Arrow-Barro Debt-GDP Dynamics

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Abstract

We build a stochastic counterpart of Barro's (1979) celebrated non-stochastic model of tax rate and debt/GDP dynamics by adding markets for trading risks along lines suggested by Arrow (1964) and Shiller (1994). These modifications preserve Barro's striking prescriptions that a government should keep its debt-GDP ratio and its tax rate constant over time. It also brings a new striking prescription: that to insure its primary surplus risk under an optimal policy the government should sell the same number of shares of a Shiller macro security each period. Relative to the widely used $r-g$ rate of Barro and Blanchard (2019), our model adds a risk premium λ to the rate at which the government should discount risk-free debt.

Keywords: Tax smoothing, Ricardian equivalence, debt-GDP dynamics, risk premium.

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1 Introduction

In endorsing different rates at which markets and governments should discount government debts, economists including Blanchard (2019), and Jiang et al. (2020, 2022) have confronted classic issues about how government cost-benefit analysis should discount risky public investment projects (e.g., Arrow (1966) and Arrow and Lind (1978)). To advocate a discount rate, it is necessary describe trading opportunities open to a government and its constituents, the government's purposes, and its taxing and borrowing policies.[1]

This paper illustrates how these matters are tied together by adding risks that a government also has to manage to Barro's 1979 non-stochastic model of optimal tax-debt policies. We also add markets in one-period ahead Arrow (1964) securities that allow both private and government decision makers to insure those risks. We show that it is sufficient for the government to trade a single Shiller (1994) security whose payoff is proportional to GDP growth.[2] We adopt a setup close to one that Lucas (2003, 1987, Sect. III) used to measure the costs of business cycles and benefits from improved countercyclical macroeconomic policy. We incorporate insights of Hansen et al. (1999), Alvarez and Jermann (2004), and Barillas et al. (2009) about how a stochastic discount factor (SDF) process contains information about costs of business cycles. Like Lucas (2003, 1987, Sect. III), the SDF process is determined outside our model. To stay close to Barro (1979), we specify our SDF process to be a simple generalization of his time-discount factor process.

Our model of optimal taxation and government debt management preserves the striking constant tax rate and constant debt-GDP ratio prescriptions of Barro (1979). Our model also rationalizes a formula for discounting risk-free government debt deployed by Jiang et al. (2020, 2022) that adds a risk premium to a formula used by Blanchard (2019) and many others. The source of the risk premium is that our government can service its risk-free bonds only if it sells enough of the Shiller security to insure its primary surplus risk. A risk premium on the Shiller security appears in the government budget constraint and in the rate at which the government should discount its risk-free debt.

[1] For a related perspective, see Bohn (1990, 1998).

[2] This is a consequence of a "dynamic trading" argument in the spirit of Harrison and Kreps (1979), Black and Scholes (1973), and Merton (1973).

2 The Setting

GDP $\{Y_t\}$ follows

$$Y_{t+1} = \exp\left(g - \frac{\sigma^2}{2} + \sigma\varepsilon_{t+1}\right) Y_t\,, \tag{1}$$

where $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$ is an i.i.d. process. Government expenditures $\Gamma_t = \gamma Y_t$ are perpetually proportional to Y_t. Government debt B_0 is due at time 0. Total tax collections $\mathcal{T}_t$ at time $t \geqslant 1$ are a measurable function of the history $\varepsilon^t = [\varepsilon_t, \varepsilon_{t-1}, \cdots, \varepsilon_0]$; $\mathcal{T}_0$ is an initial condition chosen by the government. A stochastic discount factor (SDF) process $\{M_t\}$ is determined outside our model and has multiplicative increments[3]

$$\frac{M_{t+1}}{M_t} \equiv m_{t+1} = \exp\left[-\left(r + \frac{\eta^2}{2}\right) - \eta\varepsilon_{t+1}\right]\,, \tag{2}$$

where η is the price of GDP growth risk ε_{t+1} and $M_0 > 0$ is given. To price Arrow (1964) securities, we can interpret the multiplicative increment $m(\varepsilon_{t+1})$ of the SDF process as an exogenous time t state-price density of a claim bought at time t that pays off at time $t+1$.[4] To understand the sense in which $m(\varepsilon)$ is a density, where $\Phi_t(\,\cdot\,)$ is the standardized normal cumulative distribution function, an Arrow security that pays off 1 unit of GDP at time $t+1$ whenever the realized GDP shock lies in interval $(\varepsilon_{t+1}, \varepsilon_{t+1} + d\varepsilon_{t+1})$ is priced at time t by $m(\varepsilon_{t+1})d\Phi(\varepsilon_{t+1})$. Let a function $\Pi(\,\cdot\,) : (-\infty, \infty) \to \mathbf{R}$ represent a bundle of ε_{t+1}-contingent payoffs. The price at time t of bundle Π is $\int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon)$.[5]

It is useful to view claims on GDP as a security and to use the SDF process with increments defined by (2) to price this Shiller (1994) macro security:

$$S_t = \mathbb{E}_t\left[\sum_{u=t+1}^{\infty} \frac{M_u}{M_t} Y_u\right] = \frac{e^{-\delta}Y_t}{1-e^{-\delta}}, \tag{3}$$

where $\lambda = \eta\sigma$ and $\delta = r + \lambda - g$. The one-period gross return on this Shiller security is

$$R_{t+1} \equiv \frac{S_{t+1} + Y_{t+1}}{S_t} = \exp\left(r + \lambda - \frac{\sigma^2}{2} + \sigma\varepsilon_{t+1}\right)\,, \tag{4}$$

[3] This is a counterpart to and generalization of an assumption of Barro (1979), who took a time-invariant risk-free interest rate as given.

[4] We abuse notation by using ε_{t+1} to represent both GDP shock and its realization.

[5] Consider a claim whose payoff is 1 for all ε_{t+1} at $t+1$ so that $\Pi(\varepsilon_{t+1}) = 1$. The time-$t$ price of this claim equals $\int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon) = \int m(\varepsilon)d\Phi(\varepsilon) = e^{-r}$.

with expected return

$$\mathbb{E}_t[R_{t+1}] = e^{r+\lambda}.$$

Since SDF process (2) implies that the price of a one-period risk-free bond is $\mathbb{E}_t(m_{t+1}) = e^{-r}$, $\lambda = \eta\sigma$ is the risk premium component of the continuously compounded return $(r + \lambda)$ on the Shiller security; it equals the price η of risk ε_{t+1} times the Shiller security's exposure to that risk: σ. To indicate the dependence of R_{t+1} on ε_{t+1}, we'll often write $R_{t+1} = R(\varepsilon_{t+1})$.

While appendix A describes outcomes when the government manages risks by trading Arrow securities, we'll instead assume that the government instead manages risks in a way recommended by Shiller (1994). Appendix A verifies that lets the government attain the same optimal outcomes it would attain by trading Arrow securities.[6]

3 Optimal Fiscal Policy

We provide a theory of how the government chooses stochastic processes for its tax and portfolio management policy $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$. We follow Barro (1979) and assume that raising revenues $\mathcal{T}_t$ brings distortions measured by $\Theta(\mathcal{T}_t, Y_t)$, where

$$\Theta(\mathcal{T}_t, Y_t) = \theta(\tau_t)Y_t \tag{5}$$

and the scaled deadweight loss function $\theta(\tau)$ is increasing, convex, and smooth. The positive derivative $\theta'(\cdot)$ plays a key role in inducing the government to make total tax collections $\mathcal{T}_t$ be homogeneous of degree one in GDP so that primary surplus $\mathcal{T}_t - G_t$ also becomes homogeneous of degree one in GDP and subject to the same risk ε_{t+1} that affects GDP growth. If it wants to issue risk-free bonds, the government must insure that risk. A way to do that would be to purchase or sell an appropriate package of one-period Arrow (1964) securities. The Shiller security is that appropriate package because the government faces primary surplus risk that is perfectly correlated with risk in the price of the Shiller security at time $t + 1$.[7]

We assume that at time t our government chooses to purchase Δ_t shares of the Shiller security. Consequently, starting with an initial risk-free debt balance B_0, risk-free government debt $\{B_t\}$ evolves as

$$B_{t+1} = e^r B_t + e^r(\gamma - \tau_t)Y_t - \Delta_t\,(R_{t+1} - e^r)\,S_t, \quad t \geqslant 0\,, \tag{6}$$

[6] When there is a complete set of Arrow securities at each date t, the Shiller security is redundant.

[7] See Appendix A.

where $(\gamma - \tau_t)Y_t$ is the government's primary deficit at time t and $-\Delta_t\left(R_{t+1} - e^r\right)S_t$ describes how the government's purchase of Δ_t shares of the Shiller security at time t affects B_{t+1}.

Let C_t be time-t consumption of a representative consumer and let a one-period felicity function be $U(C) = \frac{C^{1-\psi}}{1-\psi}$, where $\psi > 0$ is a coefficient of relative risk aversion.[8] A benevolent government wants a tax-portfolio policy $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$ that maximizes the indirect utility function F_0 of a representative household defined by

$$F_0 = \max_{\{C_t\}} \; \mathbb{E}_0\left[\sum_{t=0}^{\infty} e^{-\rho t} U(C_t)\right]. \tag{7}$$

The household's maximization is subject to the intertemporal budget constraint

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t C_t\right] \;\leqslant\; W_0 + \mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t\left(Y_t - \mathcal{T}_t - \Theta_t\right)\right], \tag{8}$$

where W_0 is the household's initial exogenously endowed wealth that is unaffected by forces active in our model. The household takes SDF $\{M_t\}$, tax collection $\{\mathcal{T}_t\}$, and deadweight loss $\{\Theta_t\}$ processes as given. By adapting the Lagrangian method used by Cox and Huang (1989) to allow for taxes and deadweight losses, we can deduce

$$F_0 = \frac{(\beta(W_0 + X_0))^{1-\psi}}{1-\psi}, \tag{9}$$

where $\beta = \left[1 - \exp\left(-\left((1-\frac{1}{\psi})r + \frac{\rho}{\psi} + \frac{1}{2}(1-\frac{1}{\psi})\frac{1}{\psi}\eta^2\right)\right)\right]^{-\frac{\psi}{1-\psi}}$ and

$$X_0 = \mathbb{E}_0\left[\sum_{s=0}^{\infty} \frac{M_s}{M_0}\left(Y_s - \mathcal{T}_s - \Theta_s\right)\right]. \tag{10}$$

To maximize the household's indirect utility function F_0 given in (9) over a taxation-portfolio policy $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$, it suffices for the the government to choose a joint $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$ process to maximize (10) subject to tax distortions (5), initial condition (B_0, Y_0) and the government budget constraint

$$B_0 \leqslant \mathbb{E}_0\left[\sum_{u=0}^{\infty} \frac{M_u}{M_0}\left(\mathcal{T}_u - \Gamma_u\right)\right]. \tag{11}$$

Before approaching this problem, we state:

[8]The key result that the household's utility maximization requires that the maximization of the market value of income flows holds for any well behaved increasing and concave utility function.

Proposition 3.1. *To recover findings of Barro (1974), suppose that the* $\tau(\cdot)$ *function satisfies* $\theta'(\cdot) = 0$ *for all tax rates* τ*. Then any taxation-portfolio strategy* $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$ *that satisfies the government's budget constraint solves the government's problem.*

Proposition 3.1 is Barro's (1974) Ricardian equivalence theorem. To make an optimal taxation-portfolio strategy profile determinant, Barro (1979) injected tax distortion function $\theta(\,\cdot\,)$ with $\theta'(\,\cdot\,) > 0$.

Now turning to the government's problem in the presence of an increasing, convex, and smooth $\theta(\,\cdot\,)$ function, we formulate (10) as a dynamic programming problem in which $X_0^* = P(B_0, Y_0)$ is the maximal attainable X_0 that satisfies (10). Value function $P(B_0, Y_0)$ satisfies the Bellman equation:

$$P(B_t, Y_t) \;=\; \max_{\tau_t,\,\Delta_t} \; \mathbb{E}_t \left[Y_t - \tau_t Y_t - \theta(\tau_t) Y_t + m_{t+1} P(B_{t+1}, Y_{t+1}) \right] . \tag{12}$$

Substituting (6) into (12) gives

$$\begin{aligned} P(B_t, Y_t) \;=\; & \max_{\tau_t,\,\Delta_t} Y_t - \tau_t Y_t - \theta(\tau_t) Y_t \\ & + \mathbb{E}_t \left[m_{t+1} P(e^r B_t + e^r(\gamma - \tau_t) Y_t - \Delta_t \left(R_{t+1} - e^r \right) S_t, Y_{t+1}) \right] . \end{aligned} \tag{13}$$

First-order necessary conditions for τ_t and Δ_t, respectively,[9] are:

$$1 + \theta'(\tau_t) = -\mathbb{E}_t \left[m_{t+1} e^r P_B(B_{t+1}, Y_{t+1}) \right] , \tag{14}$$

and

$$-\mathbb{E}_t \left[m_{t+1} (R_{t+1} - e^r) P_B(B_{t+1}, Y_{t+1}) \right] = 0 \, . \tag{15}$$

Let $b_t = B_t / Y_t$ denote the debt-GDP ratio. We guess and verify that $P(B_t, Y_t) = p(b_t) Y_t$ and that $b_{t+1} - b_t = 0$ for all t. First-order condition (14) for tax rate τ_t implies

$$1 + c'(\tau_t) = -p'(b_t) = -p'(b_0), \tag{16}$$

which equates the marginal cost $1 + c'(\tau_t)$ of taxing the household with the marginal benefit $-p'(b_t)$ of reducing debt. Equation (16) implies that the tax rate τ_t is constant over time.

[9] The second-order condition with respect to τ_t holds because $\theta(\tau)$ is convex. The second-order condition with respect to Δ_t is

$$\mathbb{E}_t \left[m_{t+1} (R_{t+1} - e^r)^2 P_{BB}(B_{t+1},, Y_{t+1}) \right] < 0$$

because $P_{BB} < 0$, as we show later.

Combining this outcome with the government's budget constraint (6) implies

$$\tau_t = (1 - e^{-\delta})b_t + \gamma. \tag{17}$$

The optimal tax rate is constant over time and independent of the deadweight cost function $\theta(\,\cdot\,)$. To verify $b_{t+1} - b_t = 0$ for all $t \geqslant 0$, it is necessary that

$$\frac{B_{t+1}}{Y_{t+1}} = \frac{e^r B_t + e^r(\gamma - \tau_t)Y_t - \Delta_t \left(R_{t+1} - e^r\right) S_t}{Y_{t+1}} = \frac{B_t}{Y_t}, \tag{18}$$

which implies:[10]

$$\Delta_t = -\left(1 - e^{-\delta}\right) b_t = -\left(1 - e^{-\delta}\right) b_0, \tag{19}$$

so it is optimal for the government to sell (i.e., take a short position in) the Shiller security. The random vector $[R_{t+1}, m_{t+1}]^\top$ is bivariate normal, so

$$\mathbb{E}_t\left[m_{t+1}R_{t+1}\right] = e^{r+\lambda-\frac{\sigma^2}{2}}\mathbb{E}_t\left[m_{t+1}e^{\sigma\varepsilon_{t+1}}\right] = \mathbb{E}_t\left[e^r m_{t+1}\right] . \tag{20}$$

Outcome (19) and $b_{t+1} - b_t = 0$ for all $t \geqslant 0$ confirm that the first-order condition (15) holds. In summary, we have established:

Theorem 3.2. *The optimal fiscal plan is described by $b_t = b_0$ and the following three equations:*

1. *Optimal tax rate:*
$$\tau(b_t) = \tau(b_0) = (1 - e^{-\delta})b_0 + \gamma \tag{21}$$

2. *Optimal purchase of Shiller security:*
$$\Delta_t = \Delta_0 = -\left(1 - e^{-\delta}\right) b_0 \,. \tag{22}$$

[10]Substituting $\tau_t = (1 - e^{-\delta})b_t + \gamma$ and (3) into (18), we obtain:

$$\begin{aligned}\Delta_t &= \frac{e^r B_t - e^r(1 - e^{-\delta})B_t}{\left(R_{t+1} - e^r\right) S_t} - \frac{B_t}{\left(R_{t+1} - e^r\right) S_t}\frac{Y_{t+1}}{Y_t} = \frac{(1 - e^{-\delta})e^r B_t}{\left(R_{t+1} - e^r\right) Y_t} - \frac{(1 - e^{-\delta})e^\delta B_t}{\left(R_{t+1} - e^r\right) Y_t} e^{\left(g-\frac{1}{2}\sigma^2\right)+\sigma\varepsilon_{t+1}} \\ &= -(1 - e^{-\delta})\frac{B_t}{Y_t}\left(\frac{e^{\delta+\left(g-\frac{1}{2}\sigma^2\right)+\sigma\varepsilon_{t+1}} - e^r}{R_{t+1} - e^r}\right) = -(1 - e^{-\delta})b_t \,.\end{aligned}$$

3. The GDP-scaled value X_0/Y_0 of after-tax, after tax-distortions GDP flowing to households:

$$p(b_t) = p(b_0) = \frac{1 - \tau(b_0) - \theta(\tau(b_0))}{1 - e^{-\delta}} \,. \tag{23}$$

Theorem 3.2 tells the government how to smooth taxes and to manage its debt.[11] Equations (21)-(23) can be solved recursively. Use equation (21) to compute a tax rate $\tau(b_0)$ and then set $\tau_t = \tau(b_0)$ for all $t \geqslant 0$. This tax rate suffices to fund government expenditures and to service the government's risk-free debt, including costs that arise from selling the Shiller security in the amount recommended by equation (22). The optimal Shiller security position Δ_t is negative and constant over time. Finally, $p(b_0)$ in equation (23) is the (scaled) value of revenues after taxes and after deadweight losses from taxation flowing to households: $p(b_0) = X_0/Y_0$, where X_0 is defined in (10) under the optimal policy.

4 Accounting for Costs and Benefits

By applying some asset pricing formulas, we can summarize how an optimal government policy distributes costs and benefits. The time-0 value of GDP is

$$V_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} M_t Y_t \right] = \frac{Y_0}{1 - e^{-\delta}} \,. \tag{24}$$

Our "value distribution formula" is:

$$V_0 = PV(\Gamma) + P(B_0, Y_0) + PV(\Theta) + B_0 \tag{25}$$

where

- the value of government spending is

$$PV(\Gamma) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} M_t \Gamma_t \right] = \frac{\gamma Y_0}{1 - e^{-\delta}} \,. \tag{26}$$

- the value of after-tax, after tax-distortions GDP flowing to households is

$$P(B_0, Y_0) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} M_t \left(Y_t - \tau_t Y_t - \Theta(\mathcal{T}_t, Y_t) \right) \right] = \frac{(1 - \tau_0 - \theta(\tau_0)) \, Y_0}{1 - e^{-\delta}} \,. \tag{27}$$

[11] To connect with findings of Barro (1974) and Barro (1979), note that when taxation brings no deadweight losses, i.e., when $\theta(\tau) = 0$, Ricardian equivalence holds because $p(b) = \frac{1-\gamma}{1-e^{-\delta}} - b$ and $p'(b) = -1$ for all b.

- the value of deadweight taxation loss is

$$PV(\Theta) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} M_t \Theta_t \right] = \frac{\theta(\tau_0) Y_0}{1 - e^{-\delta}} \,. \tag{28}$$

- the value of risk-free government debt is

$$B_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} M_t \left(\mathcal{T}_t - \Gamma_t \right) \right] = \frac{(\tau_0 - \gamma) Y_0}{1 - e^{-\delta}} \,. \tag{29}$$

Remark 4.1. *An increasing, convex, and smooth deadweight cost $\theta(\,\cdot\,)$ function induces the government to manage its exposure to GDP risk ε_{t+1} by keeping both the tax rate τ_t and the debt-GDP ratio b_t constant.*[12] *Investors are willing to hold risk-free one-period debt at a gross interest rate e^r. The government's risky primary surplus process $\{(\tau_0 - \gamma) Y_t; t \geqslant 0\}$ constitutes the ultimate "backing" behind all of its debts. To minimize deadweight costs of taxes, the government insures against primary surplus risk risk ε_{t+1} by selling $\left(1 - e^{-\delta}\right) b_0 Y_t$ shares of the Shiller security each period. The government pays a "portfolio management cost" in the form of the Shiller security's risk premium λ per unit of time.*

5 Concluding Remarks

By allowing the government to trade either a complete set of one-period Arrow securities or a single Shiller (1994) security, we have extended Barro's (1979) model in a way that preserves salient prescriptions: it is optimal for the government to keep its initial debt-GDP ratio constant forever and to levy a time-invariant tax rate sufficient to finance a constant ratio of its primary surplus to GDP. The government issues risk-free debt and sells a Shiller security each period. A Bellman equation that describes the value of after-tax, after tax-distortions GDP flowing to households discounts these flows at a rate $r + \lambda - g$ that includes the risk premium λ on the Shiller (1994) security in addition to the $r - g$ term that appears prominently in the analysis of Blanchard (2019).

We have retained an assumption shared by Arrow (1964) and Barro (1979) that financial contracts are perfectly enforced. We have denied government debt an additional "convenience yield" of a type often included in recent macro-finance papers. In subsequent work (Jiang et al., 2023), we plan to let the government service its debts only if and when it wants to; we'll

[12] The increasing, convex, and smooth $\theta(\,\cdot\,)$ function plays a key role via $-P_B(B_{t+1}, Y_{t+1}) = -p'(b_{t+1}) = -p'(b_0) = 1 + \theta'(\tau_0) > 1$

also to endow government debt with a convenience yield. Adding those features can shed light on circumstances that impart a positive drift to tax rates and to debt-GDP dynamics, features whose absence is a notable and often counterfactual feature of Barro (1979) that this paper's model shares.

A Arrow Securities Replace Shiller's Macro Security

Instead of using Shiller's macro asset to insure GDP shocks, the government can support the same optimal outcomes for the tax rate and risk-free government debt processes by using a complete set of one-time-ahead state-contingent Arrow securities. Both sets of financial arrangements support outcomes described by Theorem 3.2.

As noted in section 2, to obtain $\Pi(\varepsilon_{t+1})$ at time $t+1$ contingent on ε_{t+1} being realized, the government would have to pay $\int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon)$ at time t. When the government trades Arrow securities, the counterpart to equation (6) for the evolution of risk-free government debt $\{B_t\}$ is

$$B_{t+1} = e^r B_t + e^r(\gamma - \tau_t) Y_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon). \tag{30}$$

Substituting (30) into (12) gives

$$\begin{aligned} P(B_t, Y_t) = \max_{\tau_t, \Pi} \; & Y_t - \tau_t Y_t - \theta(\tau_t) Y_t \\ & + \mathbb{E}_t \left[m_{t+1} P \left(e^r B_t + e^r(\gamma - \tau_t) Y_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon), Y_{t+1} \right) \right]. \end{aligned} \tag{31}$$

Choosing $\Pi(\varepsilon_{t+1})$ for each ε_{t+1} to maximize $P(B_t, Y_t)$ is equivalent to choosing $\Pi(\varepsilon_{t+1})$ for each ε_{t+1} to maximize

$$\int \left[m(\varepsilon_{t+1}) P \left(e^r B_t + e^r(\gamma - \tau_t) Y_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon), Y_{t+1} \right) \right] d\Phi(\varepsilon_{t+1}). \tag{32}$$

The first-order necessary condition for Arrow security demand $\Pi(\varepsilon_{t+1})$ in state ε_{t+1} is

$$-m(\varepsilon_{t+1}) P_B(B_{t+1}, Y_{t+1}) d\Phi(\varepsilon_{t+1}) + [m(\varepsilon) d\Phi(\varepsilon)] \big|_{\varepsilon = \varepsilon_{t+1}} \int [m(\varepsilon_{t+1}) e^r P_B(B_{t+1}, Y_{t+1})] \, d\Phi(\varepsilon_{t+1}) = 0\,. \tag{33}$$

Once again guessing that $P(B_t, Y_t) = p(b_t) Y_t$ and that $b_{t+1} - b_t = 0$ for all t to simplify (33), we confirm that $1 = \mathbb{E}_t [m_{t+1} e^r]$.

The first-order necessary condition for τ_t agrees with (14). Combining this outcome with $b_{t+1} - b_t = 0$ for all t, we obtain (16) for τ_t. In conjunction with budget constraint (30), we obtain tax-smoothing result given in (21) To verify $b_{t+1} - b_t = 0$ for all $t \geqslant 0$, we can show that for all ε_{t+1}:[13]

$$\frac{B_{t+1}}{Y_{t+1}} = \frac{e^r B_t + e^r(\gamma - \tau_t)Y_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon)}{Y_{t+1}} = \frac{B_t}{Y_t}. \tag{34}$$

[13]After substituting the tax policy (21) into (34), we obtain:

$$\begin{aligned} & \frac{e^{r-\delta} B_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon)}{Y_t \exp\left[g - \frac{1}{2}\sigma^2 + \sigma\varepsilon_{t+1}\right]} = \frac{B_t}{Y_t} \\ \Leftrightarrow \quad & \left(e^{r-\delta} - e^{g - \frac{1}{2}\sigma^2 + \sigma\varepsilon_{t+1}}\right) B_t = \Pi(\varepsilon_{t+1}) - e^r \int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon) \\ \Leftrightarrow \quad & \left(-e^{-\delta} R(\varepsilon_{t+1}) B_t\right) - e^r \int \left(-e^{-\delta} R(\varepsilon) B_t\right) m(\varepsilon) d\Phi(\varepsilon) = \Pi(\varepsilon_{t+1}) - e^r \int \Pi(\varepsilon) m(\varepsilon) d\Phi(\varepsilon)\,, \end{aligned}$$

which implies

$$\Pi(\varepsilon_{t+1}) = -e^{-\delta} R(\varepsilon_{t+1}) B_t = \Delta_t R(\varepsilon_{t+1}) S_t\,,$$

where the second equality uses the expression for Δ_t given in (22).

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