Robust Hidden Markov LQG Problems

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Abstract

For linear quadratic Gaussian problems, this paper uses two risk-sensitivity operators defined by Hansen and Sargent (2007c) to construct decision rules that are robust to misspecifications of (1) transition dynamics for possibly hidden state variables, and (2) a probability density over hidden states induced by Bayes' law. Duality of risk-sensitivity to the multiplier min-max expected utility theory of Hansen and Sargent (2001) allows us to compute risk-sensitivity operators by solving two-player zero-sum games. That the approximating model is a Gaussian joint probability density over sequences of signals and states gives important computational simplifications. We exploit a modified certainty equivalence principle to solve four games that differ in continuation value functions and discounting of time t increments to entropy. In Games I, II, and III, the minimizing players' worst-case densities over hidden states are time inconsistent, while Game IV is an LQG version of a game of Hansen and Sargent (2005) that builds in time consistency. We describe how detection error probabilities can be used to calibrate the risk-sensitivity parameters that govern fear of model misspecification in hidden Markov models.

KEY WORDS: Misspecification, Kalman filter, robustness, entropy, certainty equivalence.

1 Introduction

To construct robust decisions for linear-quadratic-Gaussian (LQG) Markov discounted dynamic programming problems with hidden state variables, this paper solves four two-player zero-sum games that differ in their continuation valuation functions. The minimizing player helps a maximizing player design decision rules that satisfy bounds on the value of an objective function over a set of stochastic models that surround a baseline approximating model.

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Because they differ in their value functions and whether the minimizing player has an information advantage, the four games yield decision rules that are robust to misspecifications of different features of the decision maker's approximating model. The four games are linear-quadratic versions of games with more general functional forms described by Hansen and Sargent (2005, 2007c), with Game IV bringing discounting into a game analyzed by Whittle (1990) and Başar and Bernhard (1995). The LQG setting facilitates rapid computation.

To set the stage, section 2 describes the relationship between two classic problems without fear of model misspecification, namely, (a) a linear-quadratic-Gaussian discounted dynamic programming problem with fully observed state, and (b) a linear-quadratic-Gaussian discounted dynamic programming problem with partially unobserved state. Without fear of model misspecification, a certainty equivalence principle allows one to separate the problem of estimating the hidden state from the problem of making decisions conditioned on the distribution of the state. Certainty equivalence asserts that the optimal decision is a function of the conditional mean of the hidden state and that this function is independent of the conditional volatilities in the transition equation for the hidden states as well as of the variance of the hidden state about its conditional mean. This statement of certainty equivalence does not hold when we introduce fear of model misspecification, but another version of certainty equivalence does. Section 3 sets out LQG problems with partially hidden states in which the decision maker fears misspecification of either the distribution of stochastic shocks w^* to signals and the state transition dynamics conditioned on the entire state, or the distribution of the hidden state z conditional on a history of observed signals under his approximating model, or both. Sections 4, 5, and 6 describe three games in which a minimizing player helps a maximizing player design a decision rule that is robust to perturbations to the distributions of w^* and z under the assumption that the minimizing player can disregard past perturbations of the distribution of z. Game I solves an LQG version of recursions (20) and (21) of Hansen and Sargent (2007c), while Game II solves an LQG version of recursion (23) of Hansen and Sargent (2007c) and Game III solves an LQG version of the recursion in section 5.3 of Hansen and Sargent (2007c). Section 7 measures the time inconsistency of the worst-case distribution over hidden states (but not over observed signals) that emerges in Games I and II. In section 8, we analyze a Game IV that, like one analyzed by Whittle (1990) and Başar and Bernhard (1995), commits the decision maker to honor past distortions to distributions of hidden states. Key to attaining time consistency is that Game IV does not discount time t contributions to entropy, while Games I, II, and III do. By extending the detection error probabilities used by Anderson et al. (2003) and Hansen and Sargent (2007d, ch. 9) to calibrate models with fully observed state vectors, section 9 describes how to calibrate the parameters θ_1 and θ_2 that govern the decision maker's concerns about model misspecification. Section 10 describes examples and section 11 offers concluding remarks. Appendix appendix A describes a useful certainty equivalence result. B gives an alternative formulation of a robust filter under commitment. Appendix C describes a suite of Matlab programs that solve the four Games.[1]

[1]The reader who prefers to write his or her own programs and who is familiar with the deterministic discounted optimal linear regulator problem presented, for example, in Hansen and Sargent (2007d, ch. 4), will recognize how the optimal linear regulator can readily be tricked into solving games II and III.

2 Two benchmark problems

We state two classic optimization problems under full trust in a dynamic stochastic model. In the first, the decision maker observes the complete state. In the second, part of the state is hidden, impelling the decision maker to estimate it.

2.1 State fully observed, model trusted

Problem 2.1. *The state vector is* $x_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}$ *and* $\begin{bmatrix} Q & P \\ P' & R \end{bmatrix}$ *is a positive semi-definite matrix. Both* y_t *and* z_t *are observed at* t*. A decision maker chooses a state-contingent sequence of actions* $\{a_t\}_{t=0}^{\infty}$ *to maximize*

$$-\frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} a_t \\ x_t \end{bmatrix}' \begin{bmatrix} Q & P \\ P' & R \end{bmatrix} \begin{bmatrix} a_t \\ x_t \end{bmatrix} \tag{1}$$

subject to the law of motion

$$\begin{aligned} y_{t+1} &= A_{11}y_t + A_{12}z_t + B_1 a_t + C_1 w_{t+1} \\ z_{t+1} &= A_{21}y_t + A_{22}z_t + B_2 a_t + C_2 w_{t+1} \end{aligned} \tag{2}$$

where w_{t+1} *is an iid random vector distributed as* $\mathcal{N}(0, I)$*,* a_t *is a vector of actions, and* E_0 *is a mathematical expectation conditioned on known initial conditions* (y_0, z_0)*.*

Guess a quadratic optimal value function

$$V(y, z) = -\frac{1}{2} \begin{bmatrix} y \\ z \end{bmatrix}' \Omega \begin{bmatrix} y \\ z \end{bmatrix} - \omega. \tag{3}$$

Let *'s denote next period values for variables and matrices. The Bellman equation for problem 2.1 is

$$-\frac{1}{2} \begin{bmatrix} y \\ z \end{bmatrix}' \Omega \begin{bmatrix} y \\ z \end{bmatrix} - \omega = \max_a \left\{ -\frac{1}{2} \begin{bmatrix} a \\ y \\ z \end{bmatrix}' \begin{bmatrix} Q & P_1 & P_2 \\ P_1' & R_{11} & R_{12} \\ P_2' & R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} a \\ y \\ z \end{bmatrix} - E\beta \frac{1}{2} \begin{bmatrix} y^* \\ z^* \end{bmatrix}' \Omega^* \begin{bmatrix} y^* \\ z^* \end{bmatrix} - \beta\omega^* \right\} \tag{4}$$

where the maximization is subject to

$$\begin{aligned} y^* &= A_{11}y + A_{12}z + B_1 a + C_1 w^* \\ z^* &= A_{21}y + A_{22}z + B_2 a + C_2 w^* \end{aligned} \tag{5}$$

and the mathematical expectation E is evaluated with respect to $w^* \sim \mathcal{N}(0, I)$.

Proposition 2.2. *The Bellman equation (4) induces mappings from* Ω^* *to* Ω *and from* (ω^*, Ω^*) *to* ω*. The mapping from* Ω^* *to* Ω *is a matrix Riccati difference equation that converges to a unique positive semi-definite matrix* $\overline{\Omega}$ *starting from any initial matrix* Ω_0*. The fixed point* $\overline{\Omega}$ *is a matrix that is a function of* β, A, B, Q, P, R *but is independent of the volatility matrix* C *that governs the 'noise statistics', i.e., the conditional variance of* x^* *conditional* *on x. Problem (4)-(5) is an ordinary stochastic discounted optimal linear regulator problem with solution $a = -\overline{F}\begin{bmatrix} y \\ z \end{bmatrix}$, where $\overline{F}$ is independent of the volatility matrix C. The constant ω depends on C as well as on the other parameters of the problem. That $\overline{\Omega}$ and $\overline{F}$ are independent of the volatility matrix C is a manifestation of a* certainty equivalence principle *(see Hansen and Sargent (2007d, p. 29)).*

2.2 State partially unobserved, model trusted

The next problem enables us to state a classic certainty equivalence result about how estimation and decision separate, estimation of the hidden components of the state being done recursively with a Kalman filter that depends on C but is independent of the objective function parameters β, Q, R, P, the choice of a being a *deterministic* linear regulator that yields a linear decision rule with coefficients $\overline{F}$ that do not depend on the volatility matrix C.

Problem 2.3. *A decision maker observes y_t, does not observe z_t, has a prior distribution $z_0 \sim \mathcal{N}(\check{z}_0, \Delta_0)$, and observes a sequence of signals $\{s_{t+1}\}$ whose time $t+1$ component is*

$$s_{t+1} = D_1 y_t + D_2 z_t + H a_t + G w_{t+1}. \tag{6}$$

This and the following two equations constitute LQG specializations of equations (1), (2), and (3) of Hansen and Sargent (2007c):

$$\begin{aligned} y_{t+1} &= \Pi_s s_{t+1} + \Pi_y y_t + \Pi_a a_t \\ z_{t+1} &= A_{21} y_t + A_{22} z_t + B_2 a_t + C_2 w_{t+1} \end{aligned}$$

where $w_{t+1} \sim \mathcal{N}(0, I)$ is an i.i.d. Gaussian vector process. Substituting the signal (6) into the above equation for y_{t+1}, we obtain:

$$y_{t+1} = (\Pi_s D_1 + \Pi_y) y_t + \Pi_s D_2 z_t + (\Pi_s H + \Pi_a) a_t + \Pi_s G w_{t+1},$$

which gives the y-rows in the following state-space system:

$$\begin{aligned} y_{t+1} &= A_{11} y_t + A_{12} z_t + B_1 a_t + C_1 w_{t+1} \\ z_{t+1} &= A_{21} y_t + A_{22} z_t + B_2 a_t + C_2 w_{t+1} \\ s_{t+1} &= D_1 y_t + D_2 z_t + H a_t + G w_{t+1}, \end{aligned} \tag{7}$$

where

$$A_{11} \doteq \Pi_s D_1 + \Pi_y, A_{12} \doteq \Pi_s D_2, B_1 \doteq \Pi_s H + \Pi_a, C_1 \doteq \Pi_s G. \tag{8}$$

By applying Bayes' law, the decision maker constructs a sequence of posterior distributions $z_t \sim \mathcal{N}(\check{z}_t, \Delta_t), t \geq 1$, where $\check{z}_t = E[z_t | y_t, \ldots, y_1]$ for $t \geq 1$, $\Delta_t = E(z_t - \check{z}_t)(z_t - \check{z}_t)'$, and $q_t = (\check{z}_t, \Delta_t)$ is a list of sufficient statistics for the history of signals that can be expressed recursively in terms of the $(\check{z}, \Delta)$ components of the following linear system

$$\begin{aligned} y_{t+1} &= A_{11} y_t + A_{12} \check{z}_t + B_1 a_t + C_1 w_{t+1} + A_{12}(z_t - \check{z}_t) \\ \check{z}_{t+1} &= A_{21} y_t + A_{22} \check{z}_t + B_2 a_t + K_2(\Delta_t) G w_{t+1} + K_2(\Delta_t) D_2 (z_t - \check{z}_t) \\ \Delta_{t+1} &= \mathcal{C}(\Delta_t) \end{aligned} \tag{9}$$

and $K_2(\Delta)$ and Δ can be computed recursively using the Kalman filtering equations

$$\begin{aligned} K_2(\Delta) &= (A_{22}\Delta D_2' + C_2G')(D_2\Delta D_2' + GG')^{-1} && (10)\\ \mathcal{C}(\Delta) &\equiv A_{22}\Delta A_{22}' + C_2C_2' - K_2(\Delta)(A_{22}\Delta D_2' + C_2G')'. && (11)\end{aligned}$$

The decision maker's objective is the same as in problem 2.1, except that his information set is now reduced to $(y_t, \check{z}_t, \Delta_t)$ at t. The current period contribution to the decision maker's objective

$$U(y,z,a) = -\Big(\frac{1}{2}\Big)\begin{bmatrix} a\\ y\\ z\end{bmatrix}'\begin{bmatrix} Q & P_1 & P_2\\ P_1' & R_{11} & R_{12}\\ P_2' & R_{21} & R_{22}\end{bmatrix}\begin{bmatrix} a\\ y\\ z\end{bmatrix}$$

can be expressed as

$$\tilde{U}(y,\check{z},z-\check{z},a) = -\Big(\frac{1}{2}\Big)\Bigg\{\begin{bmatrix} a\\ y\\ \check{z}\end{bmatrix}'\begin{bmatrix} Q & P_1 & P_2\\ P_1' & R_{11} & R_{12}\\ P_2' & R_{21} & R_{22}\end{bmatrix}\begin{bmatrix} a\\ y\\ \check{z}\end{bmatrix} + (z-\check{z})'R_{22}(z-\check{z}) + 2(z-\check{z})'(P_2a + R_{21}y + R_{22}\check{z})\Bigg\}$$

whose expectation conditioned on current information $(y, \check{z}, \Delta)$ equals

$$-\Big(\frac{1}{2}\Big)\Bigg\{\begin{bmatrix} a\\ y\\ \check{z}\end{bmatrix}'\begin{bmatrix} Q & P_1 & P_2\\ P_1' & R_{11} & R_{12}\\ P_2' & R_{21} & R_{22}\end{bmatrix}\begin{bmatrix} a\\ y\\ \check{z}\end{bmatrix} + \mathrm{trace}\big(R_{22}\Delta\big)\Bigg\}. \qquad (12)$$

Guess that the value function is

$$V(y,\check{z},\Delta) = -\frac{1}{2}\begin{bmatrix} y\\ \check{z}\end{bmatrix}'\Omega\begin{bmatrix} y\\ \check{z}\end{bmatrix} - \omega \qquad (13)$$

and choose Ω and ω to verify the Bellman equation

$$\begin{aligned} -\frac{1}{2}\begin{bmatrix} y\\ \check{z}\end{bmatrix}'\Omega\begin{bmatrix} y\\ \check{z}\end{bmatrix} - \omega &= \max_a E\Bigg\{-\frac{1}{2}\begin{bmatrix} a\\ y\\ \check{z}\end{bmatrix}'\begin{bmatrix} Q & P_1 & P_2\\ P_1' & R_{11} & R_{12}\\ P_2' & R_{21} & R_{22}\end{bmatrix}\begin{bmatrix} a\\ y\\ \check{z}\end{bmatrix}\\ &- \frac{1}{2}\mathrm{trace}\big(R_{22}\Delta\big) - E\beta\frac{1}{2}\begin{bmatrix} y^*\\ \check{z}^*\end{bmatrix}'\Omega^*\begin{bmatrix} y^*\\ \check{z}^*\end{bmatrix} - \beta\omega^*\Bigg\} && (14)\end{aligned}$$

where the maximization is subject to the *innovation representation*

$$\begin{aligned} y^* &= A_{11}y + A_{12}\check{z} + B_1a + \Big\{C_1w^* + A_{12}(z-\check{z})\Big\}\\ \check{z}^* &= A_{21}y + A_{22}\check{z} + B_2a + \Big\{K_2(\Delta)Gw^* + K_2(\Delta)D_2(z-\check{z})\Big\}\\ \Delta^* &= \mathcal{C}(\Delta) && (15)\end{aligned}$$

with given initial conditions $(y_0, \check{z}_0)$ and where the mathematical expectation E is evaluated with respect to $w^* \sim \mathcal{N}(0, I)$ and $z - \check{z} \sim \mathcal{N}(0, \Delta)$. Notice that the systematic parts of the laws of motion (5) and (15), i.e., the parts other than the linear combinations of the shocks $w^*, (z-\check{z})$, are identical. In light of the objective functions in the two problems, this fact implies

Proposition 2.4. *Associated with Bellman equation (14) is the same matrix Riccati difference equation mapping Ω^* into Ω that characterized problem 2.1. It converges to a unique positive semi-definite solution $\overline{\Omega}$ starting from any initial value Ω_0. The fixed point $\overline{\Omega}$ is a function of β, A, B, Q, P, R but is independent of the matrices $C_1, C_2, K_2, D_2, \Delta$ that determine the volatilities of $y^*, \check{z}^*$ conditional on $y, \check{z}$. Because the Riccati equation for Ω is identical with the one associated with problem 2.1, the fixed point $\overline{\Omega}$ and the matrix $\overline{F}$ in the decision rule $a = -\overline{F}\begin{bmatrix} y \\ \check{z} \end{bmatrix}$ are identical with their counterparts in problem 2.2. These outcomes justify a separation of optimization and estimation that embody a certainty-equivalence principle. The sequence of constants $\{\omega_t\}$ depends on the sequences $\{\Delta_t\}, \text{trace}\big(R_{22}\Delta_t\big)$ and differs from its counterpart in problem (2.1).*

Remark 2.5. *The matrix $\overline{\Omega}$ in the quadratic form in the optimal value function and the matrix $\overline{F}$ in the decision rule for problems 2.1 and 2.3 can be computed by solving the same* deterministic optimal linear regulator problem *(see Hansen and Sargent (2007d, ch. 4)) that we can construct by setting to zero the volatility matrices multiplying shocks in the respective problems. After that, the constants ω can be computed by solving appropriate versions of the usual recursion mapping ω^* and other objects into ω.*

3 State partially unobserved, model *dis*trusted

We modify problem 2.3 by positing a decision maker who distrusts the joint distribution for $\{y_t, z_t\}_{t=0}^{\infty}$ that is implied by system (9), (10), and (11) and therefore wants a robust decision rule, i.e., one that attains a value that is guaranteed to exceed a bound over a set of other distributions. To formulate the problem recursively, we express his distrust in terms of two types of conditional distributions that are components of his *approximating model* (15):

1. The distribution of w^* conditional either on a complete information set $(y, z, \check{z}, \Delta)$ or on the incomplete information set $(y, \check{z}, \Delta)$.

2. The distribution of $(z - \check{z})$ conditional on the history of signals that emerges from the Kalman filter.

Following Hansen and Sargent (2005, 2007c), we compute robust decision rules by replacing the expectation operator in Bellman equation (14) with compositions of two risk-sensitivity operators, one of which adjusts continuation values for possible misspecification of the conditional densities of w^*, the other of which adjusts for possible misspecfication of $z - \check{z}$. We exploit the insight that the two risk-sensitivity operators can be viewed as indirect utility functions of malevolent players who choose distributions of w^* and $z - \check{z}$, respectively, to minimize the objective of the maximizing player.[2] By responding to the minimizing choices of probabilities, the maximizing player constructs a decision rule that is robust to perturbations to the distributions of w^* and $z - \check{z}$.

[2]This is the insight that connects robust control theory to risk-sensitivity. See Hansen and Sargent (2007d). See Cerreia et al. (2008) for a general representation of uncertainty averse preferences in terms of indirect utility functions for a minimizing player who chooses probabilities.

3.1 Robustness is context specific

A robust decision rule is *context specific* in the sense that it depends on the preference parameters in $(\beta, U(y, z, a))$ and also on details of the stochastic perturbations to the approximating model (15) that concern the decision maker.

We create alternative two-player zero-sum games that differ in either the one-period objective function $U(y, z, a)$ or the stochastic perturbations to the approximating model, in particular, the conditioning sets for the densities of w^* and $z - \check{z}$. Each game expresses concerns about robustness in terms of two penalty parameters, a θ_1 that measures the decision maker's distrust of the distribution of w^*, and a θ_2 that measures the decision maker's distrust of the distribution of $z - \check{z}$ that emerges from applying Bayes' law using the approximating model. The different games allow the decision maker to focus distrust on different aspects of the baseline stochastic model (15). Hansen and Sargent (2007c) used Games I, II, and III to generate Bellman equations that closely resemble (14). These games acquire convenient recursive structures by accepting time inconsistency in equilibrium worst-case distributions for the hidden state, as we emphasize in section 7. A linear-quadratic version of a game proposed by Hansen and Sargent (2005), called Game IV, is different and builds in time consistency of those distributions by not discounting time t contributions to entropy and by making the minimizing player choose once and for all at time 0.

The four Games differ in timing protocols and the information ascribed to the minimizing player who, by distorting probability distributions, helps the maximizing player achieve robustness. In Games I, II, and III, there are sequences of minimizing players.

- Game I (an LQG version of recursions (20) and (21) of Hansen and Sargent (2007c)) starts with a date $t+1$ value function that depends on $y_{t+1}, z_{t+1}, \check{z}_{t+1}$. The minimizing player at $t \geq 0$ distorts the distribution of w_{t+1} conditional on $y_t, z_t, \check{z}_t$, as restrained by a penalty parameter θ_1. A date t value function conditions on $y_t, z_t, \check{z}_t$ and also the distribution of $z_t - \check{z}_t$ conditional on $y_t, \check{z}_t$, as restrained by a penalty parameter θ_2.

- Game II (an LQG version of recursions (23) of Hansen and Sargent (2007c)) starts with a date $t+1$ value function that depends on $y_{t+1}, \check{z}_{t+1}$. The minimizing agent distorts the distribution of w_{t+1} conditioned on $y_t, z_t, \check{z}_t$, as restrained by a penalty parameter θ_1. Then the minimizing player distorts the distribution of $z_t - \check{z}_t$ conditional on $y_t, \check{z}_t$, as restrained by a penalty parameter θ_2. A date t value function conditions on y_t and $\check{z}_t$.

- Game III (an LQG version of the recursion in section 5.3 of Hansen and Sargent (2007c)) is a special case of Game II in which the decision maker's one-period objective function does not depend on z_t and in which $\theta_1 = \theta_2$.

The arguments of their value functions distinguish Games I and II.

Game IV (an LQG version of the game with undiscounted entropy in Hansen and Sargent (2005)) has a single minimizing player who chooses once-and-for-all at time 0.

- In Game IV, a time 0 decision maker observes $y_t, \check{z}_t$ at time t and chooses distortions of the distribution of w_{t+1} conditional on the history of y_s, z_s for $s = 0, \ldots, t$, as well as a time 0 distortion to the distribution of $z_0 - \check{z}_0$.

Each of the four games implies worst-case distortions to the mean vector and covariance matrix of the shocks w_{t+1} in (9) and to the mean and covariance $(\check{z}_t, \Delta_t)$ that emerge from the Kalman filter. The worst-case means feed back on the state in ways that help the decision maker design robust decisions.

In these games, the ordinary version of certainty equivalence does not prevail: decision rules now depend vitally on matrices governing conditional volatilities. However, a modified version of certainty equivalence described by Hansen and Sargent (2007d, p. 33) and appendix A does apply. It allows us to compute robust decisions by solving deterministic two-player zero-sum games while keeping track only of the distorted *means* of perturbed distributions together with conditional volatility matrices associated with the approximating model.

4 Game I: continuation value a function of $(y, z, \check{z}, \Delta)$

This game corresponds to an LQG version of recursions (20) and (21) of Hansen and Sargent (2007c) and lets the minimizing player but not maximizing player observe z. That information advantage induces the decision maker to explore the fragility of his decision rule with respect to misspecifications of the dynamics conditional on the entire state. Because the maximizing and minimizing players have different information sets, we solve this game in two steps. The first step conditions on information available to the maximizing player. The second step conditions on the larger information set available to the minimizing player.

Step 1: *Conditioning decisions on* $(y, \check{z}, \Delta)$ This step corresponds to solving problem (21) of Hansen and Sargent (2007c). Let $W(y^*, \check{z}^*, \Delta^*, z^*)$ be a quadratic function of next period's state variables. In terms of the state variables $(y, \check{z}, \Delta)$, the law of motion for $(y, z, \check{z}, \Delta)$ can be written as

$$\begin{aligned} y^* &= A_{11}y + A_{12}\check{z} + B_1 a + C_1 w^* + A_{12}(z - \check{z}) \\ z^* &= A_{21}y + A_{22}\check{z} + B_2 a + C_2 w^* + A_{22}(z - \check{z}) \\ \check{z}^* &= A_{21}y + A_{22}\check{z} + B_2 a + K_2(\Delta)Gw^* + K_2(\Delta)D_2(z - \check{z}) \\ \Delta^* &= \mathcal{C}(\Delta) \end{aligned} \tag{16}$$

where $w^* \sim \mathcal{N}(0, I)$ and $z - \check{z} \sim \mathcal{N}(0, \Delta)$. We replace these distributions with the distorted distributions $w^* \sim \mathcal{N}(\tilde{v}, \Sigma)$ and $z - \check{z} \sim \mathcal{N}(u, \Gamma)$. By feeding back on prior states, $\tilde{v}$ and u can represent possibly misspecified dynamics in the approximating model. At this point, we use a modified certainty equivalence result to form a law of motion for a deterministic two-player zero-sum game that will yield a decision rule that solves the stochastic two-player zero-sum game (21) of Hansen and Sargent (2007c) that interests us. We replace w^* with the distorted mean vector $\tilde{v}$ and $z - \check{z}$ with the distorted mean vector u. The modified certainty equivalence principle in Hansen and Sargent (2007d, p. 33) and appendix A asserts that we can solve (21) of Hansen and Sargent (2007c) by replacing it with a deterministic two-player zero-sum game that treats the distorted means $\tilde{v}$ and u as variables under the control of a minimizing player. Omitted stochastic terms affect constants in value functions, but not decision rules. Replacing shocks with distorted means gives us

$$y^* = A_{11}y + A_{12}\check{z} + B_1 a + C_1\tilde{v} + A_{12}u$$

$$\begin{aligned} z^* &= A_{21}y + A_{22}\check{z} + B_2 a + C_2\tilde{v} + A_{22}u \\ \check{z}^* &= A_{21}y + A_{22}\check{z} + B_2 a + K_2(\Delta)G\tilde{v} + K_2(\Delta)D_2 u \\ \Delta^* &= \mathcal{C}(\Delta). \end{aligned} \tag{17}$$

Problem 4.1. *Let θ_1 and θ_2 be positive scalars. For a quadratic value function $W(y, \check{z}, \Delta, z)$, to be computed in step 2, choose an action a and accompanying distorted mean vectors $u, \tilde{v}$ by solving Bellman equation*

$$\max_a \min_u \Big[\tilde{U}(y, \check{z}, z - \check{z}, a) + \theta_2 \frac{u'\Delta^{-1}u}{2} + \min_{\tilde{v}} \Big(\beta W(y^*, \check{z}^*, \Delta^*, z^*) + \theta_1 \frac{\tilde{v}'\tilde{v}}{2}\Big)\Big] \tag{18}$$

where the optimization is subject to the law of motion (17). Minimization over $\tilde{v}$ implements risk-sensitivity operator T^1 and minimization over u implements T^2 in the stochastic problem (21) of Hansen and Sargent (2007c). A robust decision rule attains the right side of (18) and takes the form $a = -F(\Delta)\begin{bmatrix} y \\ \check{z} \end{bmatrix}$. To make the extremization on the right side of Bellman equation (18) well posed, (θ_1, θ_2) must be large enough that

$$\begin{bmatrix} \theta_2\Delta^{-1} - R_{22} & 0 \\ 0 & \theta_1 I \end{bmatrix} - \beta \begin{bmatrix} A_{12} & C_1 \\ A_{22} & C_2 \\ K_2(\Delta)D_2 & K_2(\Delta)G \end{bmatrix}' \Omega^*(\Delta^*) \begin{bmatrix} A_{12} & C_1 \\ A_{22} & C_2 \\ K_2(\Delta)D_2 & K_2(\Delta)G \end{bmatrix}$$

is positive definite.

Remark 4.2. *The matrices $C_1, C_2, A_{12}, A_{22}, K_2(\Delta)D_2$ that determine conditional volatilities in the approximating model (15) influence the maximizing player's choice of a because they determine the minimizing player's decisions $\tilde{v}, u$ and therefore the future state.*

Step 2. *Conditioning continuation values on $(y, \check{z}, \Delta, z)$* This step constructs a continuation value function $W(y, \check{z}, \Delta, z)$ by allowing the minimizing player to condition on z as well as on $(y, \check{z}, \Delta)$. This corresponds to solving (20) of Hansen and Sargent (2007c). To facilitate conditioning on z, rewrite the law of motion as

$$\begin{bmatrix} y^* \\ z^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{21} & K_2(\Delta)D_2 & A_{22} - K_2(\Delta)D_2 \end{bmatrix} \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \\ B_2 \end{bmatrix} F(\Delta) \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ K_2(\Delta)G \end{bmatrix} v \tag{19}$$

together with

$$\Delta^* = \mathcal{C}(\Delta). \tag{20}$$

Here v is the distorted mean of w^* conditioned on $(y, z, \check{z}, \Delta)$, while $\tilde{v}$ in step 1 is the distorted mean of w^* conditional on $(y, \check{z}, \Delta)$.

Problem 4.3. *Posit a quadratic value function*

$$W(y, \check{z}, \Delta, z) = -\frac{1}{2} \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix}' \Omega(\Delta) \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix} - \omega$$

and update it via

$$W(y,\check{z},\Delta,z) = U(y,z,a) + \min_{v}\left\{\beta W^*(y^*,\check{z}^*,\Delta^*,z^*) + \theta_1 \frac{v'v}{2}\right\} \tag{21}$$

where the minimization is subject to the law of motion (19),(20). For the minimization problem on the right side of Bellman equation (21) to be well posed, we require that θ_1 *be large enough that* $\theta_1 I - \beta\bar{C}(\Delta)'[\Omega^* \circ \mathcal{C}(\Delta)]\bar{C}(\Delta)$ *is positive definite, where* $\bar{C}(\Delta) = \begin{bmatrix} C_1 \\ C_2 \\ K_2(\Delta)G \end{bmatrix}$.

Remark 4.4. *We use the modified certainty-equivalence principle described by Hansen and Sargent (2007d, ch. 2) and appendix A. After we compute the worst-case conditional means,* v, u, *it is easy to compute the corresponding worst-case conditional variances* $\Sigma(\Delta), \Gamma(\Delta)$ *as*

$$\Sigma(\Delta) \doteq \left(I - \frac{\beta}{\theta_1}\bar{C}(\Delta)'[\Omega^* \circ \mathcal{C}(\Delta)]\bar{C}(\Delta)\right)^{-1} \tag{22}$$

and

$$\Gamma(\Delta) \doteq \left(\Delta^{-1} - \frac{1}{\theta_2}R_{22} - \frac{1}{\theta_2}\begin{bmatrix}0 & I & 0\end{bmatrix}\Omega(\Delta)\begin{bmatrix}0 \\ I \\ 0\end{bmatrix}\right)^{-1}, \tag{23}$$

provided that this matrix is positive definite.

Remark 4.5. *The decision rule* $a = -F(\Delta)\begin{bmatrix} y \\ \check{z}\end{bmatrix}$ *that solves the infinite horizon problem also solves a stochastic counterpart that is formulated in terms of the* T^1 *and* T^2 *risk-sensitivity operators (equation (21) of Hansen and Sargent (2007c)).*

5 Game II: value function depends on $(y, \check{z}, \Delta)$

This game withdraws the Game I information advantage from the minimizing player and works with a transition law for only $(y, \check{z})$. The game solves an LQG version of recursion (23) of Hansen and Sargent (2007c). To exploit the modified certainty equivalence principle of appendix A, we replace w^* with $\tilde{v}$ and $(z - \check{z})$ with u in the stochastic law of motion (15) to obtain

$$\begin{bmatrix} y^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a + \begin{bmatrix} C_1 \\ K_2(\Delta)G \end{bmatrix}\tilde{v} + \begin{bmatrix} A_{12} \\ K_2(\Delta)D_2 \end{bmatrix} u. \tag{24}$$

Problem 5.1. *Guess a quadratic value function*

$$V(y,\check{z}) = -\frac{1}{2}\begin{bmatrix} y \\ \check{z} \end{bmatrix}'\Omega(\Delta)\begin{bmatrix} y \\ \check{z} \end{bmatrix} - \omega$$

and form the Bellman equation

$$V(y,\check{z}) = \max_{a}\min_{u,\tilde{v}}\left[\tilde{U}(y,\check{z},z-\check{z},a) + \theta_2\frac{u'\Delta^{-1}u}{2} + \beta V(y^*,\check{z}^*) + \theta_1\frac{\tilde{v}'\tilde{v}}{2}\right] \tag{25}$$

where the optimization is subject to the law of motion (24). This can be formulated as a deterministic optimal linear regulator *problem. For the extremization problem on the right side of the Bellman equation to be well posed, we require that* (θ_1, θ_2) *be large enough that*

$$\begin{bmatrix} \theta_2 \Delta^{-1} - R_{22} & 0 \\ 0 & \theta_1 I \end{bmatrix} - \beta \begin{bmatrix} A_{12}' & D_2' K_2(\Delta)' \\ C_1' & G' K_2(\Delta)' \end{bmatrix} \Omega(\Delta^*) \begin{bmatrix} A_{12} & C_1 \\ K_2(\Delta) D_2 & K_2(\Delta) G \end{bmatrix}$$

is positive definite. After Bellman equation (25) has been solved, worst-case variances $\Sigma(\Delta)$ *and* $\Gamma(\Delta)$ *of* w^* *and* $z - \check{z}$*, respectively, can be computed using standard formulas.*

6 Game III: $\theta_1 = \theta_2$ and no hidden states in objective

Game III solves an LQG version of the recursion described in section 5.3 of Hansen and Sargent (2007c). Game III is a special case of Game II that features situations in which[3]

1. The current period objective function depends on (y, a) but not on z.
2. As in Game II, the decision maker and the minimizing player both have access to the reduced information set $(y, \check{z}, \Delta)$.
3. The multipliers $\theta_1 = \theta_2 = \theta$.

The one period objective is

$$\hat{U}(y, a) = -\frac{1}{2} \begin{bmatrix} a \\ y \end{bmatrix}' \begin{bmatrix} Q & P \\ P' & R \end{bmatrix} \begin{bmatrix} a \\ y \end{bmatrix}.$$

The law of motion for the stochastic system is

$$\begin{bmatrix} y^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a + \left\{ \begin{bmatrix} C_1 \\ K_2(\Delta) G \end{bmatrix} w^* + \begin{bmatrix} A_{12} \\ K_2(\Delta) D_2 \end{bmatrix} (z - \check{z}) \right\}.$$

Using a Cholesky decomposition of the covariance matrix of the composite shock in braces, we represent it in terms of a new normalized composite shock $\tilde{w}$ as

$$\begin{bmatrix} y^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a + \tilde{C} \tilde{w}^*$$

where $\tilde{w}^* \sim \mathcal{N}(0, I)$ and $\tilde{C}\tilde{w}^* = \left\{ \begin{bmatrix} C_1 \\ K_2(\Delta) G \end{bmatrix} w^* + \begin{bmatrix} A_{12} \\ K_2(\Delta) D_2 \end{bmatrix} (z - \check{z}) \right\}$. To form an appropriate deterministic optimal linear regulator problem, we replace $\tilde{w}^*$ with the distorted mean $\tilde{v}$ to form the law of motion

$$\begin{bmatrix} y^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a + \tilde{C} \tilde{v}. \tag{26}$$

[3] Hansen et al. (2002, eqns. (45)-(46)) used a version of Game III to study the effects of robust filtering on asset pricing.

Problem 6.1. *For a quadratic value function*

$$W(y, \check{z}) = -\frac{1}{2} \begin{bmatrix} y \\ \check{z} \end{bmatrix}' \Omega \begin{bmatrix} y \\ \check{z} \end{bmatrix} - \omega,$$

solve the Bellman equation

$$W(y, \check{z}) = \max_a \min_{\tilde{v}} \left\{ \hat{U}(y, a) + \beta W^*(y^*, \check{z}^*) + \theta \frac{\tilde{v}'\tilde{v}}{2} \right\} \tag{27}$$

where the optimization is subject to (26) and $\Delta^* = \mathcal{C}(\Delta)$. A robust decision rule is $a = -F(\Delta) \begin{bmatrix} y \\ \check{z} \end{bmatrix}$ and the worst-case mean of $\tilde{w}^*$ is $\tilde{v} = H(\Delta) \begin{bmatrix} y \\ \check{z} \end{bmatrix}$; the worst-case variance of $\tilde{w}^*$ can be found in the usual way indicated in Hansen and Sargent (2007d, ch. 3). For the minimization part of the problem on the right side of Bellman equation (27) to be well posed, we require that θ be large enough that

$$\theta I - \beta \tilde{C}(\Delta)'[\Omega^* \circ \mathcal{C}(\Delta)]\tilde{C}(\Delta)$$

is positive definite.

Remark 6.2. *Problem 6.1 is an ordinary robust control problem with an observed state (see Hansen and Sargent (2007d, ch. 2)) that does not separately consider perturbations to the distributions of the random shocks w^* and $z - \check{z}$ that contribute to $\tilde{v}$.*

7 Time inconsistency of worst-case z distributions

While Game III focuses on possible misspecification of a composite shock and so does not distinguish between errors coming from misspecifications of the separate distributions w^* and $z - \check{z}$, Games I and II do explicitly separate misspecification of the w^* shock from the $z - \check{z}$ shock. They allow the minimizing player at time t to distort the distribution of $z - \check{z}$ anew each period and to disregard the distortions to the distribution of $z - \check{z}$ that are implied by prior distortions to the distributions of past w's and $z - \check{z}$'s.[4] This feature of Games I and II makes the worst-case distributions for z time inconsistent.

We illustrate the time inconsistency of worst-case beliefs about z in the context of Game I. For Game I, the worst-case distribution of w^* conditioned on $(y, \check{z}, \Delta)$ is $\mathcal{N}(v, \Sigma(\Delta))$ where v is a linear function of $y, \check{z}$ that can be expressed as

$$v = \beta \left[\theta_1 I - \beta \bar{C}(\Delta)' \left[\Omega^* \circ \mathcal{C}(\Delta) \right] \bar{C}(\Delta) \right]^{-1} \bar{C}(\Delta)' \left[\Omega^* \circ \mathcal{C}(\Delta) \right] \bar{A} \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix}, \tag{28}$$

and $\Sigma(\Delta)$ is given by (22). The worst-case mean of z is $\check{z} + u$ where u is given by an expression of the form $u = -\tilde{F}_{21}(\Delta)y - \tilde{F}_{23}(\Delta)\check{z}$ and the worst-case covariance of z is given

[4] For more about this feature, see Hansen and Sargent (2007d, ch. 17).

by (23). Using these in $z^* = A_{21}y + A_{22}z + B_2a + C_2w^*$ shows that the mean of z^* implied by the current worst-case conditional distribution is

$$\tilde{z}^* = A_{21}y + A_{22}(\check{z} + u) + B_2a + K_2\Big(\Gamma(\Delta)\Big)(s^* - D_1y - D_2(\check{z} + u) - Ha) \tag{29}$$

and that the implied covariance of z^* is

$$\tilde{\Delta}^* = \Big(A_{22} - K_2(\Gamma(\Delta))D_2\Big)\Gamma(\Delta)\Big(A_{22} - K_2(\Gamma(\Delta))D_2\Big)' + C_2\Sigma(\Delta)C_2'. \tag{30}$$

Equations (29) and (30) give the mean and covariance of z^* conditioned on s^* and the history of s implied by the current worst-case model. Without commitment, these do not equal the mean and covariance, respectively, of the worst-case distribution that our decision maker synthesizes next period by adjusting the sufficient statistics $(\check{z}^*, \Delta^*)$ for the conditional distribution of z^* that emerges from the ordinary Kalman filter. Thus,

$$\begin{aligned} \tilde{z}^* &\neq (\check{z}^* + u^*) \\ \tilde{\Delta}^* &\neq \mathsf{C}[\Gamma(\Delta)]. \end{aligned} \tag{31}$$

Gaps between the left and right sides indicate time inconsistency in the worst-case distributions of z_t.[5]

As stressed in Hansen and Sargent (2005), time-inconsistency of the worst-case distributions of z is a consequence of our decision to set up Games I and II by following Hansen and Sargent (2007c) in discounting time t contributions both to entropy and to utility.[6] In the next section, we follow Hansen and Sargent (2005) by not discounting entropy and positing a single minimizing player who chooses distortions once-and-for-all at time 0.

8 Game IV: commitment

Games I, II, and III adopt sequential timing protocols that give a time t decision maker the freedom to distort afresh the distribution $\mathcal{N}(\check{z}_t, \Delta_t)$ that emerges from Bayes' law as applied to the approximating model and to disregard distortions to the distribution of z_t that are implied by the approximating model's transition law for z together with distortions to the distribution of earlier z's. We now solve a Game IV that implements a linear-quadratic version of a game of Hansen and Sargent (2005) that imposes commitment to prior distortions.

[5] Time inconsistency does not appear in the worst-case distribution for signals given the signal history and manifests itself only when we unbundle the distorted signal distribution into separate components coming from distortions to the w^* and $z - \check{z}$ distributions.

[6] Epstein and Schneider (2003, 2007) also tolerate such inconsistencies in their recursive formulation of multiple priors. As in our Game II, they work with continuation values that depend on signal histories; and they distort the distribution of the future signal conditional on the hidden states, not just the distribution conditional on the signal history.

8.1 The problem

The limited information problem under commitment is[7]

$$\max_{\{a_t\}} \min_{h_0 \in \mathcal{H}_0} \min_{\{m_{t+1} \in \mathcal{M}_{t+1}\}} \quad E \quad \sum_{t=0}^{\infty} M_t \left(-\beta^t \left(\frac{1}{2}\right) \begin{bmatrix} a_t \\ x_t \end{bmatrix}' \begin{bmatrix} Q & P \\ P' & R \end{bmatrix} \begin{bmatrix} a_t \\ x_t \end{bmatrix} + \theta m_{t+1} \log m_{t+1} | \mathcal{S}_0 \right)$$
$$+ \quad \theta E (h_0 \log h_0 | \mathcal{S}_0) \tag{32}$$

subject to

$$\begin{aligned} x_{t+1} &= Ax_t + Ba_t + Cw_{t+1} \\ s_{t+1} &= Dx_t + Ha_t + Gw_{t+1} \\ M_{t+1} &= m_{t+1} M_t \\ M_0 &= h_0, \end{aligned} \tag{33}$$

where h_0 is a nonnegative random variable that is measurable with respect to y_0, z_0 and whose mean is 1; and m_{t+1} a nonnegative random variable that is measurable with respect to the history of $y_s, z_s, s = t+1, t, t-1, \ldots, 0$ and whose expectation conditioned on the history up to t is 1. The decision maker's choice of h_0 at time 0 distorts the prior distribution of z_0, while his distortions of the distribution of z_t for future t's are implied by his time 0 choice of the sequence $\{m_{t+1}\}_{t=0}^{\infty}$. This captures how this game builds in commitment to prior distortions. Hansen and Sargent (2005) show that this problem can be solved in the following two steps.

8.2 Step 1: solve a problem with observed states and without random shocks

We first solve the following two-player zero-sum game with no uncertainty. The problem is

$$\max_{\{a_t\}} \min_{\{v_t\}} \left(\frac{1}{2}\right) \sum_{t=0}^{\infty} \left(-\beta^t \begin{bmatrix} a_t \\ x_t \end{bmatrix}' \begin{bmatrix} Q & P \\ P' & R \end{bmatrix} \begin{bmatrix} a_t \\ x_t \end{bmatrix} + \theta |v_t|^2 \right) \tag{34}$$

subject to

$$x_{t+1} = Ax_t + Ba_t + Cv_t. \tag{35}$$

Notice that one-period utilities are discounted, but increments $|v_t|^2$ to entropy are not.

[7] The distortion m_{t+1} is a likelihood ratio that changes the distribution of w_{t+1} from a normal distribution with mean zero and covariance matrix I to a normal distribution with a mean v_t that is given by the second equation of (36) and a covariance matrix Υ_t, where $\Upsilon_t^{-1} = I - \frac{1}{\theta} C' \Omega_{t+1} C \beta^{t+1}$. The distortion m_{t+1} equals

$$m_{t+1} = \exp \left[-\frac{1}{2} (w_{t+1} - v_t)' (\Upsilon_t)^{-1} (w_{t+1} - v_t) + \frac{1}{2} w_{t+1} \cdot w_{t+1} - \frac{1}{2} \log \det \Upsilon_t \right],$$

A simple calculation shows that

$$E (m_{t+1} \log m_{t+1} | \mathcal{X}_t) = \frac{1}{2} \left[|v_t|^2 + \text{trace}(\Upsilon_t - I) - \log \det \Upsilon_t \right]$$

where the component terms $\frac{1}{2}|v_t|^2$ and $\text{trace}(\Upsilon_t - I) - \log \det \Upsilon_t$ are both nonnegative.

For sufficiently large values of θ, the Markov perfect equilibrium gives rise to a date t value function that is quadratic. Inclusive of discounting,[8] we denote it

$$-\frac{\beta^t}{2}(x_t)'\Omega_t x_t.$$

Define

$$\begin{aligned}
\tilde{Q}_t &\doteq \begin{bmatrix} Q & 0 \\ 0 & -\beta^{-t}\theta I \end{bmatrix} \\
\tilde{P} &\doteq \begin{bmatrix} P \\ 0 \end{bmatrix} \\
\tilde{R} &\doteq R - \tilde{P}'(\tilde{Q}_t)^{-1}\tilde{P} = R - P'Q^{-1}P \\
\tilde{B} &\doteq \begin{bmatrix} B & C \end{bmatrix} \\
\tilde{A} &\doteq A - \tilde{B}(\tilde{Q}_t)^{-1}\tilde{P} = A - BQ^{-1}P.
\end{aligned}$$

The robust a_t and the worst-case v_t are

$$\begin{aligned}
\begin{bmatrix} a_t \\ v_t \end{bmatrix} &= -\begin{bmatrix} Q + B'\beta\Omega_{t+1}B & \beta B'\Omega_{t+1}C \\ \beta C'\Omega_{t+1}B & \beta C'\Omega_{t+1}C - \beta^{-t}\theta I \end{bmatrix}^{-1} \begin{bmatrix} \beta B'\Omega_{t+1}A + P \\ \beta C'\Omega_{t+1}A \end{bmatrix} x_t \\
&= -\left(\beta\left[\tilde{Q}_t + \beta\tilde{B}'\Omega_{t+1}\tilde{B}\right]^{-1}\tilde{B}'\Omega_{t+1}\tilde{A} + (\tilde{Q}_t)^{-1}\tilde{P}\right) x_t, \qquad (36)
\end{aligned}$$

where the matrix Ω_t in the value function satisfies the Riccati equation

$$\Omega_t = \tilde{R} + \beta\tilde{A}'\Omega_{t+1}\tilde{A} - \beta\tilde{A}'\Omega_{t+1}\tilde{B}\left[\tilde{Q}_t + \beta\tilde{B}'\Omega_{t+1}\tilde{B}\right]^{-1}\tilde{B}'\Omega_{t+1}\tilde{A}. \qquad (37)$$

(Also see Başar and Bernhard (1995, p. 272).)

When $\beta < 1$, as $t \to +\infty$, the solution for Ω_t converges to one that would be obtained under a no-robustness ($\theta = \infty$) specification, v_t converges to zero, and the limiting control law converges to that associated with $\theta = \infty$ (i.e., the one associated with no fear of model misspecification). When $\theta < +\infty$, the decision maker is concerned about robustness, but that concern diminishes over time. The dissipation of concerns about robustness with the passage of time is a direct consequence of the different discounting of one-period returns (they are discounted) and one-period entropies (they are not discounted).

8.3 Step 2: given $\{\Omega_t\}$, compute the filter

Hansen and Sargent (2005) derive the following recursions for the robust estimates. Starting from the sufficient statistics $(\check{z}_0, \check{\Delta}_0)$ that describe the decision maker's prior $z_0 \sim \mathcal{N}(\check{z}_0, \check{\Delta}_0)$,

[8] This problem is well posed only for sufficiently large values of θ. See Lemma 3.1 of Başar and Bernhard (1995).

for $t \geq 0$ iterate on[9]

$$\hat{z}_t = \check{z}_t + \left[(\check{\Delta}_t)^{-1} - \frac{\beta^t}{\theta}\begin{bmatrix}0 & I\end{bmatrix}\Omega_t \begin{bmatrix}0\\ I\end{bmatrix}\right]^{-1} \frac{\beta^t}{\theta}\begin{bmatrix}0 & I\end{bmatrix}\Omega_t \begin{bmatrix}y_t\\ \check{z}_t\end{bmatrix} \tag{38}$$

$$a_t = -\begin{bmatrix}I & 0\end{bmatrix}\left[\beta\left(\tilde{Q}_t + \beta\tilde{B}'\Omega_{t+1}\tilde{B}\right)^{-1}\tilde{B}'\Omega_{t+1}\tilde{A} + (\tilde{Q}_t)^{-1}\tilde{P}\right]\begin{bmatrix}y_t\\ \hat{z}_t\end{bmatrix} \tag{39}$$

$$(\tilde{\Delta}_t)^{-1} = (\check{\Delta}_t)^{-1} - \frac{\beta^t}{\theta}R_{22} \tag{40}$$

$$\tilde{z}_t = \check{z}_t + \frac{\beta^t}{\theta}\tilde{\Delta}_t\left[{P_2}'a_t + {R_{12}}'y_t + R_{22}\check{z}_t\right] \tag{41}$$

$$\check{z}_{t+1} = \mathsf{M}(y_t, \tilde{z}_t, a_t, s_{t+1}, \tilde{\Delta}_t) \tag{42}$$

$$\check{\Delta}_{t+1} = \mathsf{C}(\tilde{\Delta}_t) \tag{43}$$

where

$$\mathsf{M}(y, \check{z}, a, s^*, \Delta) \doteq A_{21}y + A_{22}\check{z} + B_2 a + K_2(\Delta)(s^* - D_1 y - D_2\check{z} - Ha). \tag{44}$$

Here $\check{z}^* = \mathsf{M}(y, \check{z}, a, s^*, \Delta) \doteq A_{21}y + A_{22}\check{z} + B_2 a + K_2(\Delta)(s^* - D_1 y - D_2\check{z} - Ha)$ would be the update of $\check{z}$ associated with the usual Kalman filter. When hidden states appear in the one-period utility function, the commitment feature of the problem induces adjustment (40) to the estimates coming from the Kalman filter. This adjustment vanishes when the utility function contains no hidden states.[10]

Following the robust control literature (e.g., Başar and Bernhard (1995) and Whittle (1990)), Hansen and Sargent (2005) interpret this recursive implementation of the commitment problem as one in which as time unfolds the decision maker's benchmark model changes in ways that depend on actions that affected past values of the one-period objective function. That reflects the feature that the Kalman filtering equations (10)-(11) are backward-looking.

The wish to acquire robust estimators leads one to explore the utility consequences of distorting the evolution of hidden states. Under commitment, the date zero utility function is the relevant one for inducing robustness via exponential twisting of probabilities of hidden states. The change in benchmark models represented in steps (38) and (41) captures this.

As Whittle (1990) emphasized, the decision rule (39) has forward-looking components that come from 'control' and backward-looking components that come from 'filtering under commitment'. The sufficient statistic $\check{z}_t$ used as a benchmark in state estimation is backward-looking. When hidden state variables enter the one-period utility function, $\check{z}_t$ can deviate from the state estimate obtained by direct application of the Kalman filter. The forward-looking component comes from the control component of the problem through the matrices Ω_{t+1} and Ω_t in (37). We combine both components to express a robust action partly as a function of a distorted estimate of the hidden state $\hat{z}_t$.

[9]In Hansen and Sargent (2005) there are typos in equations (50) ($\tilde{\Delta}_{k-1}$ should be included as an argument of the function on the right) and in the expression for $\hat{z}_t$ (Δ_t should be $\check{\Delta}_t$) that correspond to equations (40) and (38), respectively. Appendix B describes an alternative formulation of these recursions.

[10]The distortion associated with $\check{m}_j$ implies a step in updating beliefs that is in addition to the updating associated with the ordinary Kalman filter defined in (9), (10), and (11) to update the hidden state conditional mean of the hidden state. Since $\check{m}_j$ is an exponential of a quadratic function of z_j, these distortions are computed using the normal density and a complete the square argument.

Example 8.1. *(Hidden state not in objective)*

Suppose that $P_2 = 0$, $R_{12} = 0$, and $R_{22} = 0$, so that the one-period objective does not depend on the hidden state. In this case, there is an alternative way to solve the robust control problem that first solves the filtering problem and then computes an ordinary robust control for the reduced information configuration associated with the innovations representation (9), (10), and (11).[11]

Write the solution to the ordinary (non-robust) filtering problem as:

$$\bar{z}_{t+1} = A_{21}y_t + A_{22}\bar{z}_t + B_2 a_t + K_2(\Delta_t)\bar{w}_{t+1}$$

where the innovation

$$\bar{w}_{t+1} = D_2(z_t - \bar{z}_t) + G w_{t+1}$$

is normal with mean zero and covariance matrix

$$D\Sigma_t D' + GG'$$

conditioned on $\mathcal{S}_t$. Instead of distorting the joint distribution (w_{t+1}, x_t), we can distort the distribution of the innovation $\bar{w}_{t+1}$ conditioned on $\mathcal{S}_t$. It suffices to add a distortion $\bar{v}_t$ to the mean of $\bar{w}_{t+1}$ with entropy penalty

$$\theta \bar{v}_t'(D\Sigma_t D' + GG')^{-1}\bar{v}_t,$$

and where $\bar{v}_t$ is restricted to be a function of the signal history. While the conditional covariance is also distorted, certainty equivalence allows us to compute the mean distortion by solving a deterministic zero-sum two-player game. As in the robustness problem with full information, discounting causes concerns about robustness to wear off over time.

Remark 8.2. *Under the example 8.1 assumptions, the only difference between Games III and IV is that Game III discounts time t contributions to entropy while Game IV does not. When the objective function satisfies the special conditions of example 8.1 and $\beta = 1$, outcomes of Games III and IV coincide.*

Remark 8.3. *Example 10.1 is a special case of example 8.1.*

Example 8.4. *(Pure estimation)*

The state is exogenous and unaffected by the control. The objective is to estimate $-Px_t$. The control is an estimate of $-Px_t$. To implement this specification, we set $B = 0, Q = I$, and $R = P'P$. For this problem, the solution of (37) for Game IV is $\Omega_t = 0$ for all $t \geq 0$ because $a = -Px$ sets the full information objective to zero. The solution to the estimation problem is $a_t = -P\check{x}_t$ where $\check{x}_t = \begin{bmatrix} y_t \\ \check{z}_t \end{bmatrix}$ and $\tilde{z}_t = \check{z}_t = \hat{z}_t$. In this case, the Game IV recursions (38)–(43) collapse to

$$\check{z}_t = \mathsf{M}\Big(y_{t-1}, \check{z}_{t-1}, a_{t-1}, s_t, D(\check{\Delta}_{t-1})\Big) \quad (45)$$

$$\check{\Delta}_t = \mathsf{C} \circ \mathsf{D}(\check{\Delta}_{t-1}) \quad (46)$$

where $\mathsf{D}(\check{\Delta}_t) = \left[(\check{\Delta}_t)^{-1} - \frac{\beta^k}{\theta}R_{22}\right]^{-1}$ is the operator affiliated with (40). For $\beta = 1$, it can be verified that these are the recursions described by Hansen and Sargent (2007d, ch. 17).

[11] This approach is used in the asset pricing applications in Hansen and Sargent (2007a).

9 Calibrating (θ_1, θ_2) with detection error probabilities

In this section, we describe how to calibrate (θ_1, θ_2) using the detection error probabilities advocated earlier by Hansen et al. (2002), Anderson et al. (2003), and Hansen and Sargent (2007d, ch. 9).

An equilibrium of one of our two-player zero-sum games can be represented in terms of the following law of motion for $(y, \check{z}, s)$:

$$\begin{aligned} y^* &= A_{11}y + A_{12}\check{z} + B_1 a + C_1 w^* + A_{12}(z - \check{z}) \\ \check{z}^* &= A_{21}y + A_{22}\check{z} + B_2 a + K_2(\Delta)Gw^* + K_2(\Delta)D_2(z - \check{z}) \\ s^* &= D_1 y + D_2\check{z} + Ha + Gw^* + D_2(z - \check{z}) \\ \Delta^* &= \mathcal{C}(\Delta) \end{aligned} \tag{47}$$

where under the approximating model

$$w^* \sim \mathcal{N}(0, I) \text{ and } z - \check{z} \sim \mathcal{N}(0, \Delta) \tag{48}$$

and under the worst-case model associated with a (θ_1, θ_2) pair

$$w^* \sim \mathcal{N}(\tilde{v}, \Sigma(\Delta)) \text{ and } z - \check{z} \sim \mathcal{N}(u, \Gamma(\Delta)). \tag{49}$$

We have shown how to compute decision rules for $a, \tilde{v}, u, \Sigma(\Delta), \Gamma(\Delta)$ for each our zero-sum two-player games; $a, \tilde{v}, u$ are linear functions of $y, \check{z}$. Evidently, under the approximating model

$$s^* \sim \mathcal{N}(\bar{D}_1 y + \bar{D}_2 \check{z}, \Omega_a) \tag{50}$$

where $\Omega_a(\Delta) = GG' + D_2 \Delta D_2'$ and the $(\bar{\cdot})$ over a matrix indicates that the feedback rule for a has been absorbed into that matrix; while under the worst-case model

$$s^* \sim \mathcal{N}(\hat{D}_1 y + \hat{D}_2 \check{z}, \Omega_w) \tag{51}$$

where $\Omega_w(\Delta) = G\Sigma G' + D_2 \Gamma(\Delta) D_2'$ and the $(\hat{\cdot})$ over a matrix indicates the feedback rules for a and the conditional means $\tilde{v}, u$ have been absorbed into that matrix.

Where N is the number of variables in s_{t+1}, conditional on $y_0, \check{z}_0$, the log likelihood of $\{s_{t+1}\}_{t=0}^{T-1}$ under the approximating model is

$$\log L_a = -\frac{1}{T}\sum_{t=0}^{T}\left[\frac{N}{2}\log(2\pi) + \frac{1}{2}\log|\Omega_a(\Delta_t)| + \frac{1}{2}\Big(s_{t+1} - \bar{D}_1 y_t - \bar{D}_2 \check{z}_t\Big)' \Omega_a(\Delta_t)^{-1}\Big(s_{t+1} - \bar{D}_1 y_t - \bar{D}_2 \check{z}_t\Big)\right] \tag{52}$$

and the log likelihood under the worst-case model is

$$\log L_w = -\frac{1}{T}\sum_{t=0}^{T}\left[\frac{N}{2}\log(2\pi) + \frac{1}{2}\log|\Omega_w(\Delta_t)| + \frac{1}{2}\Big(s_{t+1} - \hat{D}_1 y_t - \hat{D}_2 \check{z}_t\Big)' \Omega_w(\Delta_t)^{-1}\Big(s_{t+1} - \hat{D}_1 y_t - \hat{D}_2 \check{z}_t\Big)\right] \tag{53}$$

By applying procedures like those described in Hansen et al. (2002) and Anderson et al. (2003), we can use simulations in the following ways to approximate a detection error probability:

- Repeatedly simulate $\{y_{t+1}, \check{z}_{t+1}, s_{t+1}\}_{t=0}^{T-1}$ under the approximating model. Use (52) and (53) to evaluate the log likelihood functions of the approximating model and worst case model for a given (θ_1, θ_2). Compute the fraction of simulations for which $\log L_w > \log L_a$ and call it r_a. This approximates the probability that the likelihood ratio says that the worst-case model generated the data when the approximating model actually generated the data.

- Repeatedly simulate $\{y_{t+1}, \check{z}_{t+1}, s_{t+1}\}_{t=0}^{T-1}$ under the worst-case model affiliated with a given (θ_1, θ_2) pair. Use (52) and (53) to evaluate the log likelihood functions of the approximating and worst case models. Compute the fraction of simulations for which $\log L_a > \log L_w$ and call it r_w. This approximates the probability that the likelihood ratio says that the approximating model generated the data when the worst-case model generated the data.

- As in Hansen et al. (2002) and Anderson et al. (2003), define the overall detection error probability to be

$$p(\theta_1, \theta_2) = \frac{1}{2}(r_a + r_w). \tag{54}$$

9.1 Practical details

The detection error probability $p(\theta_1, \theta_2)$ in (54) can be used to calibrate the pair (θ_1, θ_2) jointly. This seems to be the appropriate procedure for Game II, especially when z does not appear in the objective function. However, for Game I, we think that the the following sequential procedure makes sense.

1. First pretend that y, z are both observable. Calibrate θ_1 by calculating detection error probabilities for a system with an observed state vector using the approach of Hansen et al. (2002) and Hansen and Sargent (2007d, ch. 9).

2. Then having pinned down θ_1 in step 1, use the approach leading to formula (54) to calibrate θ_2.

This procedure takes the point of view that θ_1 measures how difficult it would be to distinguish one model of the partially hidden state from another if we were able to observe the hidden state, while θ_2 measures how difficult it is to distinguish alternative models of the hidden state. The probability $p(\theta_1, \theta_2)$ measures both sources of model uncertainty.

10 Examples

Example 10.1. *(Permanent income model with hidden Markov labor earnings)*
A consumer orders streams of consumption a_t by the mathematical expectation of

$$-\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left[(a_t - by_{1t})^2 + \eta k_t^2\right] \tag{55}$$

subject to

$$
\begin{aligned}
y_{1,t+1} &= 1 \\
k_{t+1} &= (r+1)k_t + e_t - a_t \\
e_{t+1} &= z_{1t} + z_{2t} + c_e w_{t+1} \\
z_{1t+1} &= z_{1t} + c_1 w_{t+1} \\
z_{2t+1} &= \rho z_{2t} + c_2 w_{t+1}
\end{aligned} \tag{56}
$$

where b is a 'bliss' rate of consumption, $z_{1,0} \sim \mathcal{N}(\check{z}_{1,0}, \sigma_1^2)$ and $z_{2,0} \sim \mathcal{N}(\check{z}_{2,0}, \sigma_2^2)$ are independent priors, w_{t+1} is a 3×1 normalized Gaussian random vector, e_t is labor income, and k_t is the consumer's financial wealth at the beginning of period t. We assume that the consumer observes k_{t+1}, e_{t+1} at $t+1$ and that $\beta(r+1) = 1$, where $r > 0$. Here $\eta > 0$ is a very small cost of caring for assets that we include to enforce a stable solution (see Hansen and Sargent (2007b, ch. 4) for details).

Set $y_t = \begin{bmatrix} 1 \\ k_t \\ e_t \end{bmatrix}$, $z_t = \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}$. It is enough to take $s_{t+1} = e_{t+1}$ (because k_{t+1} is an exact function of time t information). To map this into our general problem, we set

$$
A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (r+1) & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix}
$$

$$
B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$$
C_1 = \begin{bmatrix} 0 \\ 0 \\ c_e \end{bmatrix}, \quad C_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},
$$

$$
D_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad H = 0, \quad G = c_e
$$

$$
\begin{bmatrix} Q & P_1 & P_2 \\ P_1' & R_{11} & R_{12} \\ P_2' & R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} 1 & -b & 0 & 0 & 0 & 0 \\ -b & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

and

$$
\Delta_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{bmatrix}.
$$

Example 10.2. *(Forecasting a geometrically discounted sum)*[12]

Suppose that

$$p_t = E_t \sum_{j=0}^{\infty} \beta^j d_{t+j} \tag{57}$$

where

$$\begin{aligned} \mathcal{D}d_{t+1} &= z_{1t} + z_{2t} + \sigma_3 \epsilon_{3,t+1} &(58)\\ z_{1,t+1} &= z_{1t} + \sigma_1 \epsilon_{1,t+1} &(59)\\ z_{2,t+1} &= \rho z_{2t} + \sigma_2 \epsilon_{2,t+1}, \quad |\rho| < 1, &(60)\\ y_t &= 1 &(61) \end{aligned}$$

where $\mathcal{D}$ is the first difference operator and $z_{10} \sim \mathcal{N}(\hat{z}_{10}, \Delta_{11}), z_{20} \sim \mathcal{N}(\hat{z}_{20}, \Delta_{22})$. Note that (57) can be rewritten as

$$p_t = \frac{1}{1-\beta} d_t + \frac{1}{1-\beta} E_t \sum_{j=1}^{\infty} \beta^j \mathcal{D}d_{t+j}. \tag{62}$$

The signal at t is $\mathcal{D}d_t$ and the observed history at t is $d_t, \mathcal{D}d_t, \mathcal{D}d_{t-1}, \ldots, \mathcal{D}d_1$. The unobserved state is $z_t = \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}$ and the observed state is $y_t = 1$. The objective function (62) involves one-period returns that can be expressed as cross-products of y_t and $\mathcal{D}d_{t+1}$. To implement this example, set

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix},$$

$$D_1 = 0, \quad D_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & \sigma_3 \end{bmatrix}, \quad H = 0,$$

$$Q = 0, \qquad R = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & 0 \\ .5 & 0 & 0 \end{bmatrix}, \quad P' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example 10.3. *(Muth revisited)*

In this special case of example 8.4, a scalar hidden state evolves as

$$z_{t+1} = \rho z_t + \sigma_1 \epsilon_{1,t+1}$$

and a signal is related to z_t by

$$s_{t+1} = z_t + \sigma_2 \epsilon_{2,t+1}$$

where $\epsilon_{1,t+1}$ and $\epsilon_{2,t+1}$ are orthogonal i.i.d. random sequences distributed $\mathcal{N}(0, I)$ and $z_0 \sim \mathcal{N}(\check{z}_0, \Delta_0)$. The decision maker wants to estimate the hidden state z_t. Muth (1960) studied a non-robust version of this problem for $\rho = 1$. We proceed as for example 8.4 and set $a_t = -Px_t,\ P = -1, R = 1, Q = 1, A = \rho, B = 0, C = \begin{bmatrix} \sigma_1 & 0 \end{bmatrix}, D = 1, H = 0, G = \begin{bmatrix} 0 & \sigma_2 \end{bmatrix}$.

[12] See the model of stock prices created by Lewis and Whiteman (2008) for an example with a related structure.

Example 10.4. *(A monopolist estimates demand)*

A monopolist faces the inverse demand function $p_t = z_{1t} + z_{2t} - \alpha y_t$ *where* p_t *is the price of the product,* y_t *is the quantity sold by the monopolist,* $\alpha > 0$ *is minus the slope of the inverse demand curve, and*

$$\begin{aligned} z_{1t+1} &= z_{1t} + \sigma_1 \epsilon_{t+1} \\ z_{2t+1} &= \rho z_{2t} + \sigma_2 \epsilon_{t+1} \end{aligned}$$

here σ_1, σ_2, *and* σ_y *are each* 1×3 *vectors and* ϵ_{t+1} *is an i.i.d.* 3×1 *vector distributed* $\mathcal{N}(0, I)$. *Time* $t+1$ *output* y_{t+1} *is controlled via*

$$y_{t+1} = y_t + a_t + \sigma_y \epsilon_{t+1}$$

where σ_y *is a* 1×3 *vector.*

A monopolist has prior $z_t \sim \mathcal{N}(\check{z}_0, \Delta_0)$ *over the hidden states* z_t. *The monopolist knows* α *and observes* p_t *and* y_t, *and therefore* $z_{1t} + z_{2t}$, *but not* z_{1t} *and* z_{2t} *separately. The signal equation is*

$$p_{t+1} = z_{1t} + \rho z_{2t} - \alpha y_t - \alpha a_t + (\sigma_1 + \sigma_2 - \alpha \sigma_y)\epsilon_{t+1}$$

where $s_{t+1} \equiv p_{t+1}$. *The monopolist faces period* t *adjustment costs* $d a_t^2$ *and chooses a strategy for* a *to maximize the expectation of*

$$\sum_{t=0}^{\infty} \beta^t \Big\{ (z_{1t} + z_{2t} - \alpha y_t) y_t - d a_t^2 \Big\}.$$

To implement this example, set

$$Q = -d, \quad P' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad R = \begin{bmatrix} -\alpha & .5 & .5 \\ .5 & 0 & 0 \\ .5 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \sigma_y \\ \sigma_1 \\ \sigma_2 \end{bmatrix},$$

$$D = \begin{bmatrix} -\alpha & 1 & \rho \end{bmatrix}, \quad H = -\alpha, \quad G = \begin{bmatrix} -\alpha \sigma_y + \sigma_1 + \sigma_2 \end{bmatrix}.$$

11 Concluding remarks

We can step slightly outside the LQG structure of this paper to consider more general settings in which an additional hidden Markov state indexes a finite set of LQG submodels, for example, as in the models without fear of model misspecification analyzed by Svensson and Williams (2008).[13] It would be possible to use the ideas and computations in this paper to adapt the Svensson and Williams structure to incorporate fears of misspecification of the submodels and of the distribution over submodels. That would involve calculations

[13] The periodic models of Hansen and Sargent (2007b, ch. 17) are closely related to the structures of Svensson and Williams (2008).

closely related to ones that Hansen and Sargent (2007a) use to model countercyclical uncertainty premia and that Cogley et al. (2008) use to design a robust monetary policy when part of the problem is to affect the evolution of submodel probabilities through purposeful experimentation.

A Modified certainty equivalence

Connections between the outcomes of the following two problems allow us to compute the T^1 operator easily.

Problem A.1. *Let $V(x) = -\frac{1}{2}x'\Omega x - \omega$, where Ω is a positive definite symmetric matrix. Consider the control problem*

$$\min_v V(x^*) + \frac{\theta}{2}|v|^2$$

subject to a linear transition function $x^* = Ax + Cv$. If θ is large enough that $I - \theta^{-1}C'\Omega C$ is positive definite, the problem is well posed and has solution

$$v = Kx \tag{63}$$

$$K = [\theta I - C'\Omega C]^{-1}C'\Omega A. \tag{64}$$

The following problem uses (63) and (64) to compute the T^1 operator:

Problem A.2. *For the same value function $V(x) = -\frac{1}{2}x'\Omega x - \omega$ that appears in problem A.1, let the transition law be*

$$x^* = Ax + Cw^*$$

where $w^* \sim \mathcal{N}(0, I)$. The T^1 operator gives the indirect utility function of the following minimization problem:

$$\min_{m^*} E[m^*V(x^*) + \theta m^* \log m^*].$$

The minimizer is

$$\begin{aligned} m^* &\propto \exp\left(\frac{-V(x^*)}{\theta}\right) \\ &= \exp\left[-\frac{1}{2}(w^* - v)'\Sigma^{-1}(w^* - v) + \frac{1}{2}w^* \cdot w^* - \frac{1}{2}\log\det\Sigma\right] \end{aligned}$$

where v is given by (63)-(64) from problem A.1, the worst-case variance $\Sigma = (I - \theta^{-1}C'\Omega C)^{-1}$, and the entropy of m^* is

$$Em^* \log m^* = \frac{1}{2}\Big[|v|^2 + \text{trace}(\Sigma - I) - \log\det\Sigma\Big].$$

Therefore, we can compute the objects (v, Σ) needed to form T^1 by solving the deterministic problem A.1.

B Alternative formulation

When Δ_t is nonsingular, recursions (40)–(43) can be implemented with the following recursions that are equivalent to the formulation of Başar and Bernhard (1995). Let

$$
\begin{aligned}
\check{A} &\doteq A_{22} - C_2 G'(GG')^{-1} D_2 \\
\check{N} &\doteq C_2 C_2' - C_2 G'(GG')^{-1} G C_2{}'
\end{aligned}
$$

Then we can attain $\check{z}_t$ directly via the recursions:

$$
\check{\Delta}_{t+1} = \check{A}\left[(\check{\Delta}_t)^{-1} - \frac{\beta^t}{\theta} R_{22} + D_2{}'(GG')^{-1} D_2\right] \check{A}' + \check{N} \tag{65}
$$

and

$$
\begin{aligned}
\check{z}_{t+1} \;=\;& A_{21} y_t + \check{A}\check{z}_t + B_2 a_t \\
&+ \check{A}\left[(\check{\Delta}_t)^{-1} - \frac{\beta^t}{\theta} R_{22} + D_2{}'(GG')^{-1} D_2\right]^{-1} D_2'(GG')^{-1}(s_{t+1} - D_1 y_t - D_2 \check{z}_t) \\
&+ \frac{\beta^t}{\theta}\check{A}\left[(\check{\Delta}_t)^{-1} - \frac{\beta^t}{\theta} R_{22} + D_2{}'(GG')^{-1} D_2\right]^{-1} (P_2{}' a_t + R_{12}{}' y_t + R_{22}\check{z}_t)\,.
\end{aligned} \tag{66}
$$

C Matlab programs

This appendix describes how to use Matlab programs that solve and simulate outcomes of our four games.

Four object oriented programs named

```
PreData_ComputeGameOne.m
PreData_ComputeGameTwo.m
PreData_ComputeGameThree.m
PreData_ComputeGameFour.m
```

compute objects in the respective games that do not depend on data. After these objects have been computed, the programs

```
ComputeGameOne_dataparts.m
ComputeGameTwo_dataparts.m
ComputeGameThree_dataparts.m
ComputeGameFour_dataparts.m
```

generate time series of decisions and filtered estimates.

A sample driver files illustrates two different ways of using these Game solving functions to solve example 10.1. The program `example_permanent_income.m` simulates data from the approximating model then solves Game I. Then, in order to show how to use an external data set, it saves and loads the simulated data and proceeds to solve each of Games II , III and IV . Of course it is also possible to simulate new data set each time you call a game solving function. Examples of this last feature are included as comments in the driver file. Once the functions mentioned above have computed the objects of interest, you can extract them for further analysis. Both the the accompanying readme file and the permanent income driver file show you ways to extract these results. Finally, we have provided a couple of files that plot subsets of these results in the hope it will facilitate your analysis.

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