of Lagrange multipliers on the implementability constraints that confront the planner and the continuation planners. Subsection 4.3 conducts a reverse engineering exercise that reveals mechanics that underlie our implementation. We work backwards from decision rules that express the tax rates chosen by the Ramsey planner and continuation Ramsey planners as functions of period-i government expenditures and debt payments as well as the Lagrange multipliers for their respective implementability constraints. While the functional forms of these decision rules are identical, for the Ramsey planner and for continuation Ramsey planners, different arguments enter these functions. For a Ramsey plan to be implemented, these arguments must be aligned appropriately. This logic puts restrictions on debt management policies and Lagrange multipliers that imply the "short route" debt management policy together with the invariance of the respective Lagrange multipliers.

Section 5 studies Debortoli et al.'s counterexamples in which Lucas and Stokey's debt management policy fails to implement Ramsey plans when initial debts are too high. We show how our modified timing protocol and debt management policy implement a Ramsey plan regardless of whether Ramsey tax rates fall above or below the peak of a Laffer curve. It is invariance of Lagrange multipliers for Ramsey and continuation plans that is essential for implementability (see subsection 5.2). Subsection 5.3 shows that for some initial debt structures, but not for others, policies that issue both short and longer term debts also implement a Ramsey plan. Using both short- and long-term debt can fail to implement a Ramsey plan, for example, when initial debts are too high.

Section 6 describes additional examples that alter initial debt structures and government expenditure patterns. We designed these examples to illustrate covariation of outcomes under different exogenous government expenditure and initial term debt structures. By slightly enriching initial debt structures in Debortoli et al.'s counterexamples, subsection 6.1 shows that whether Lucas-Stokey's implementation works does not depend on whether the Ramsey planner sets a tax rate on the 'bad' side of the Laffer curve. Subsection 6.2 studies Ramsey tax plans and debt management policies that implement them when initial debt payments decline exponentially toward zero, while subsection 6.3 periodic initial debt sequences, and subsection 6.4 studies periodic government spending sequences. Associated optimal plans display different patterns of government accumulations of one-period liabilities and consequent issues of one-period debt that illustrate how our implementation works.

Section 7 discusses possible extensions. A technical appendix ends our paper.

2 Setting

There is no uncertainty.[8] Time t_i is discrete with increment $\Delta > 0$, so $\{t_i = i\Delta, i = 0, 1, \cdots\}$. Period i means time t_i. A representative household and a benevolent government participate in a complete set of perfectly competitive markets. The representative household supplies a labor supply sequence $\{N_{0,i}\Delta\}_{i=0}^{\infty}$ that produces a sequence of a single

[8] Lucas and Stokey (1983, ftnt. 1) said that many, if not most, of the issues analyzed in their paper could be analyzed in a context without uncertainty, like the setting analyzed by Turnovsky and Brock (1980). Lucas and Stokey's footnote 2 about the consequences of Turnovsky and Brock's restricting continuation governments to issue only short term debt is especially interesting in view of the substantial difference between Lucas and Stokey's debt management policy and ours.

nonstorable good that must be allocated between a consumption sequence $\{C_{0,i}\Delta\}_{i=0}^{\infty}$ and an exogenous government expenditure sequence $\{G_i\Delta\}_{i=0}^{\infty}$:

$$C_{0,i}\Delta + G_i\Delta = N_{0,i}\Delta\,, \text{for all}\, i \geqslant 0\,, \tag{1}$$

where a first subscript indicates a time that a variable is chosen and a second subscript indicates a time that a variable is realized. The representative household orders consumption and labor supply streams $\{C_{0,i}\}_{i=0}^{\infty}$ and $\{N_{0,i}\}_{i=0}^{\infty}$ according to

$$\sum_{i=0}^{\infty} e^{-\rho i\Delta} U(C_{0,i}, N_{0,i})\Delta\,, \tag{2}$$

where $\rho > 0$ and $U(\cdot,\cdot)$ is strictly increasing in consumption C, strictly decreasing in labor supply N, globally concave, and continuously differentiable. Along with Debortoli et al. (2021), we assume that labor supply $\{N_{0,i}\}$ has no upper bound.

In period 0, the government inherits an initial term-debt structure $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$ that it must service: the government must pay its creditors an amount $b_{-1,i}\Delta$ during time interval $[i\Delta, (i+1)\Delta)$. Let $\tau_{0,i}$ denote a period i flat tax rate that the government sets at period $i = 0$ and let $q_{0,i}$ be the period 0 value of a zero-coupon bond with a unit payoff at time i. The representative household faces a single intertemporal budget constraint:

$$\sum_{i=0}^{\infty} \big(C_{0,i}\Delta\big) \cdot q_{0,i} \leqslant \sum_{i=0}^{\infty} \big(b_{-1,i}\Delta\big) \cdot q_{0,i} + \sum_{i=0}^{\infty} \big((1-\tau_{0,t_i})N_{0,i}\Delta\big) \cdot q_{0,i}\,. \tag{3}$$

The left side of (3) is the present value of the household's consumption stream $\{C_{0,i}\}_{i=0}^{\infty}$ and the right side is the sum of the household's financial wealth and its human wealth, the present value of after-tax labor income.

Given tax rate sequence $\{\tau_{0,i}\}_{i=0}^{\infty}$ and bond price sequence $\{q_{0,i}\}_{i=0}^{\infty}$, the household chooses $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$ to maximize (2) subject to constraint (3). First-order necessary conditions for $N_{0,i}$ and $C_{0,i}$ are

$$1 - \tau_{0,i} = -\frac{U_{N,i}}{U_{C,i}} \tag{4}$$

$$q_{0,i} = e^{-\rho i\Delta}\frac{U_{C,i}}{U_{C,0}}\,. \tag{5}$$

Let

$$S_{0,i}\Delta = (\tau_{0,i}N_{0,i} - G_i)\,\Delta \tag{6}$$

denote the primary government budget surplus over time interval $[i\Delta, (i+1)\Delta)$. Constraint (3) becomes[9]

$$\sum_{i=0}^{\infty} \left(S_{0,i}\Delta\right) \cdot q_{0,i} \geqslant \sum_{i=0}^{\infty} \left(b_{-1,i}\Delta\right) \cdot q_{0,i}\,. \tag{7}$$

The left side is the time 0 value of primary government budget surplus $\{S_{0,i}\Delta\}_{i=0}^{\infty}$ and the right side is the time 0 value of the government debt payout sequence $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$.

Definition 1. *Given the government's initial term-debt structure $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$ and expenditure sequence $\{G_i\Delta\}_{i=0}^{\infty}$, a competitive equilibrium is a feasible allocation $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$, a flat-rate tax sequence $\{\tau_{0,i}\}_{i=0}^{\infty}$, and a bond price sequence $\{q_{0,i}\}_{i=0}^{\infty}$ for which*

- *the government's budget constraint* (1) *is satisfied, and*
- *given government spending, tax, and bond price sequences, the allocation solves the household's optimization problem*

We follow Lucas and Stokey (1983) and use first-order conditions (4) and (5) together with feasibility constraint (1) to eliminate tax rates and bond prices from the government budget constraint (7) and thereby to deduce the following implementability constraint on competitive equilibrium allocations $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$:

$$\sum_{i=0}^{\infty} e^{-\rho i\Delta}\left[C_{0,i}U_{C,i} + (C_{0,i} + G_i)U_{N,i} - b_{-1,i}U_{C,i}\right] \cdot \Delta \geqslant 0\,. \tag{8}$$

Proposition 1. *Given an initial term-debt structure $\{b_{-1,i}\Delta\}_{i=0}^{\infty}$ and government spending sequence $\{G_i\Delta\}_{i=0}^{\infty}$, an allocation $\{C_{0,i}\Delta, N_{0,i}\Delta\}_{i=0}^{\infty}$ is a competitive equilibrium allocation if and only if it satisfies (8) at $i = 0$ and (1) for all $i \geqslant 0$.*

3 Ramsey and Continuation Ramsey Plans

A Ramsey planner and a sequence of continuation Ramsey planners are linked together by a structure of 'partial commitments' like those described by Lucas and Stokey (1983, p. 69) in which debts incurred by previous planners must be paid or rescheduled at equilibrium prices, but sequences of tax rates can be redesigned each period.

[9] Competitive equilibrium and Walras' Law underlies this assertion.

3.1 Ramsey Plan

A Ramsey planner chooses the best competitive equilibrium. It can accomplish this by choosing a sequence $\{C_{0,i}\Delta\}_{i=0}^{\infty}$ that maximizes (2) subject to resource constraint (1) and implementability constraint (8). Given that sequence, we can use first-order necessary conditions (4) and (5) to compute associated equilibrium tax rate and bond price sequences.

We denote competitive equilibrium objects associated with a Ramsey plan as

$$\{C_{0,i}^{*}\Delta, N_{0,i}^{*}\Delta, \tau_{0,i}^{*}, q_{0,i}^{*}\}_{i=0}^{\infty}.$$

An equivalence class $\mathcal{D}$ of continuation debt structures $\{b_{0,i}\}_{i=1}^{\infty}$ supports a Ramsey tax plan.[10]

Characterizing Ramsey plan. Attach a nonnegative multiplier Φ_0 to implementability constraint (8) and form a Lagrangian:

$$\mathcal{L}_0 = \sum_{i=0}^{\infty} e^{-\rho i\Delta}\left[U(C_{0,i}, C_{0,i}+G_i) + \Phi_0\left(C_{0,i}U_{C,i} + (C_{0,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\right]\Delta\,. \tag{9}$$

The Ramsey planner maximizes the right side of (9) over consumption plans $\{C_{0,i}\}_{i=0}^{\infty}$ and minimizes over multiplier Φ_0. Call the extremizing values: $\{C_{0,i}^{*}\}_{i=0}^{\infty}$ and Φ_0^{*}. For a given Φ_0, the optimal consumption plan satisfies

$$C(\Phi_0; b_{-1,i}, G_i) := \arg\max_{C_{0,i}}\left[U(C_{0,i}, C_{0,i}+G_i) + \Phi_0\left(C_{0,i}U_{C,i} + (C_{0,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\right]. \tag{10}$$

Substituting (10) into implementability condition (8), we obtain

$$\sum_{i=0}^{\infty} e^{-\rho i\Delta}\big[C(\Phi_0; b_{-1,i}, G_i)U_{C,i} + (C(\Phi_0; b_{-1,i}, G_i) + G_i)U_{N,i} - b_{-1,i}U_{C,i}\big]\Delta \geqslant 0\,. \tag{11}$$

Substituting $C(\Phi_0; b_{-1,i}, G_i)$ into the right side of (9) lets us write $\mathcal{L}_0$ as a function $\mathcal{L}_0(\Phi_0)$. An optimal Φ_0^{*} that maximizes $\mathcal{L}_0(\Phi_0)$ must satisfy (11). The Ramsey allocation can be represented with two functions, namely, $C_{0,i}^{*} = C(\Phi_0^{*}; b_{-1,i}, G_i)$ and $N_{0,i}^{*} = C_{0,i}^{*} + G_i = N(\Phi_0^{*}; b_{-1,i}, G_i)$.

[10] In complete market structures like ours, this is a typical portfolio irrelevance outcome.

Proposition 2. *Under regularity conditions 1 and 2 described in Appendix A, (11) holds with equality and there exists a positive root Φ_0^* of (11) that solves*

$$\Phi_0^* = \arg\min_{\Phi_0} \ \mathcal{L}_0(\Phi_0)\,,$$

where the Ramsey allocation, tax rate, and bond price sequence satisfy

$$C_{0,i}^* = C(\Phi_0^*; b_{-1,i}, G_i); \qquad N_{0,i}^* = C_{0,i}^* + G_i; \tag{12}$$

$$\tau_{0,i}^* = 1 + \frac{U_{N,i}(C_{0,i}^*, N_{0,i}^*)}{U_{C,i}(C_{0,i}^*, N_{0,i}^*)}; \qquad q_{0,i}^* = e^{-\rho i \Delta} \frac{U_{C,i}(C_{0,i}^*, N_{0,i}^*)}{U_{C,0}(C_{0,0}^*, N_{0,0}^*)}\,. \tag{13}$$

The value $\mathcal{L}_0^*$ of a Ramsey plan is $\sum_{i=0}^{\infty} e^{-\rho i \Delta} U(C_{0,i}^*, C_{0,i}^* + G_i)\Delta$.

See Appendix A for a proof and supporting technical details.

3.2 Continuation Ramsey Plans

There is a sequence of continuation Ramsey planners indexed by $j \in \{1, 2, \cdots\}$, each of whom manipulates a continuation representative household. A period j continuation planner confronts a debt structure $\{b_{j-1,i}\}_{i=j}^{\infty}$ chosen by the preceding planner and the tail government spending rate process $\{G_i\}_{i=j}^{\infty}$. The period j continuation planner faces an intertemporal budget constraint:

$$\sum_{i=j}^{\infty} \big(S_{j,i}\Delta\big) \cdot q_{j,i} \geqslant \sum_{i=j}^{\infty} \big(b_{j-1,i}\Delta\big) \cdot q_{j,i}\,, \tag{14}$$

where $S_{j,i}\Delta = (\tau_{j,i} N_{j,i} - G_i)\,\Delta$ is the primary surplus for period j government over time interval $[i\Delta, (i+1)\Delta)$. The period j continuation planner seeks continuation sequences $\{C_{j,i}\,, N_{j,i}\}_{i=j}^{\infty}$ that maximize

$$\sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} U(C_{j,i}, N_{j,i})\Delta \tag{15}$$

subject to continuation implementability constraint (CIC):

$$\sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} \left(C_{j,i} U_{C,i} + (C_{j,i} + G_i) U_{N,i} - b_{j-1,i} U_{C,i}\right) \Delta \geqslant 0\,. \tag{16}$$

Our period j continuation planner confronts the tail government spending rate process of $\{G_i\}_{i=j}^{\infty}$ and a debt structure $\{b_{j-1,i}\}_{i=j}^{\infty}$ inherited from the preceding planner. The continuation planner cannot reset $\tau_{j-1,j}$. The period j planner sets a flat-rate tax $\tau_{j,j+1}$ that the period $j+1$ continuation planner cannot reset and sets (actually, just "recommends") a continuation tax plan $\{\tau_{j,i}\}_{i=j+2}^{\infty}$ that a period $j+1$ continuation planner is free to reset. Knowing the period j continuation government's tax plan, a period j continuation household orders $\{C_{j,i}\,, N_{j,i}\}_{i=j}^{\infty}$ according to the continuation utility functional (15).

Definition 2. *A continuation of a Ramsey plan in period j is the tail of the Ramsey plan for $i \geqslant j$: $\{C^*_{0,i}\Delta, N^*_{0,i}\Delta, \tau^*_{0,i}, q^*_{0,i}\}_{i=j}^{\infty}$. A continuation Ramsey planner in period j must administer tax rate $\tau_{j-1,j}$ for period j and finance an inherited continuation debt structure $\{b_{j-1,i}\Delta\}_{i=j}^{\infty}$ but is free to choose a continuation tax rate plan $\{\tau_{j,i}\}_{i=j+1}^{\infty}$. It also passes a rescheduled continuation debt structure $\{b_{j,i}\Delta\}_{i=j+1}^{\infty}$ on to a period $i+1$ continuation planner. A Ramsey plan is said to be **implemented** if each continuation Ramsey planner confirms the Ramsey planner's tax rate plan $\{\tau^*_{0,i}\}_{i=0}^{\infty}$*

Constructing continuation Ramsey plan. Attach a nonnegative multiplier Φ_j to CIC (16) and form a Lagrangian:

$$\mathcal{L}_j = \sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} \left[U(C_{j,i}, C_{j,i} + G_i) + \Phi_j \left(C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{j-1,i}U_{C,i}\right)\right] \Delta \,. \tag{17}$$

Call the extremizing values: $\{C^*_{j,i}\}_{i=j}^{\infty}$ and Φ^*_j. Note that under our timing protocol, continuation households' period j consumption satisfies $C^*_{j,j} = C^*_{j-1,j}$ because of the FOCs. For a given Φ_j, the optimal consumption plan for $i \geqslant j+1$ satisfies

$$C(\Phi_j; b_{j-1,i}, G_i) := \underset{C_{j,i}}{\arg\max} \left[U(C_{j,i}, C_{j,i} + G_i) + \Phi_j \left(C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{j-1,i}U_{C,i}\right)\right] . \tag{18}$$

Substituting (10) into the CIC (16), we obtain

$$\begin{aligned}\sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta}&\left[C(\Phi_j; b_{j-1,i}, G_i)U_{C,i} + (C(\Phi_j; b_{j-1,i}, G_i) + G_i)U_{N,i} - b_{j-1,i}U_{C,i}\right]\Delta \\ &+ \left[C^*_{j,j}U_{C,j} + (C^*_{j,j} + G_j)U_{N,j} - b_{j-1,j}U_{C,j}\right]\Delta \geqslant 0\,.\end{aligned} \tag{19}$$

Similarly, we obtain the continuation Ramsey plan:

Proposition 3. *Under regularity conditions 1 and 3 described in Appendix A and timing protocol given in Definition 2, (19) holds with equality and there exists a positive root* Φ_j^* *of (19) that solves*

$$\Phi_j^* = \arg\min_{\Phi_j} \ \mathcal{L}_j(\Phi_j)\,,$$

where the continuation Ramsey allocation and tax rate sequence for $i \geqslant j+1$ *satisfy*

$$C_{j,i}^* = C(\Phi_j^*; b_{j-1,i}, G_i); \quad N_{j,i}^* = C_{j,i}^* + G_i; \quad \tau_{j,i}^* = 1 + \frac{U_{N,i}(C_{j,i}^*, N_{j,i}^*)}{U_{C,i}(C_{j,i}^*, N_{j,i}^*)};\ . \tag{20}$$

The value $\mathcal{L}_j^*$ *of a continuation Ramsey plan is* $\sum_{i=j}^{\infty} e^{-\rho(i-j)\Delta} U(C_{j,i}^*, C_{j,i}^* + G_i)\Delta$.

3.3 Lucas and Stokey's Timing Protocol

In Lucas and Stokey's (1983) timing protocol, a period j planner sets $\tau_{j,j}$ as well as a continuation tax plan $\{\tau_{j,i}\}_{i=j+1}^{\infty}$ that a period $j+1$ continuation planner can reset. For instance, Lucas and Stokey (1983) constructed examples of government expenditure processes and restructured debt processes $\{b_{0,i}\Delta\}_{i=1}^{\infty}$ that induce period 1 continuation planner to adhere to continuations of a Ramsey tax plan $\{\tau_{0,i}^*\}_{i=1}^{\infty}$ and its associated allocation and price system $\{C_{0,i}^*\Delta, N_{0,i}^*\Delta, q_{0,i}^*\Delta\}_{i=1}^{\infty}$. Debortoli, Nunes, and Yared (2021) constructed examples in which continuation term debt structures $\{b_{0,i}\Delta\}_{i=1}^{\infty}$ like those recommended by Lucas and Stokey (1983) fail to induce continuation Ramsey planners to continue a Ramsey plan. In order to implement an optimal plan, we shall modify Lucas and Stokey's timing protocol and construct a different debt-management strategy.

4 Implementation

4.1 Debt Management

A sequence of primary surplus $S_{0,i}^*\Delta$ for period i given in (6) is an important component of our implementation of a Ramsey plan. Another key component is a scalar Π_j^* that tracks unpaid government liabilities accumulated under a Ramsey plan from period 0 to $j-1$, conditional on no initial term debts before period $j-1$, $\{b_{-1,i}\}_{i\leqslant j-1}$, having been rescheduled. It is defined by the following equation:

$$\Pi_j^* = \sum_{i=0}^{j-1} \frac{q_{0,i}^*}{q_{0,j}^*} \left(b_{-1,i} - S_{0,i}^*\right) \Delta \quad \text{with} \quad \Pi_0^* = 0\,. \tag{21}$$

Note that $\left(b_{-1,i} - S_{0,i}^*\right)$ describes the part of the initial term debt in period i, $b_{-1,i}$, that is not paid by the contemporaneous period's primary surplus $S_{0,i}^*$. Evaluating it at bond price ratio $\frac{q_{0,i}^*}{q_{0,j}^*}$, we obtain its period-$j$ value. The stock Π_j^* sums period-j values of these unpaid liabilities from periods 0 through $j-1$. The time $j-1$ continuation planner must finance Π_j^*. Let $r_{0,j}$ denote the equilibrium interest rate in period j. Along the Ramsey plan, $r_{0,j}^* = \frac{1}{\Delta}\log(q_{0,j}^*/q_{0,j+1}^*)$ and Π_j^* follows a recursion:

$$\Pi_{j+1}^* = \left(\Pi_j^* + (b_{-1,j} - S_{0,j}^*)\Delta\right) e^{r_{0,j}^*\Delta}\,. \tag{22}$$

We describe a debt management policy that implements the Ramsey plan under our timing protocol. First, this policy leaves term debts with periods to maturity weakly larger than two unchanged, so that

$$b_{j-1,i}\Delta = b_{-1,i}\Delta, \quad j \geqslant 1, i \geqslant j+1\,. \tag{23}$$

Second, to finance the sum of (i) the accumulated liability Π_j^* given in (21) and (ii) the initial term debt $b_{-1,j}\Delta$ due in period j, the debt management strategy tells the period $j-1$ government to issue just enough one-period (short-term) debt that will fall due in period j. Consequently, one-period debt issuance satisfies

$$b_{j-1,j}\Delta = \Pi_j^* + b_{-1,j}\Delta\,, \quad j \geqslant 1\,. \tag{24}$$

Equation (24) evidently "takes the short route."

4.2 Implementation Proposition

We use forward induction starting with the first continuation Ramsey planner. Assume that the period $j-1$ planner confirmed the Ramsey tax plan by setting $\{\tau_{j-1,i} = \tau_{0,i}^*\}_{i=j}^{\infty}$. Consider the period $j \geqslant 1$ continuation problem. The period j government inherits longer term debt $\{b_{j-1,i}\Delta = b_{-1,i}\Delta\}_{i=j+1}^{\infty}$ and one-period debt $b_{j-1,j}\Delta$. Because the period j continuation planner cannot reset the period j tax rate $\tau_{j-1,j}$, the period j's government's tax rate equals $\tau_{j-1,j} = \tau^*_{0,j}$ and consequently $C_{j,j} = C_{j-1,j} = C^*_{0,j}$. The period j continuation planner is free to choose a new continuation consumption sequence $\{C_{j,i}\Delta\}_{i=j+1}^{\infty}$ that need not equal $\{C^*_{0,i}\Delta\}_{i=j+1}^{\infty}$. Substituting the debt management policy described in equations (23) and (24) into CIC (16), we obtain:

$$\sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} \left(C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \\ + \left(C^*_{0,j}U_{C,j} + (C^*_{0,j} + G_j)U_{N,j} - b_{-1,j}U_{C,j}\right)\Delta - \Pi^*_j U_{C,j} \geqslant 0\,. \tag{25}$$

Attach a nonnegative Lagrange multiplier Φ_j to (25), take the period j continuation government's objective function (15), and form a continuation Lagrangian $\mathcal{L}_j$:

$$\mathcal{L}_j = \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} \left[U(C_{j,i}, N_{j,i}) + \Phi_j \left(C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\right]\Delta \\ + \left[U(C^*_{0,j}, N^*_{0,j}) + \Phi_j \left(C^*_{0,j}U_{C,j} + (C^*_{0,j} + G_j)U_{N,j} - b_{-1,j}U_{C,j}\right)\right]\Delta - \Phi_j \Pi^*_j U_{C,j}\,. \tag{26}$$

The continuation planner maximizes (26) with respect to $\{C_{j,i}\}_{i=j+1}^{\infty}$ and minimizes it with respect to Φ_j. For $i \geqslant j+1$, the continuation planner sets $C_{j,i}$ to solve

$$\max_{C_{j,i}} \left\{U(C_{j,i}, C_{j,i} + G_i) + \Phi_j \left(C_{j,i}U_{C,i} + (C_{j,i} + G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\right\}\,, \tag{27}$$

so the maximizer takes the form $C_{j,i} = C(\Phi_j; b_{-1,i}, G_i)$ for $i \geqslant j+1$, where $C(\cdot;\cdot,\cdot)$ is given in (10).

Proposition 4. *Under regularity conditions 1 and 4 described in Appendix A, (25) holds with equality and $I_j(\Phi^*_j) = 0$ where*

$$I_j(\Phi_j) = \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} \left(C(\Phi_j; b_{-1,i}, G_i)U_{C,i} + (C(\Phi_j; b_{-1,i}, G_i) + G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \\ + \left(C^*_{0,j}U_{C,j} + (C^*_{0,j} + G_j)U_{N,j} - b_{-1,j}U_{C,j}\right)\Delta - \Pi^*_j U_{C,j}\,. \tag{28}$$

The continuation Ramsey allocation for period j satisfies

$$C^*_{j,i} = C(\Phi^*_j; b_{-1,i}, G_i);\ \ N^*_{j,i} = C^*_{j,i} + G_i;\ i \geqslant j+1 \tag{29}$$

See Appendix A for a proof.

Lemma 1. *The Lagrange multiplier Φ_0^* associated with the Ramsey plan satisfies $I_j(\Phi_0^*) = 0$.*

See Appendix A for a proof.

Problem (27) that confronts a period j continuation planner who is choosing $C_{j,i}$ for $i \geqslant j+1$ is static, as is problem (10) that had confronted the Ramsey planner who chose $C_{0,i}$ for $i \geqslant 0$. Consequently, $C_{j,i} = C_{0,i}$ for $i \geqslant j+1$ if and only if Φ_j in (27) equals Φ_0^* in (10).

Theorem 1. *Suppose that*

$$U(C,N) = \log C - \eta \frac{N^\gamma}{\gamma} \quad \textit{with} \quad \eta > 0 \quad \textit{and} \quad \gamma \geqslant 1\,. \tag{30}$$

A Ramsey plan exists under conditions 1 and 4 described in Appendix A. Under the timing protocol described in Definition 2, the plan can be implemented by using the debt management strategy provided in (23) *and* (24)*. Lagrange multipliers of continuation planners equal the Lagrange multiplier of the Ramsey planner: $\Phi_j^* = \Phi_0^*$ for $j \geqslant 1$.*

Proof. We use forward induction. We suppose that the Ramsey plan has been implemented up to period $j-1$ and then verify that it can also be implemented in period j. We begin by verifying that (30) satisfies Condition 1. Together with Condition 4, Proposition 4 implies that the continuation planner's optimal Lagrange multiplier $\Phi_j^* > 0$ is positive and that the CIC (25) must hold with equality so that $I_j(\Phi_j^*) = 0$. Substituting (30) into (27), we obtain $C_{j,i} = C(\Phi_j; b_{-1,i}, G_i)$ for $i \geqslant j+1$, where $C(\Phi; b, G)$ satisfies the following first-order necessary condition for the problem defined on the right side of equation (27):

$$F(C,\Phi_j) := \frac{1}{C} - \eta(C+G)^{\gamma-1} + \Phi_j \left(\frac{b}{C^2} - \eta\gamma(C+G)^{\gamma-1} \right) = 0\,. \tag{31}$$

The period j continuation planer confronts the tail of the initial term debt service sequence $\{b_{-1,i}\Delta; i \geqslant j+1\}$ as well as a short-term debt balance of $b_{j-1,j}\Delta = \Pi_j^* + b_{-1,j}\Delta$. Using $N(\Phi_j; b_{-1,i}, G_i) = C(\Phi_j; b_{-1,i}, G_i) + G_i$ to simplify the CIC (28), we obtain:

$$I_j(\Phi_j) = \left(1 - \eta(C_{j,j}+G_j)^\gamma - \frac{b_{-1,j}}{C_{j,j}}\right)\Delta - \frac{\Pi_j^*}{C_{j,j}} + \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} \left(1 - \eta(C_{j,i}+G_i)^\gamma - \frac{b_{-1,i}}{C_{j,i}}\right)\Delta\,, \tag{32}$$

where $C_{j,j} = C_{j-1,j} = C^*_{0,j}$ under our timing protocol.

To show that $I_j(\Phi_j) = 0$ has a unique positive root where $I_j(\Phi_j) = 0$ is defined by CIC (32), it is necessary and sufficient to show that $\sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta}\left(1 - \eta(C_{j,i} + G_i)^{\gamma} - \frac{b_{-1,i}}{C_{j,i}}\right)$, the last term in (32), is strictly monotone in Φ_j. Denote $P(\Phi_j; b, G) = 1-\eta\left(C(\Phi_j; b, G) + G\right)^{\gamma} - b/C(\Phi_j; b, G)$. It is then sufficient to prove that $P(\Phi_j; b, G)$ is a strictly monotone function of Φ_j. Recognizing that C is a function of Φ_j and differentiating $F(C, \Phi_j) = 0$ given in (31), we obtain

$$\frac{\partial F}{\partial C}\frac{\partial C}{\partial \Phi_j} + \frac{\partial F}{\partial \Phi_j} = 0\,, \tag{33}$$

where $\frac{\partial F}{\partial C} = -1/C^2 - \eta(\gamma-1)(C+G)^{\gamma-2} - \Phi_j \eta\gamma(\gamma-1)(C+G)^{\gamma-2} - 2\Phi_j b/C^3 < 0$ and $\frac{\partial F}{\partial \Phi_j} = \left(b/C^2 - \eta\gamma(C+G)^{\gamma-1}\right)$. Differentiate $P(\Phi_j; b, G)$ and use (33) and $\frac{\partial F}{\partial C} < 0$ to obtain

$$\frac{\partial P}{\partial \Phi_j} = \left[b/C^2 - \eta\gamma(C+G)^{\gamma-1}\right]\frac{\partial C}{\partial \Phi_j} = -\frac{\partial F}{\partial C}\left(\frac{\partial C}{\partial \Phi_j}\right)^2 > 0\,. \tag{34}$$

We conclude that $P(\Phi_j; b, G)$ and therefore $\sum_{i=j+1}^{\infty} P(\Phi_j; b_{-1,i}, G_i)$ are both strictly monotone in Φ_j. Since $I_j(\Phi_j)$ is strictly monotone in Φ_j, $I_j(\Phi_j) = 0$ has a unique root. Applying Lemma 1 lets us conclude that the Ramsey plan is implemented.

□

4.3 Mechanics

Our implementation induces a period $j \geqslant 1$ planner to set continuation allocation $\{C^*_{j,i}\}_{i \geqslant j}$ to the tail of the Ramsey allocation $\{C^*_{0,i}\}_{i \geqslant j}$, so continuation tax rates, asset prices, and debt management plans align to support the Ramsey tax plan and associated outcomes. To understand mechanics underlying these outcomes, first recall from Propositions 2 and 4 that for $i \geqslant j+1$

$$C^*_{0,i} = C(\Phi^*_0; b_{-1,i}, G_i) \quad \text{and} \quad C^*_{j,i} = C(\Phi^*_j; b_{j-1,i}, G_i). \tag{35}$$

In (35), different *arguments* in the same *function* determine $C^*_{0,i}$ and $C^*_{j,i}$. A good way to understand our implementation is to find arguments of (35) that guarantee that $C^*_{0,i} = C^*_{j,i}$ for all $j \geqslant 1$. To do that, we reverse engineer (1) continuation debt-management policies that satisfy $b_{j-1,i} = b_{-1,i}$ for $i \geqslant j+1$, and (2) assure that $\Phi^*_j = \Phi^*_0$ for all $j \geqslant 1$.

If we impose reverse engineering specification (1) and also assume that the Ramsey plan has been followed up to period $j-1$, at the beginning of period $j-1$ the government has accumulated Π_j^* given by (21), evaluated at time j prices. The period $j-1$ planner must finance both Π_j^* and the initial term debt due $b_{-1,j}\Delta$. To do that, the period $j-1$ planner issues one-period debt equal to $b_{j-1,j}\Delta = \Pi_j^* + b_{-1,j}\Delta$. This is the short-term-debt-only policy given in (24).

To establish that $\Phi_j^* = \Phi_0^*$, recall that period j's continuation planner's Lagrange multiplier Φ_j^* satisfies

$$\sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta} \left(C(\Phi_j^*; b_{-1,i}, G_i)U_{C,i} + (C(\Phi_j^*; b_{-1,i}, G_i) + G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta + \left(C_{j,j}^* U_{C,j} + (C_{j,j}^* + G_j)U_{N,j} - b_{-1,j}U_{C,j}\right)\Delta - \Pi_j^* U_{C,j} = 0 \tag{CIC}$$

and that the Ramsey planner's Lagrange multiplier Φ_0^* satisfies

$$\sum_{i=j+1}^{\infty} e^{-i\rho\Delta} \left(C(\Phi_0^*; b_{-1,i}, G_i)U_{C,i} + (C(\Phi_0^*; b_{-1,i}, G_i) + G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta + \left(C_{0,j}^* U_{C,j} + (C_{0,j}^* + G_j)U_{N,j} - b_{-1,j}U_{C,j}\right)\Delta - \Pi_j^* U_{C,j} = 0\,, \tag{IC}$$

Setting $C_{j,j}^* = C_{0,j}^*$ makes the second term in (CIC) for period j's continuation planner identical to the second term in (IC) for the Ramsey planner. Furthermore, if $\Phi_j^* = \Phi_0^*$, the first term in (CIC) for period j's continuation planner equals the first term in (IC) for the Ramsey planner. Thus, both terms agree, so both equations hold.

We can now verify that $\Phi_j^* = \Phi_0^*$ and $C_{j,j}^* = C_{0,j}^*$ for all $j \geqslant 1$ under our timing protocol for setting taxes, which requires that the period j continuation planner must administer tax rate $\tau_{j-1,j}$ for period j which is set by the period $j-1$ planner. This induces continuation households in period j to choose $C_{j,j}^* = C_{j-1,j}^* = C_{0,j}^*$, which follows from the household's first-order necessary condition $1 - \tau_{j-1,j} = -U_{N,j}(C_{j,j}, C_{j,j} + G_j)/U_{C,j}(C_{j,j}, C_{j,j} + G_j)$.[11] Thus, taken together, our timing protocol and debt management strategy imply $\Phi_j^* = \Phi_0^*$.[12] We have thus reverse engineered a "take the short route" debt-management policy that implements the Ramsey plan.

[11]Note that for $j = 1$, our timing protocol implies $C_{1,1}^* = C_{0,1}^*$, which ensures the forward induction.

[12]This follows from Lemma 1, which states that Ramsey planner's Lagrange multiplier Φ_0^* is a root of (CIC), and from Theorem 1, which states that (CIC) has a unique root for the utility function given in (30).

5 Debortoli et al.'s Examples

5.1 Analysis

Debortoli et al. constructed examples in which Lucas and Stokey's debt management policy does not implement the Ramsey plan under their timing protocol. These examples feature an initial debt structure that takes the form

$$b_{-1,0} = b > 0 \text{ and } b_{-1,i} = 0 \text{ for all } i \geqslant 1 \tag{36}$$

a high value of b, a constant government spending process $G_i = G$ for all $i \geqslant 0$, and the utility function (30). After first reviewing why Lucas and Stokey's debt management policy fails to implement the Ramsey plan in Debortoli et al.'s examples under their timing protocol, we explain how our debt management policy succeeds in implementing the Ramsey plan under our timing protocol.

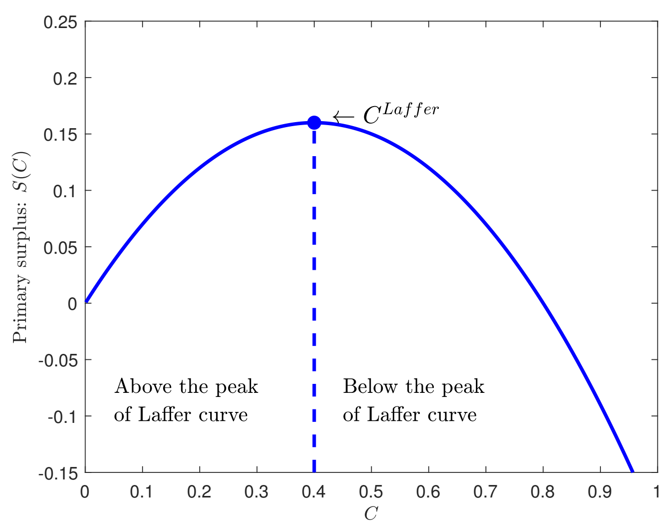


Figure 1: **Laffer curve.** Parameter values: $\gamma = 1$, $\eta = 1$, and $G = 0.2$

To begin, let $\{C^*_{0,0} = C^*_0, C^*_{0,i} = C^*_1; i \geqslant 1\}$ denote the Ramsey plan and let C^{Laffer} denote consumption at the peak of the Laffer curve (see Figure 1.) Debortoli et al. show (i) that there exists $b^* \in (0, \bar{b})$ such that the Ramsey planner sets $C^*_1 > C^{\text{Laffer}}$ if $b < b^*$

and $C_1^* < C^{\text{Laffer}}$ if $b > b^*$ (their Proposition 1);[13] and (ii) that if $b < b^*$, then Lucas and Stokey's debt management policy implements the Ramsey plan, but that if $b > b^*$, then Lucas and Stokey's debt management policy does not implement the Ramsey plan (their Proposition 2.) Debortoli et al. highlight a coincidence that occurs between C_1^* being tied to a tax rate that is above the peak of the Laffer curve and the failure of Lucas and Stokey's debt management policy to implement the Ramsey plan.

Does that coincidence persist for other initial debt structures under Lucas and Stokey's timing protocol? Subsection 6.1 shows that the answer is no. Does that coincidence persist under our timing protocol and debt management policy? The following proposition shows that the answer to this question is also no.

Proposition 5. *For all $b \in (0, \bar{b})$, the following debt management policy implements the Ramsey plan under our Definition 2 timing protocol:*

$$b_{0,1}\Delta = \Pi_1^* \quad \textit{and} \quad b_{0,i}\Delta = b_{-1,i}\Delta = 0; \; i \geqslant 2\,, \tag{37}$$

where $\Pi_1^* = \frac{e^{\rho\Delta}C_1^*}{C_0^*}\left(b - C_0^*\left(1 - \eta(C_0^* + G)^\gamma\right)\right)\Delta$. The Lagrange multiplier for the continuation planner's problem equals the Lagrange multiplier for the Ramsey problem: $\Phi_1^* = \Phi_0^*$.

We can establish this proposition directly by applying Theorem 1, but to highlight the irrelevance for implementability of being on the bad side of the Laffer curve, we instead provide the following alternative proof.

Proof. Under our timing protocol, the Ramsey planner sets $C_{1,1} = C_1^*$. Confronting debt structure (37), the period 1 continuation planner chooses $\{C_{1,i}, i \geqslant 2\}$ to maximize

$$\sum_{i=1}^{\infty} e^{-\rho(i-1)\Delta}\left(\log C_{1,i} - \eta\frac{N_{1,i}^\gamma}{\gamma}\right)\Delta$$

subject to the CIC:

$$\left(1 - \eta(C_{1,1} + G)^\gamma - \frac{b_{0,1}}{C_{1,1}}\right)\Delta + \sum_{i=2}^{\infty} e^{-\rho(i-1)\Delta}\left(1 - \eta(C_{1,i} + G)^\gamma - \frac{b_{0,i}}{C_{1,i}}\right)\Delta = 0\,. \tag{38}$$

[13] Debortoli et al. show that a Ramsey plan exists if and only if $b < \bar{b}$, where

$$\bar{b} \;=\; \max_{\tilde{C}} \quad \tilde{C}\left[[1 - \eta(\tilde{C} + G)^\gamma] + \frac{e^{-\rho\Delta}}{1 - e^{-\rho\Delta}}\left(1 - \eta G^\gamma\right)\right].$$

The associated Lagrangian is extremized by setting $C_{1,i} = C_{1,2}$ for all $i \geqslant 3$. Consequently, the CIC (38) and the debt management strategy (37) jointly imply a quadratic equation for $C_{1,2}$:

$$\frac{1}{1-e^{-\rho\Delta}}\frac{1}{C_{1,2}}\big(S(C_{1,2})-b_{-1,2}\big) = -\left[\frac{e^{2\rho\Delta}}{C^*_{0,0}}\big(S(C^*_{0,0})-b_{-1,0}\big)+\frac{e^{\rho\Delta}}{C^*_{0,1}}\big(S(C^*_{0,1})-b_{-1,1}\big)\right], \quad (39)$$

where $S(C) = C\,(1-\eta(C+G)^{\gamma})$ is the primary surplus rate. Equation (13) for setting taxes confirms that a quadratic equation $S(C)$ implies a Laffer curve.

Now rewrite the Ramsey plan's IC as

$$\frac{1}{1-e^{-\rho\Delta}}\frac{1}{C^*_{0,2}}\big(S(C^*_{0,2})-b_{-1,2}\big) = -\left[\frac{e^{2\rho\Delta}}{C^*_{0,0}}\big(S(C^*_{0,0})-b_{-1,0}\big)+\frac{e^{\rho\Delta}}{C^*_{0,1}}\big(S(C^*_{0,1})-b_{-1,1}\big)\right]. \quad (40)$$

Since the right sides of (39) and (40) are identical, their left sides must be identical too. Consequently, $C_{1,2} = C^*_1$ is a root of the quadratic equation (40). It must be the larger root because if it were not, there would exist another root of (39), $\widehat{C}_1$, larger than C^*_1. But if that were true, the Ramsey planner would have chosen $\{C_{0,0} = C^*_0, C_{0,1} = C^*_1, C_{0,i} = \widehat{C}_1; i \geqslant 2\}$, contradicting the hypothesis that $\{C_{0,0} = C^*_0, C_{0,1} = C^*_1, C_{0,i} = C^*_1; i \geqslant 2\}$ is the Ramsey plan and thus verifying Proposition 5. $\square$

The validity of Proposition 5 does not depend on whether the Ramsey planner sets the tax rate on the good or the bad side of the Laffer curve. Even though a high period 1 tax rate and associated low period 1 consumption are associated with large accumulated government liabilities, under our timing protocol the period 1 continuation planner cannot reset period 1 consumption. This makes the right side of the CIC (39) equal to the right side of the IC (40). By managing only short-term debt and leaving inherited debts maturities weakly larger than two periods unchanged, the period 1 continuation planner continues the Ramsey tax plan, aligning the left side of its CIC (39) with the IC (40) under the Ramsey plan.

5.2 Three Lagrange Multipliers

Lagrange multipliers of the Ramsey planner and continuation Ramsey planners behave differently under our timing protocol and debt management policy, on the one hand, and under Lucas and Stokey's, on the other hand. Here we focus on the $\eta = \gamma = 1$ case, in which case the Lagrange multiplier for the Ramsey plan satisfies

$$\Phi_0^* = \frac{C_1^* - (C_1^*)^2}{(C_1^*)^2 - b_{-1,i}}\,. \tag{41}$$

For both Lucas and Stokey's setup and our setup, the Lagrangian for a period-1 continuation planner is

$$\mathcal{L}_1 = \sum_{i=1}^{\infty} e^{-\rho(i-1)\Delta} \left[\log C_{1,i} - (C_{1,i} + G_i) + \Phi_1 \left(1 - (C_{1,i} + G_i) - \frac{b_{0,i}}{C_{1,i}} \right) \right] \Delta\,, \tag{42}$$

where $\{b_{0,i}\}_{i\geqslant 1}$ is the debt structure that the Ramsey planner passes on to the period 1 continuation planner. The period 1 continuation planner's first-order condition for $C_{1,i}$ is

$$\frac{1}{C_{1,i}} - 1 + \Phi_1 \left(\frac{b_{0,i}}{C_{1,i}^2} - 1 \right) = 0. \tag{43}$$

We first study outcomes under Lucas and Stokey's timing protocol and debt management policy. Debortoli et al. showed that the Ramsey planner sets

$$b_{0,i} = \frac{(C_1^*)^2 - C_1^*}{\Phi_1} + (C_1^*)^2 \quad \text{for all } \; i \geqslant 1, \tag{44}$$

an equation that can be obtained by substituting $C_{1,i} = C_{0,i}^* = C_1^*$, which has to hold for the Ramsey plan to be implemented, into the first-order necessary condition (43). Substituting (44) into the CIC (38), we can obtain what Debortoli et al. call a 'constructed' Lagrange multiplier associated with the period 1 continuation plan, namely,

$$\widehat{\Phi}_1 = \frac{1}{2} \frac{1 - C_1^*}{C_1^* - C^{\text{Laffer}}}\,, \tag{45}$$

where $C^{\text{Laffer}} = (1 - G)/2$. The Lagrange multiplier Φ_0^* associated with the Ramsey plan given by (41) does not equal the "constructed" Lagrange multiplier $\widehat{\Phi}_1$ associated with the period 1 continuation Ramsey plan given by (45).

Equation (45) implies that $\widehat{\Phi}_1 < 0$ if and only if $C_1^* < C^{\text{Laffer}}$, which by virtue of (13) means that the Ramsey plan puts $\tau_{0,1}^*$ on the 'wrong' side of the Laffer curve. When $\widehat{\Phi}_1 < 0$, it is not optimal for the continuation planner to continue the Ramsey plan. In this way, Debortoli et al. (2021) establish that the period 1 continuation planner would never choose a tax rate $\tau_{1,1}^*$ above the peak of the Laffer curve. This verifies that if the initial debt is too high, then under Lucas and Stokey's timing protocol and debt management strategy, the Ramsey plan is not implemented. When $\widehat{\Phi}_1 < 0$, the Lagrange multiplier Φ_1^* associated with the continuation Ramsey plan does not equal the constructed Lagrange multiplier $\widehat{\Phi}_1$ associated with the period 1 continuation Ramsey plan given by (45), an inequality that indicates that the Ramsey plan will not be implemented under this timing protocol and debt management policy.

Turning now to our timing protocol given in Definition 2, the period 1 continuation Ramsey planner sets $C_{1,1} = C_{0,1}^*$ and does not adjust term debts with times to maturity (weakly) larger than two, so $b_{0,i} = b_{-1,i}$ for all $i \geqslant 2$. Substituting these two conditions into (43) delivers the following 'constructed' Lagrange multiplier for the period 1 planner:

$$\widehat{\Phi}_1 = \frac{C_1^* - (C_1^*)^2}{(C_1^*)^2 - b_{-1,i}}\,,\; i \geqslant 2. \tag{46}$$

Evidently, $\widehat{\Phi}_1$ is positive and equals the Ramsey planner's Lagrange multiplier Φ_0^* given in (41), so $\widehat{\Phi}_1 = \Phi_0^* > 0$. Consequently, the Lagrange multiplier Φ_1^* associated with the continuation Ramsey plan equals the constructed Lagrange multiplier $\widehat{\Phi}_1$ associated with the period 1 continuation Ramsey plan given by (46).

Taking stock, under Lucas and Stokey's timing protocol, when b is too high, $\widehat{\Phi}_1 < 0$ and the Ramsey plan is not implemented. However, under our timing protocol, $\widehat{\Phi}_1 = \Phi_0^* > 0$ and our "short route" debt management policy implements the Ramsey plan.

5.3 Other Debt-Management Policies Can Also Work

If a Ramsey plan exists, our short-term debt management strategy implements it. For some initial debt structures, but not for others, debt management policies that issue a mixture of short and some longer term debts can also implement a Ramsey plan. To understand this claim, suppose that a Ramsey planner finances a fraction $\alpha \in [0, 1]$ of accumulated liabilities up to period 1, Π_1^*, by issuing one-period debt

$$b_{0,1}\Delta = \alpha \Pi_1^* + b_{-1,1}\Delta\,, \tag{47}$$

and finances fraction $(1-\alpha)$ of Π_1^* by issuing longer-term debts, $\{b_{0,i}\}_{i\geqslant 2}$. Substituting (47) into CIC (38) gives

$$b_{0,i}\Delta = (1-\alpha)(e^{\rho\Delta} - 1)\Pi_1^* + b_{-1,i}\Delta\,,\; i \geqslant 2\,. \tag{48}$$

Use the continuation planner's optimality condition (43) to form the 'constructed' Lagrange multiplier:

$$\widehat{\Phi}_1 = \frac{1 - C_1^*}{((1-\alpha)e^{\rho\Delta} + 1)C_1^* - (1-\alpha)e^{\rho\Delta}(1-G)}\,. \tag{49}$$

If $\alpha = 1$, then the debt management strategy corresponds to our short-term debt only strategy, since $b_{-1,i} = 0$ for $i \geqslant 2$, (49) simplifies to (46). If $\alpha = 1 - e^{-\rho\Delta}$, then the debt management strategy corresponds to Lucas and Stokey's consol-based implementation strategy in which $b_{0,1} = b_{0,i}$ for $i \geqslant 2$ and $\widehat{\Phi}_1$ given in (49) simplifies to (45).

Sometimes using both short- and long-term debt fails to implement a Ramsey plan. A necessary condition for a successful implementation is $\widehat{\Phi}_1 > 0$. This condition requires that $((1-\alpha)e^{\rho\Delta} + 1)C_1^* - (1-\alpha)e^{\rho\Delta}(1-G) > 0$ or equivalently

$$C_1^* > \frac{(1-\alpha)e^{\rho\Delta}(1-G)}{(1-\alpha)e^{\rho\Delta} + 1}\,. \tag{50}$$

Note that C_1^* is strictly decreasing in b (see Debortoli et al.) and that the right side of (50) is increasing in α. Consequently, there exists a threshold $\tilde{b}(\alpha)$ that is an increasing function of α for all $\alpha \leqslant 1$ such that inequality (50) prevails if $b < \tilde{b}(\alpha)$. Thus, for a given α, $\widehat{\Phi}_1 > 0$ and a mixed short- and long-term debt issuance policy (47)-(48) implements a Ramsey plan only when $b < \tilde{b}(\alpha)$.[14]

5.4 Quantitative Example

We turn briefly to Debortoli et al.'s numerical example of a situation in which the Ramsey plan exists but is not implemented. Debortoli et al. (2021) set an initial debt structure: $b_{-1,0} = 0.6$ and $b_{-1,i} = 0$ for $i \geqslant 1$ with $G = 0.2$ and $\eta = \gamma = 1$.[15] The Ramsey plan is $C_{0,0}^* = 0.8172, \tau_{0,0}^* = 0.1828$ and $C_{0,i}^* = 0.3125, \tau_{0,0}^* = 0.6875$ for all $i \geqslant 1$. At the peak of the Laffer curve, $C^{\text{Laffer}} = 0.4$ and $\tau^{\text{Laffer}} = 0.6$, so the period 0 Ramsey tax rate is on the 'right' side of the Laffer curve and the period 1 Ramsey tax rate is on the 'wrong' side of the Laffer curve. The Ramsey plan's Lagrange multiplier is $\Phi_0^* = 2.19$.

Lucas and Stokey's Ramsey planner leaves the time 1 continuation planner with $b_{0,i} = 0.1523$ for all $i \geqslant 1$, which differs from the tail $b_{-1,i} = 0$ for all $i \geqslant 1$ of the initial debt structure. Consequently, the continuation planner's 'constructed' Lagrange multiplier $\widehat{\Phi}_1 = -3.93 < 0$. Facing debt structure $\{b_{0,i} = 0.1523\}_{i=1}^{\infty}$, the continuation planner sets

[14] For the two polar special cases discussed in the text, $\tilde{b}(1) = \bar{b}$ and $\tilde{b}(1 - e^{-\rho\Delta}) = b^*$.

[15] See their section 3 and their figure 3.

$C_{1,i} = 0.4874$ instead of continuing the Ramsey plan $C^*_{0,i} = 0.3125$ for all $i \geqslant 1$. The associated Lagrange multiplier for the continuation planner is $\Phi^*_1 = 2.93 > 0$, which differs from both the 'constructed' Lagrange multiplier $\widehat{\Phi}_1 = -3.93$ and the Ramsey planner's Lagrange multiplier $\Phi^*_0 = 2.19$.

Turning to our timing protocol and debt management policy, the Ramsey planner sets $b_{0,1} = 0.3871$ and $b_{0,i} = b_{-1,i} = 0$ for all $i \geqslant 2$, leaving term debts with time to maturities (weakly) than two unchanged. The continuation planner sets $C_{1,i} = C^*_{0,i} = 0.3125$ for $i \geqslant 1$, confirming the tail of the Ramsey plan. The associated 'constructed' Lagrange multiplier for the continuation planner now equals the resulting Lagrange multiplier for the continuation planner, $\Phi^*_1 = 2.19 > 0$, which equals the Lagrange multiplier for Ramsey planner: $\widehat{\Phi}_1 = \Phi^*_1 = \Phi^*_0 = 2.19$. Thus, as we would expect, Debortoli et al.'s quantitative example confirms equality of Lagrange multipliers for the Ramsey plan and the continuation Ramsey plan as a tell-tale sign of implementability.

6 More Examples

This section adds quantitative examples to those presented in subsection 5.4. Subsection 6.1 studies the consequences of modestly perturbing Debortoli et al.'s initial debt structure (36) while retaining their assumption that government expenditures are constant. Our experiments in this setting show that an optimal tax rate on the wrong side of the Laffer curve is not a tell-tale sign that Lucas and Stokey's debt management policy fails to implement the Ramsey plan, but that variations in pertinent Lagrange multipliers on the Ramsey and continuation problems are a tell-tale sign. Remaining subsections turn to looking under the hood of our implementation, in the same spirit of the suite of examples provided in Lucas and Stokey (1983, Sec. 3). Subsection 6.2 studies implementable debt and tax sequences that emerge when government expenditures are constant but initial debt payments $\{b_{-1,i}\}$ decline exponentially. Subsection 6.3 studies implementable debt and tax sequences that emerge when government expenditures are constant but initial debt payments $\{b_{-1,i}\}$ fluctuate periodically. Subsection 6.4 studies implementable debt and tax sequences that emerge when initial debt payments $\{b_{-1,i}\}$ are constant but government expenditures fluctuate periodically. Like example 4 of Lucas and Stokey (1983, sec. 3), this example presents an optimal tax sequence and an equilibrium interest rate sequence that matches prescriptions and other aspects of Barro (1979).

6.1 Perturbing DNY's Example

We now parameterize a class of initial debt structures in the following way:

$$b_{-1,0} = b_0,\ b_{-1,1} = b_1,\ b_{-1,i} = b_2,\ i \geqslant 2. \tag{51}$$

Comparing debt structure (51) with Debortoli et al.'s initial debt structure (36) that we studied in section 5, notice that now $b_{-1,i} = b_2 \neq b_1$ for periods $i \geqslant 2$. Let $\{C^*_{0,0} = C^*_0, C^*_{0,1} = C^*_1, C^*_{0,i} = C^*_2; i \geqslant 2\}$ denote the Ramsey plan.

Under Lucas and Stokey's timing protocol and debt management policy, a Ramsey planner that confronts initial debt structure (51) restructures debt according to

$$b_{0,1} = \frac{(C^*_1)^2 - C^*_1}{\Phi_1} + (C^*_1)^2 \quad \text{and} \quad b_{0,i} = \frac{(C^*_2)^2 - C^*_2}{\Phi_1} + (C^*_2)^2,\ i \geqslant 2\,. \tag{52}$$

We have reverse engineered this restructuring plan from choices that the period 1 continuation planner must make to continue the Ramsey plan. We must verify that Lagrangian (42) is maximized when the consumption sequence $\{C_{1,i}\}_{i\geqslant 1}$ satisfies $C^*_{0,1} = C^*_1$ and $C^*_{0,i} = C^*_2$ for $i \geqslant 2$.[16] Substituting (52) into the CIC (38), we obtain the following 'constructed' Lagrange multiplier for the period 1 continuation planner:

$$\widehat{\Phi}_1 = \frac{1}{2} \frac{(1 - C^*_1) + (1 - C^*_2)\,\frac{1}{1-e^{-\rho\Delta}}}{(C^*_1 - C^{\text{Laffer}}) + (C^*_2 - C^{\text{Laffer}})\,\frac{1}{1-e^{-\rho\Delta}}}, \tag{53}$$

where $C^{\text{Laffer}} = (1-G)/2$. Compare (53) under Debortoli et al.'s initial debt structure (51) with (45) under the debt structure (36) and notice that the sign of the 'constructed' Lagrange multiplier $\widehat{\Phi}_1$ shapes Laffer curves for all periods $i \geqslant 1$. Evidently, $\frac{1}{1-e^{-\rho\Delta}}$ represents a discounted value of $\left(C^*_2 - C^{\text{Laffer}}\right)$ from period 2 onward.

If $\widehat{\Phi}_1 > 0$, the period 1 continuation planner chooses to continue the Ramsey tax plan and sets $\Phi^*_1 = \widehat{\Phi}_1$. However, if $\widehat{\Phi}_1 < 0$, the period 1 continuation planner chooses not to continue the Ramsey plan but instead chooses a new plan that extremizes its Lagrangian (42) and provides a positive period 1 continuation plan Lagrange multiplier $\Phi^*_1 > 0 > \widehat{\Phi}_1$.

If the Ramsey planner sets all tax rates on the 'wrong' side of the Laffer curve, so that $C^*_1 < C^{\text{Laffer}}$ and $C^*_2 < C^{\text{Laffer}}$, then $\widehat{\Phi}_1 < 0$. If $C^*_i > C^{\text{Laffer}}$ for all $i \geqslant 1$, $\widehat{\Phi}_1 > 0$.

Depending on values of the initial debt structure parameters b_0, b_1, b_2 in specification

[16] Subsection 5.2 describes an analogous reverse-engineering exercise for Debortoli et al.'s initial debt structure (36).

(51), a Ramsey planner may choose to set some tax rates on the 'wrong' side of the Laffer curve and other tax rates on the 'good' side, while $\widehat{\Phi}_1$ can be either positive or negative. Whether a Ramsey plan's tax rates are on the 'wrong' side of a Laffer curve does not indicate the plan can be implemented. Table 1 illustrates this. Case I sets $b_{-1,0} = 0.3$, $b_{-1,1} = 0$, and $b_{-1,i} = 0.15$ for $i \geqslant 2$, while case II increases $b_{-1,0}$ to 0.4 and keeps all future debts at their case I values. Under Lucas and Stokey's timing protocol and debt management policy, the Ramsey plan is implemented in case I but not in case II. With our timing protocol, our debt management policy implements a Ramsey plan in both cases.

Table 1: Lucas-Stokey's and implementation and ours

Parameter values: $\eta = \gamma = 1, G = 0.2, \rho = 0.5, \Delta = 1$: $C^{\text{Laffer}}_{\text{Max}} = 0.4$.

Case	I			II		
Period i	0	1	$\geqslant 2$	0	1	$\geqslant 2$
$b_{-1,i}$	0.3	0	0.15	0.4	0	0.15
A. Ramsey plan						
$C^*_{0,i}$	0.6297	0.2930	0.5035	0.6459	0.0702	0.4102
LM Φ^*_0	2.41			13.24		
B. LS implementation and continuation Ramsey plan						
$b_{0,i}$	n.a.	0.0710	0.2357	n.a.	0.0272	0.2509
LM $\widehat{\Phi}_1$	13.96			-2.92		
$C^*_{1,i}$	n.a.	0.2930	0.5035	n.a.	0.3877	0.6116
LM Φ^*_1	13.96			1.92		
Works?	Yes: $\widehat{\Phi}_1 = \Phi^*_1$			No: $\widehat{\Phi}_1 \neq \Phi^*_1$		
C. Our implementation						
Π^*_i	0	0.1478	-0.0018	0	0.0538	0.0251
$b_{0,i}$	n.a.	0.1478	0.15	n.a.	0.0538	0.15
LMs	$\widehat{\Phi}_1 = \Phi^*_1 = \Phi^*_0 = 2.41$			$\widehat{\Phi}_1 = \Phi^*_1 = \Phi^*_0 = 13.24$		
Works?	Yes			Yes		

Panel A in Table 1 reports Ramsey allocations and associated Ramsey Lagrange multipliers $\Phi^*_0 > 0$ for both cases. The peak of the Laffer curve $C^{\text{Laffer}} = 0.4$ so that $\tau^{\text{Laffer}} = 0.6$ and the Ramsey allocation in period 1 is on the 'wrong' side of the Laffer curve, while the

Ramsey plan consumptions for all future periods ($i \geqslant 2$) are on the 'right' side of the Laffer curve. This opens the possibility that the 'constructed' Lagrange multiplier $\widehat{\Phi}_1$ for the two cases under Lucas and Stokey's implementation can be either positive or negative, as Panel B of Table 1 confirms. In case I, the Ramsey planner restructures debt to $b_{0,1} = 0.0710$ and $b_{0,i} = 0.2357$ for all $i \geqslant 2$. Facing this debt structure, the continuation planner chooses to continue the Ramsey plan and $\Phi_1^* = \widehat{\Phi}_1 = 13.96$. Although the period 1 Ramsey allocation is on the 'wrong' side of the Laffer curve, the continuation planner chooses to continue the Ramsey plan because it recognizes that the benefits of devaluing its future debts, $b_{0,i}$ for all periods $i \geqslant 2$, outweighs the costs of being on the 'wrong' side of the Laffer curve in period 1. The Lagrange multipliers $\Phi_1^* = \widehat{\Phi}_1 > 0$ express how the continuation Ramsey planner accepts this tradeoff and continues the Ramsey plan. Consequently, Lucas and Stokey's debt management policy implements the Ramsey plan in case I.

In case II, the Ramsey planner restructures initial debts $\{b_{-1,i}\}_{i \geqslant 1}$ to $b_{0,1} = 0.0272$ and $b_{0,i} = 0.2509$ for all $i \geqslant 2$. The planner backloads its term debt more than that in case I: $b_{0,1}$ decreases from 0.0710 in case I to 0.0272 and $b_{0,i}$ for all $i \geqslant 2$ increases from 0.2357 in case I to 0.2509. The period 1 continuation planner's 'constructed' Lagrange multiplier is negative in case II: $\widehat{\Phi}_1 = -2.92$. Facing the restructured term debt, instead of continuing the Ramsey plan the continuation planner sets $C_{1,1} = 0.3877$ and $C_{1,i} = 0.6116$ for $i \geqslant 2$. The resulting Lagrange multiplier for the continuation planner is $\Phi_1^* = 1.92$. Since $\Phi_1^* > 0 > \widehat{\Phi}_1$, Lucas and Stokey's debt management policy does not implement the Ramsey plan in case II. Notice that although it doesn't continue the Ramsey plan, the continuation planner chooses to stay on the 'wrong' side of the Laffer curve in period 1: $C_{1,1} = 0.3877 < C^{\text{Laffer}}$. Relative to case I, in case II future (restructured) debts $b_{0,i} = 0.2509$ for all $i \geqslant 2$ are so high that keeping $C_{1,1}$ on the 'wrong' side of the Laffer curve is optimal because that reduces the value of those debts.

Panel C Table 1 confirm that under our timing protocol, our debt management policy implements the Ramsey plan in both cases. By keeping term debts with times to maturity (weakly) than two unchanged, i.e., $b_{0,i} = b_{-1,i} = 0.15$ for all $i \geqslant 2$, while using short-term (one-period) debt to track accumulated liabilities so that $b_{0,1} = \Pi_1^* = 0.1478$ in case I and $b_{0,1} = \Pi_1^* = 0.0538$ in case II, the period 1 continuation planner's Lagrange multiplier equals its 'constructed' Lagrange multiplier, which also equals Ramsey planner's Lagrange multiplier: $\Phi_1^* = \widehat{\Phi}_1 = \Phi_0^*$.

In summary, the quantitative examples in Table 1 show first, that tax rates on the 'wrong' side of the Laffer curve are not universally coincident symptoms of failure of Lucas and Stokey's debt management policy to implement a Ramsey plan under their timing protocol; and second, that under our timing protocol and debt management policy, a Ramsey plan can always be implemented, even if the Ramsey planner sets some tax rates above the peak of the Laffer curve.

6.2 Declining $\{b_{-1,i}\}$ bring increasing $\{\tau_{0,i}\}$

Here government spending unfolds at a constant rate $G_i = G = 0.2$ for all $i \geqslant 0$ and the initial debt structure $\{b_{-1,i}\}_{i\geqslant 0}$ starts with $b_{-1,0} = b_0 = 0.2$ in period 0 and decays exponentially at a rate of $\theta = 0.5$ each period. Figure 2 reports the Ramsey outcome and our implementation. As $b_{-1,i}$ declines monotonically as time period i advances (panel A), the tax rate $\tau^*_{0,i}$ increases (panel B.) Since $C^*_{0,i} = 1 - \tau^*_{0,i}$, by setting a higher tax rates when debt $b_{-1,i}$ is higher, the Ramsey planner makes the marginal utility $U_C(C^*_{0,i})$ higher. The Ramsey planner thereby minimizes the time 0 discounted value of $\{b_{-1,i}\}_{i\geqslant 1}$. Consumption growth rate $g^*_{0,i} = \log(C^*_{0,i+1}/C^*_{0,i})$ is negative,[17] but becomes less negative over time and approaches zero. Coincidentally, the equilibrium interest rate $r^*_{0,i} = \rho + g^*_{0,i}$ increases over time and approaches the discount rate $\rho = 0.5$ (see panel C). By keeping the interest rate low in early periods when debt is high, the planner lowers the market value of its liabilities.

The Ramsey planner front loads consumption so much that increases in output $N^*_{0,i} = C^*_{0,i} + G$ overwhelm decreases in the tax rate $\tau^*_{0,i} = 1 - C^*_{0,i}$, making the primary surplus $S^*_{0,i} = \tau^*_{0,i} N^*_{0,i} - G = (1 - C^*_{0,i})(C^*_{0,i} + G) - G$ increase over time. Since $b_{-1,i}$ decreases over time, $S^*_{0,i} - b_{-1,i}$ is negative at $i = 0$, turns positive at $i = 2$, and converges to 15% (see panel D).[18]

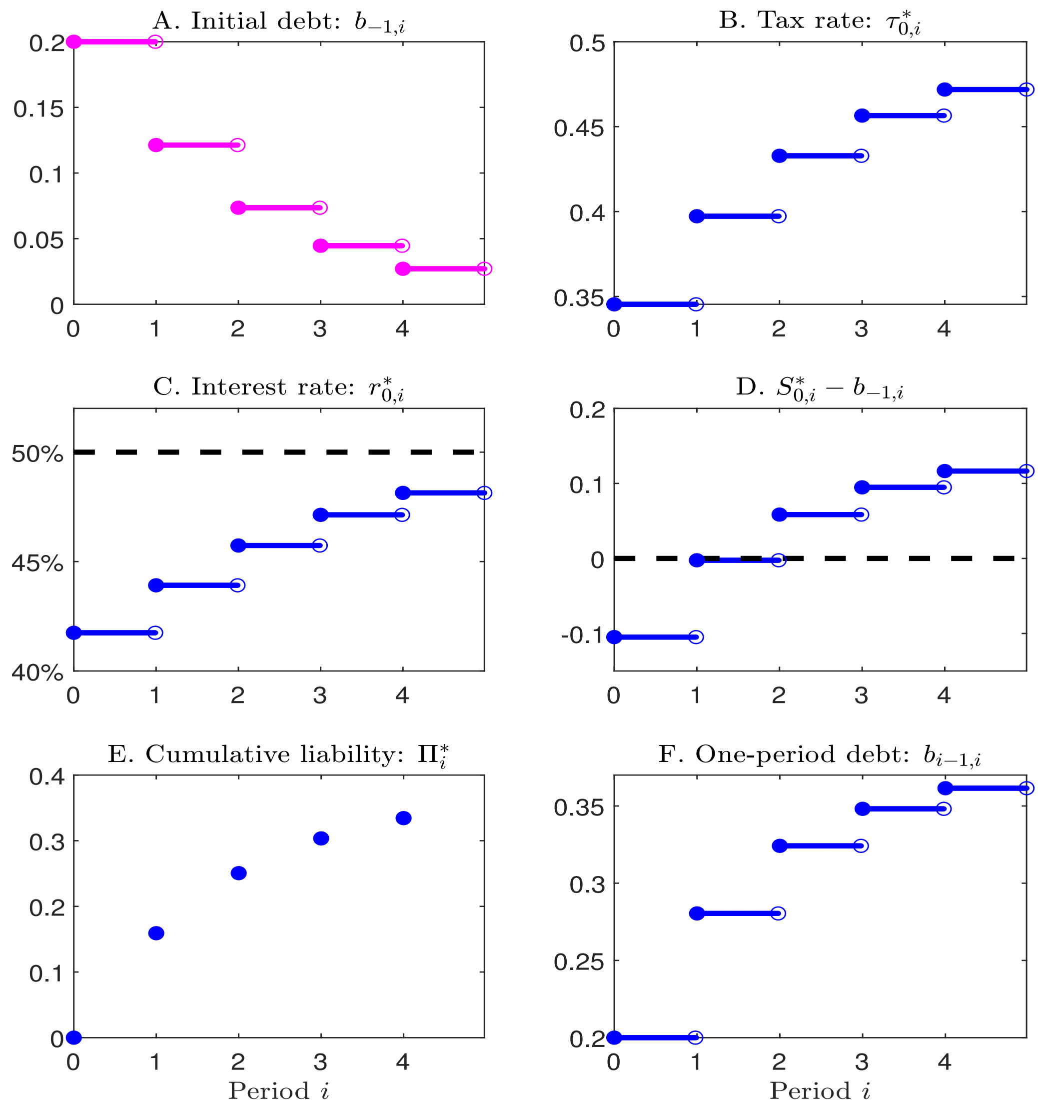
Government liabilities accumulate according to $\Pi^*_{i+1} = \left(\Pi^*_i + (b_{-1,i} - S^*_{0,i})\Delta\right) e^{r^*_{0,i}\Delta}$, starting from $\Pi^*_0 = 0$. Panel E shows that Π^*_i increases over time and converges to 0.38 as $i \to \infty$.[19] The government honors its liabilities by issuing one-period debt $b_{i-1,i}\Delta$ at time $i - 1$ that equals the sum of cumulative liability Π^*_i (in panel E) and the initial debt due

[17] In this example, $C^*_{0,i} = 2b_0 e^{-\theta i\Delta}\Phi^*_0/(\sqrt{1 + 4b_0 e^{-\theta i\Delta}\Phi^*_0(1+\Phi^*_0)} - 1)$. Therefore, the consumption growth rate is negative: $g^*_{0,t} = \log(C^*_{0,i+1}/C^*_{0,i}) < 0$ because $dC^*_{0,i}/di = -b_0\Phi^*_0\theta\Delta e^{-\theta i\Delta}/\sqrt{1 + 4b_0 e^{-\theta i\Delta}\Phi^*_0(1+\Phi^*_0)} < 0$.

[18] As $i \to \infty$, $b_{-1,i} \to 0$. We can show that $C^*_{0,i} \to \frac{1}{1+\Phi^*_0}$ and $\lim_{i\to\infty} S^*_{0,i} - b_{-1,i} = \frac{1}{1+\Phi^*_0}\left(1 - \frac{1}{1+\Phi^*_0} - G\right) = 15\%$, using $G_i = 0.2$ and $\Phi^*_0 = 0.99$.

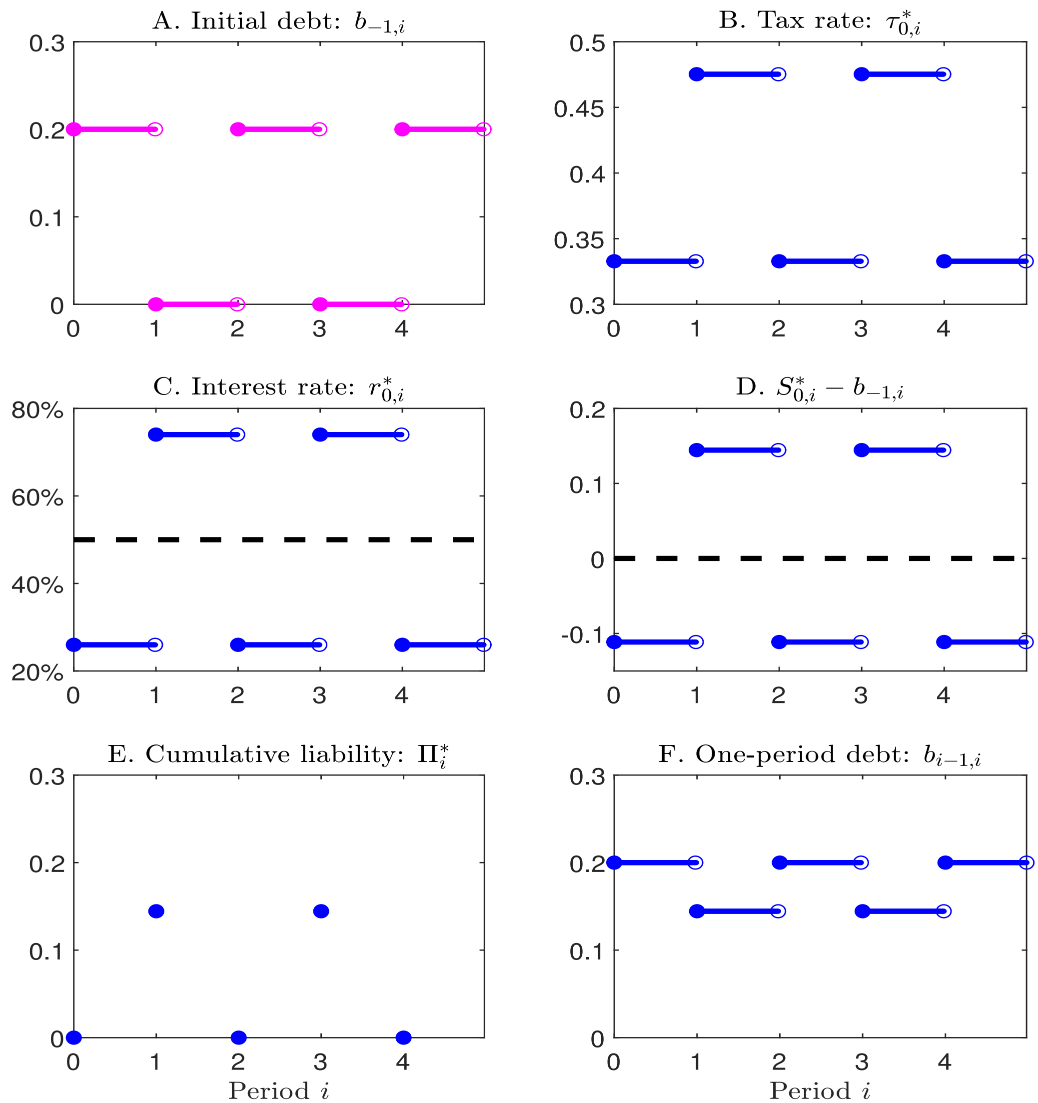
[19] By letting $i \to \infty$ in $\Pi^*_i = \left(\Pi^*_{i-1} + (b_{-1,i-1} - S^*_{0,i-1})\Delta\right) e^{r^*_{0,i-1}\Delta}$, we obtain $\lim_{i\to\infty}\Pi^*_i = \left(\lim_{i\to\infty}\Pi^*_i + \lim_{i\to\infty}(b_{-1,i-1} - S^*_{0,i-1})\Delta\right) e^{\rho\Delta}$ and thus $\lim_{i\to\infty}\Pi^*_i = \frac{\Delta e^{\rho\Delta}}{e^{\rho\Delta}-1}\lim_{i\to\infty}\left(S^*_{0,i-1} - b_{-1,i-1}\right) = \frac{\Delta e^{\rho\Delta}}{e^{\rho\Delta}-1}\frac{1}{1+\Phi^*_0}\left(1 - \frac{1}{1+\Phi^*_0} - G\right)$.

Figure 2: Ramsey plan and our debt management for exponentially decaying debt structure: $b_{-1,i} = b_0 e^{-\theta i \Delta}$. Parameter values: $b_0 = b^H = 0.2$, $\theta = 0.5, \eta = \gamma = 1, \rho = 0.5, G = 0.2$, and $\Delta = 1$.



in period i: $b_{-1,i}\Delta$. As $i \to \infty$, $b_{-1,i}\Delta \to 0$, $\lim_{i\to\infty} b_{i-1,i}\Delta = \lim_{i\to\infty} \Pi^*_i = 0.38$.

Figure 3: Ramsey plan and debt management in an example where $b_{-1,i}$ is high (b^H) in even periods and is low (b^L) in odd periods. Parameter values: $b^H = 0.2, b^L = 0, \eta = \gamma = 1, \rho = 0.5, G = 0.2$, and $\Delta = 1$.



6.3 Periodic $\{b_{-1,i}\}$ brings periodic $\{\tau_{0,i}\}$

Here initial debts $b_{-1,i}$ oscillate between two levels: $b_{-1,i} = b^H$ in even periods and $b_{-1,i} = b^L < b^H$ in odd periods. We set $b^H = 0.2$ and $b^L = 0$, start with $b_{-1,0} = b^H$. Figure 3 plots outcomes under the Ramsey plan and our timing protocol and debt management policy.

The tax rate $\tau^*_{0,i}$ takes a high value when $b_{-1,i}$ is low. Panel B of Figure 3 indicates that $\tau^*_{0,i} = \tau^H = 0.47$ in odd periods and $\tau^*_{0,i} = \tau^L = 0.33$ in even periods. Since $C^*_{0,i} = 1 - \tau^*_{0,i}$, the Ramsey planner sets marginal utility $U_C(C^*_{0,i})$ to be low in high debt periods and high in low debts period. This policy makes the time 0 value of the government's debt payments be low.

The equilibrium interest rate $r^*_{0,i}$ equals the sum of ρ and the consumption growth rate $g^*_{0,i} = \log(C^*_{0,i+1}/C^*_{0,i})$. This sum moves in lock step with $g^*_{0,i}$. Evidently, $g^*_{0,i}$ moves inversely with $b_{-1,i}$ because $g^*_{0,i} < 0$ when consumption is high and consumption is high when initial debt due is high. Thus, $r^*_{0,i}$ is high in odd periods: $r^H = 0.74$ when $b_{-1,i} = b^L$, but is low ($r^L = 0.26$) in even periods (see panel C).

Panel D shows that $S^*_{0,i} - b_{-1,i}$ is $S^*_{0,i} - b_{-1,i} = -0.11 < 0$ in even periods when debt is high ($b^H = 0.2$) and $S^*_{0,i} - b_{-1,i} = 0.14 > 0$ in odd periods when $b_{-1,i} = b^L = 0$. Panel E reports cumulative liabilities $\Pi^*_i = \left(\Pi^*_{i-1} + (b_{-1,i-1} - S^*_{0,i-1})\Delta\right) e^{r^*_{0,i-1}\Delta}$, starting from $\Pi^*_0 = 0$. It shows that Π^*_i is 0.14 in an odd periods and zero in an even periods. A high level of Π^*_i accompanies high debt ($b_{-1,i-1} = b^H = 0.2$) and low surplus ($S^*_{0,i-1} = 0.09$) in period $i-1$. A zero level of Π^*_i is an outcome of zero debt ($b_{-1,i-1} = b^L = 0$) and a high surplus ($S^*_{0,i-1} = 0.14$) in period $i-1$. Here the government policy uses the surplus to finance all of its accumulated liabilities $\Pi^*_{i-1} = 0.14$.

Finally, in our implementation, the government honors its liabilities by issuing one-period debt $b_{i-1,i}\Delta$ that equals the sum of its cumulative liabilities Π^*_i (in panel E) and initial debt term debt due in period i: $b_{-1,i}\Delta$. In an even period i, the government simply issues one-period debt to pay for its initial debt $b_{-1,i}$, since $\Pi^*_i = 0$. After that, in odd periods i, the government issues one-period debt to pay for its cumulative liability $\Pi^*_i = 0.14$, since $b_{-1,i} = 0$. Consequently, $b_{i-1,i}$ oscillates between 0.2 in even periods and 0.14 in odd periods (panel F.)

6.4 Periodic $\{G_i\}$ and constant $\{b_{-1,i}\}$ brings constant $\{\tau_{0,i}\}$

This example assumes a cyclical government spending process G_i that oscillates between two levels: $G_i = G^H$ in even periods and $G_i = G^L < G^H$ in odd periods. We set $G_0 = G^H$

and $G^H = 0.3$ and $G^L = 0.1$, fix $b_{-1,i} = b = 0.1$ for all i, and plot Ramsey outcomes and continuation debt structures under our implementation in Figure 4.

The Ramsey planner sets a constant flat tax rate $\tau^*_{0,i} = 61\%$ for all i, since $\tau^*_{0,i} = 1 - C^*_{0,i}$ and $C^*_{0,i} = 0.39$ (panel B). Outcomes resemble the tax smoothing in Barro (1979), where the the interest rate is an exogenous constant, and example 4 of Lucas and Stokey (1983, sec. 3) , where the interest rate is endogenous. In our example the interest rate is endogenous but, because consumption $C^*_{0,i}$ is constant over time, turns out to be constant: $r^*_{0,i} = \rho + \log(C^*_{0,i+1}/C^*_{0,i}) = \rho$ (see panel C).

Since $\tau^*_{0,i}$ and $N^*_{0,i} = G_i + C^*_{0,i}$ are constant, the primary surplus $S^*_{0,i} = \tau^*_{0,i} N^*_{0,i} - G_i = C^*_{0,i}\left(1 - C^*_{0,i} - G_i\right)$ moves inversely to G_i. Consequently, $S^*_{0,i} - b_{-1,i}$ is also cyclical, attaining a high value when spending is low ($G_i = G^L$) and a low value when spending is high ($G_i = G^H$). Panel E reports the government's accumulated liabilities Π^*_i (with $\Pi^*_0 = 0$), which It shows that Π^*_i oscillate between 0.08 in an odd periods and zero in even periods. A high Π^*_i accompanies high spending (G^H) and a negative $S^*_{0,i-1} - b_{-1,i-1}$. Similarly, if government spending G_{i-1} is low, the government manages its surplus in period $i-1$ to cover all of its accumulated liabilities Π^*_{i-1}, making $\Pi^*_i = 0$.

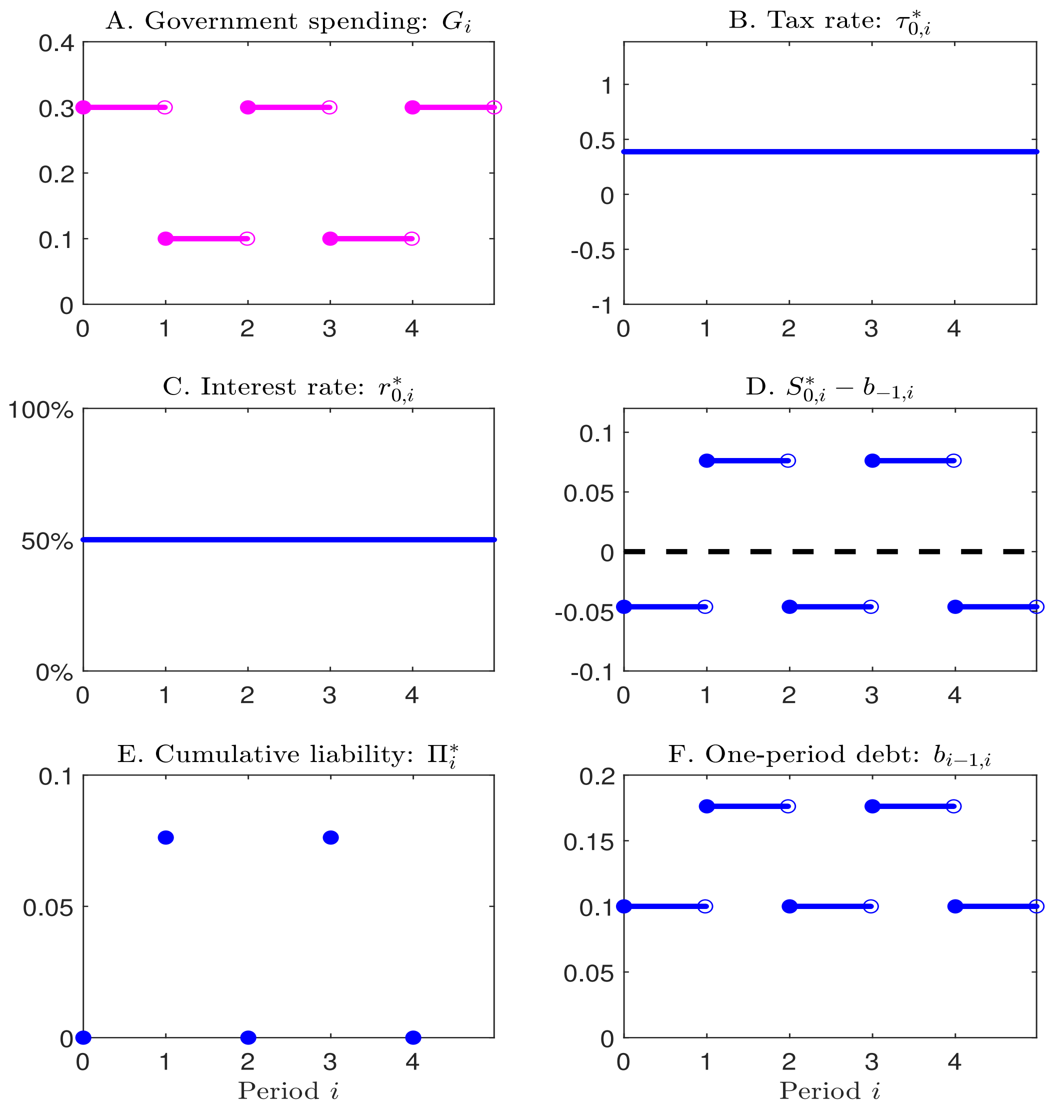
Turning to the government's debt management strategy under our implementation, since $b_{-1,i} = b = 0.1$ for all $i \geqslant 0$, in even periods i, the government issues one-period debt to finance $b = 0.1$ and $\Pi^*_i = 0$, but in odd periods i, the government issues one-period debt to finance the sum of its accumulated liabilities $\Pi^*_i = 0.08$ and its time i initial debt obligation $b = 0.1$. Consequently, $b_{i-1,i}$ oscillates between 0.1 in even periods and 0.18 in odd periods (panel F) .

7 Concluding Remarks

Debortoli et al. (2021) showed that Lucas and Stokey's (1983) way of implementing a Ramsey plan for flat-rate taxes on labor fails for a simple initial government debt structure in which debt is too big. Our paper shows how to implement that Ramsey plan by slightly modifying Lucas and Stokey's timing protocol for setting tax rates and by using a "short route" debt management strategy instead of Lucas and Stokey's debt management plan that uses consols. Our implementation always works, regardless of initial debt levels or whether the Ramsey plan sets tax rates above or below peaks of Laffer curves.

Our innovations involve two complementary modifications to Lucas and Stokey's analysis. First, we expand continuation Ramsey planners' authority by allowing them to pre-set

Figure 4: Ramsey plan and debt management in an example where government spending G_i is high (G^H) in even periods and is low (G^L) in odd periods, and $b_{-1,i} = b$ for all $i \geqslant 0$. Parameter values: $G^H = 0.3, G^L = 0.1, b = 0.1, \eta = \gamma = 1, \rho = 0.5$, and $\Delta = 1$.



a one period-ahead tax rate, while also shrinking their authority by preventing them from resetting current-period tax rates. Second, we propose a debt management strategy that leaves longer-term debts unchanged while using only short-term debt to finance accumulated unpaid government liabilities. This "short route" debt management policy implements all Ramsey plans and equalizes Lagrange multipliers that appear in the constrained optimum problems faced by the Ramsey planner and all continuation planners. We have explained how equality of those Lagrange multipliers is decisive evidence for implementability.

The leading role of short-term debt in our implementation opens a way to formulate the Ramsey problem recursively. By constructing a "state" variable based on accumulated government liabilities Π_i^* and establishing the invariance of Lagrange multipliers across time, our approach makes it possible to formulate a tractable Bellman equation for continuation planners. Such a recursive representation facilitates quantitative versions of richer variants of our model, potentially bridging gaps between theoretical fiscal policy design and practical implementation.[20]

Our analysis opens other research possibilities. A useful extension would be to incorporate uncertainty.[21] This would let us study how stochastic shocks to government spending, productivity, or other economic fundamentals affect optimal tax policies and debt management strategies under limited commitment. The invariance of Lagrange multipliers and associated recursive formulations promise to provide insights into implementation conditions under uncertainty as well. Further, by computing limits as the time increment Δ approaches zero, we can develop continuous-time formulations of our model that should bring insights and convenient mathematical expressions.[22]

[20]Kydland and Prescott (1980) described a concise way of representing some Ramsey plans recursively. That paper is silent about implementation and did nothing to damped their skepticism about the plausibility of Ramsey plans that they expressed in Kydland and Prescott (1977).

[21]Again, see Lucas and Stokey (1983, ftnt. 1).

[22]Jiang et al. (2024) analyzed a continuous time analysis that interprets the timing protocol that restricts the current tax rate as a covenant on government debt. We prefer the interpretation in the present paper. Jiang et al. presented a recursive formulation of the Ramsey problem and an associated HJB equation.

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Appendix

A Technical Details

We state four regularity conditions that guarantee that a Ramsey plan and a continuation Ramsey plan exist.

Condition 1. *For a given utility function $U(C, C+G)$ that is concave and differentiable, the function $(CU_C + (C+G)U_N - bU_C)$ is also concave and differentiable in C when $G > 0, b \geqslant 0$.*

Condition 2. *For a given initial debt structure $\{b_{-1,i}\}_{i=0}^{\infty}$, there exists allocations $\{C_{0,i}\Delta\}_{i=0}^{\infty}$ that strictly satisfy the IC (8).*

Condition 3. *For period j ($j \geqslant 1$) continuation planner that inherits a debt structure $\{b_{j-1,i}\}_{i=j}^{\infty}$ and is subject to new timing protocol given in Definition 2, there exists allocations $\{C_{j,i}\Delta\}_{i=j}^{\infty}$ that strictly satisfy the CIC (16).*

Condition 4. *For a period j ($j \geqslant 1$) continuation planner who inherits a debt structure* (23)-(24) *and is subject to new timing protocol given in Definition 2, there exists allocations $\{C_{j,i}\Delta\}_{j=i}^{\infty}$ that strictly satisfy the CIC (25).*

Condition 1 ensures both Ramsey problem and continuation Ramsey problem being concave problems. Conditions 2, 3, and 4 are Slater's conditions for the Ramsey problem and continuation Ramsey problem, respectively. For example, the economy studied in Debortoli et al. (2021) satisfies these four regularity conditions when initial debt is not too high.

Proof of Proposition 2. Note that the first condition states the Ramsey problem is a concave problem, and the second condition is a Slater's condition that ensures the strong duality for this concave problem (i.e., Boyd and Vandenberghe, 2004). Then, for the Ramsey plan $\{C^*_{0,i}\}_{i=0}^{\infty}$, there must exist a Φ^*_0 such that the following Karush-Kuhn-Tucker condition holds:

$$
\begin{aligned}
&C_{0,i}^{*} = \arg\max_{C_{0,i}} \left[U(C_{0,i}, C_{0,i}+G_i) + \Phi_0^{*}\left(C_{0,i}U_{C,i} + (C_{0,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\right] \\
&\Phi_0^{*} \cdot \left(\sum_{i=0}^{\infty} e^{-\rho i\Delta}\left[C_{0,i}^{*}U_{C,i} + (C_{0,i}^{*}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right]\cdot\Delta\right) = 0 \\
&\Phi_0^{*} \geqslant 0 \\
&\sum_{i=0}^{\infty} e^{-\rho i\Delta}\left[C_{0,i}^{*}U_{C,i} + (C_{0,i}^{*}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right]\cdot\Delta \geqslant 0
\end{aligned}
$$

Hence, optimal consumption satisfies $C_{0,i}^{*} = C(\Phi_0^{*}; b_{-1,i}, G_i)$, where $C(\cdot;\cdot,\cdot)$ is defined in (10).

Lastly, we show that the optimal Lagrange multiplier must be positive, $\Phi_0^{*} > 0$, and thus the implementability condition (11) must hold with equality. We prove it by the method of contradiction. Assume $\Phi_0^{*} = 0$. Then optimality requires the optimal consumption $C_{0,i}^{*}$ must satisfy

$$U_C(C_{0,i}^{*}, C_{0,i}^{*}+G_i) + U_N(C_{0,i}^{*}, C_{0,i}^{*}+G_i) = 0$$

and thus the utility $U(C_{0,i}^{*}, C_{0,i}^{*}+G_i)$ achieves its global maximum. Contradiction then arises because the implementability condition (8) does not hold because

$$\sum_{i=0}^{\infty} e^{-\rho i\Delta}\left[C_{0,i}^{*}U_{C,i} + (C_{0,i}^{*}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right]\cdot\Delta = -\sum_{i=0}^{\infty}\left(G_i + b_{-1,i}\right)U_{C,i}\Delta < 0\,.$$

□

Proof of Proposition 4. Given the inherited short-term balance $b_{j-1,j}\Delta = \Pi_j^{*} + b_{-1,j}\Delta$ and the tail of the initial term debt $\{b_{-1,i}\Delta; i \geqslant j+1\}$, the period i continuation Ramsey planner's Lagrangian is given by (26), and the optimal consumptions for period $i \geqslant j+1$, given a Lagrange multiplier Φ_j, satisfy (27).

We then show that optimality requires $\Phi_j^{*} > 0$ and the continuation implementability condition binds, $I_j(\Phi_j^{*}) = 0$, where $I_j(\,\cdot\,)$ is given by (28). We prove this by the method of contradiction. If $\Phi_j^{*} = 0$, the optimal consumption $C_{j,i}^{*}$ defined in (27) must satisfy

$$U_C(C_{j,i}^{*}, C_{j,i}^{*}+G_i) + U_N(C_{j,i}^{*}, C_{j,i}^{*}+G_i) = 0, \quad i \geqslant j+1\,, \tag{A-1}$$

and $U(C_{j,i}^{*}, C_{j,i}^{*}+G_i)$ achieves the global maximum. On the other hand, using (21), the continuation implementatibity condition (25), if holds, implies that

$$\begin{aligned}
&\sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta}\left(C^*_{j,i}U_{C,i} + (C^*_{j,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \\
&\qquad + \left(C^*_{0,j}U_{C,j} + (C^*_{0,j}+G_j)U_{N,j} - b_{-1,j}U_{C,j}\right)\Delta - \Pi^*_j U_{C,j} \\
&= \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta}\left(C^*_{j,i}U_{C,i} + (C^*_{j,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \\
&\qquad + \sum_{i=0}^{j} e^{-\rho(i-j)\Delta}\left(C^*_{0,i}U_{C,i} + (C^*_{0,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \geqslant 0\,.
\end{aligned}$$

We then obtain an allocation $\{C^*_{0,0},\cdots,C^*_{0,j},C^*_{j,j+1},C^*_{j,j+2},\cdots,\}$ for the Ramsey planner than satisfies the implementability condition (8). Note that $U(C^*_{j,i},C^*_{j,i}+G_i) > U(C^*_{0,i},C^*_{0,i}+G_i); i \geqslant j+1$ because $U(C^*_{j,i},C^*_{j,i}+G_i)$ achieves the global maximum. As a result, the households utility (2) under this constructed allocation is larger than that given in Proposition 2. Contradiction arises.

□

Proof of Lemma 1. Let $\Phi_j = \Phi^*_0$, then $C_{j,i} = C(\Phi_j; b_{-1,i}, G_i) = C^*_{0,i}$ for $i \geqslant j+1$. Substituting into (28) and using (21) , we obtain

$$\begin{aligned}
I_j(\Phi^*_0) &= \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta}\left(C^*_{0,i}U_{C,i} + (C^*_{0,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \\
&\qquad + \left(C^*_{0,j}U_{C,j} + (C^*_{0,j}+G_j)U_{N,j} - b_{-1,j}U_{C,j}\right)\Delta - \Pi^*_j U_{C,j}\,. \\
&= \sum_{i=j+1}^{\infty} e^{-\rho(i-j)\Delta}\left(C^*_{0,i}U_{C,i} + (C^*_{0,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \\
&\qquad + \sum_{i=0}^{j} e^{-\rho(i-j)\Delta}\left(C^*_{0,i}U_{C,i} + (C^*_{0,i}+G_i)U_{N,i} - b_{-1,i}U_{C,i}\right)\Delta \\
&= e^{j\Delta} I_0(\Phi^*_0) = 0\,.
\end{aligned}$$

When $I_j(\,\cdot,\,) = 0$ has a unique root, the period j planner chooses $\Phi^*_j = \Phi^*_0$ and thereby confirms the Ramsey plan. □