Notes

1. The uniqueness of this decomposition requires some qualification. Elements of L^2 are only defined up to an equivalence of functions that are equal almost everywhere. Hence from the vantage point of L^2, the construction of ϕ^+ and ϕ^- at a particular point, say $t = 0$, is inconsequential.

2. We take the right side of equation (3.1) as the definition of $x(t)$. Alternatively, for particular classes of γ we could define $x(t)$ using finite sum approximations for the middle integral.

3. In Examples 1 and 3 it is also possible to allow for exponential growth in the second moments of y as long as $\psi \exp(-\sigma t)$ is in L^2_+ for some σ satisfying $0 < \sigma < \delta$. The transform analysis now applies to the narrower strip $\mathbf{C}_\delta^- \cap \mathbf{C}_\sigma^+$.

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Identification of Continuous Time Rational Expectations Models from Discrete Time Data

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1. Introduction

This paper proves two propositions about identification in a continuous time version of a linear stochastic rational expectations model. The model is a continuous time version of Lucas and Prescott (1971), in which the equilibrium can be interpreted as the solution of a stochastic control problem, either of a collection of private agents or of a fictitious *social planner*. Estimation is directed toward isolating the parameters of the *agent's* objective function and of the stochastic processes of the forcing functions that the agent faces. This approach has been advocated by Lucas (1967, 1976), Lucas and Prescott (1971), and Lucas and Sargent (1981) as offering the potential to analyze an interesting class of policy interventions promised by *structural* models, while meeting the criticisms of most econometric policy evaluation methods that were made by Lucas (1976). At the same time, inspired by the work of Sims (1971), Geweke (1978), and P.C.B. Phillips (1972, 1973, 1974), we want to estimate models in which optimizing economic agents make decisions at finer time intervals than the interval of time between the observations used by the econometrician. We adopt a continuous time theoretical framework both because it is an interesting limiting case, and because it has received extensive attention in the theoretical and the econometric literatures.

Identification of the parameters of a continuous time model from discrete time data must confront the aliasing problem (see, e.g., Phillips 1973). In general, there is an uncountable infinity of continuous time models that are consistent with a collection of discrete time observations. However, with finite parameter continuous time models, the aliasing problem, while still present, is less severe. The dimensions of the aliasing identification problem for the particular class of finite parameter models treated in this paper have been studied in earlier papers by Phillips (1973) and Hansen and Sargent (1983b). In these finite parameter models, there is a finite number of observationally equivalent continuous time models that are consistent with the discrete time observations. To achieve identification of the continuous time model, an additional source of prior restrictions is needed. This paper shows how the non-linear cross-equation restrictions implied by rational expectations achieve identification of the continuous time model.

We consider a linear rational expectations model that gives rise to systems of stochastic differential and difference equations that resemble the forms of Phillips' (1973) systems. However, we analyze identifying restrictions of a different variety than those studied by Phillips. As Lucas (1976), Lucas and Sargent (1981), and Hansen and Sargent (1980a, 1981a, 1981d) have pointed out in several related contexts, even rational expectations models that are linear in the variables typically are characterized by sets of highly nonlinear cross-equation restrictions, which to a large extent replace the linear (usually exclusion, usually within-equation) restrictions used to identify many existing time series models.

The intuition underlying our results is as follows. In dynamic rational expectations models, agents' decisions partly depend on their expectations of all future values of other variables in the model. When agents are acting in continuous time, a discrete time record of agents' decisions contains information about their forecasts of other variables in the model for all instants in the future. Under rational expectations, the hints about agents' views of the future contained in their decisions at discrete points in time restrict the actual behavior of these other variables as stochastic processes in continuous time. These hints are the source of identification that we propose to utilize.

We prove identification propositions under two alternative sets of conditions. The first set of conditions severely restricts the serial correlations of the unobservable disturbance term, although it does not require that the right-hand-side observables be strictly exogenous. The second set of conditions leaves the serial correlations of the disturbance unrestricted but imposes that the right-hand-side variables must be strictly exogenous in continuous time and that they have a rational spectral density matrix. Identification is then achieved from the restrictions that the theory imposes between the projections of the en-

dogenous on the exogenous variables, on the one hand, and the spectral density matrix of the exogenous variables, on the other hand. This second set of conditions thus uses an approach to identification in the spirit of that used by Hatanaka (1975) in the context of discrete time models. Our results exhibit a tradeoff between the strength of strict exogeneity and serial correlation assumptions that are sufficient for identification. A similar tradeoff occurs in discrete time series models.

2. The Continuous Time Model

The model studied is a continuous time, linear-quadratic version of a Lucas-Prescott model of investment under uncertainty. This model has a variety of possible interpretations, applications, and extensions (for example, see Hansen and Sargent 1981a, Eckstein 1984, and Eichenbaum 1983). For the identification propositions proved here, a single factor model involving a single dynamic decision variable is used. In the appendix, we briefly indicate how the results might be extended to prove identification of continuous time, interrelated factor models from discrete time data.[1]

Consider a firm or fictitious social planner that maximizes over strategies for $K(t)$ the criterion

$$E_0 \int_0^\infty J[K(t), DK(t), t, z_1(t), y(t)]dt \tag{1}$$

where

$$\begin{aligned} &J[K(t), DK(t), t, z_1(t), y(t)] \\ &\quad = \Big\{ y(t)K(t) - \beta K(t)^2 - z_1(t)DK(t) - \alpha[DK(t)]^2 \Big\} e^{-rt} , \end{aligned}$$

where D is the time derivative operator, and where E_t is the expectations operator conditioned on information available at time period t. Here $K(t)$ is the capital stock, $z_1(t)$ is the relative price of investment goods, $y(t)$ is a random shock to productivity, all at time period t, α and β are positive constants, and r is a fixed discount rate. The variables $z_1(t)$ and $y(t)$ are elements in a vector stochastic process of forcing variables. Using results from Hansen and Sargent (1981d) and Chapter 7, the Euler equation for the certainty equivalent version of the firm's maximization problem is

$$\begin{aligned} &- \alpha D^2 K(t) + r\alpha DK(t) + \beta K(t) \\ &= -(1/2)\,[rz_1(t) - y(t) - Dz_1(t)] \end{aligned} \tag{2}$$

For simplicity, we assume that the discount rate is zero.[2] The characteristic polynomial for the Euler equation (2) can be factored

$$-\alpha s^2 + \beta = (\rho - s)\ (\rho + s)\alpha$$

where

$$\rho = \sqrt{\frac{\beta}{\alpha}}\ .$$

The solution to the Euler equation (2) that maximizes (1) is

$$(3)\quad DK(t) = -\rho K(t) - (1/2\alpha)E_t \int_0^\infty e^{-\rho\tau}[Dz_1(t+\tau) + y(t+\tau)]d\tau\ .$$

We seek to identify ρ, α, and the parameters of the stochastic processes of the forcing variables from discrete time data.[3] To provide an interpretation of the error term in equations fit by an econometrician, we assume that $y(t)$ is observed by private agents but not by the econometrician. Let $z(t)' = [z_1(t), z_2(t)']$, where $z_2(t)$ is a list of additional variables which are seen by both private agents and the econometrician and which help predict future z_1's. The econometrician knows the discrete time covariogram and cross-covariogram of the (K, z) process and from these moments seeks to identify the parameters ρ and α that characterize the continuous time objective function (1) and the parameters of the continuous time stochastic process governing (z, y). We study this identification problem using two alternative specifications of the continuous time stochastic process (z, y).

3. Identification Where (K, z) Is a First-Order Markov Process

In this section we make a special assumption about the forcing variables that is sufficient to imply that (K, z) is a covariance stationary, first-order Markov process. Specifically,

Assumption 1: The forcing variables $y(t)$ and $z_1(t)$ are governed by[4]

$$y(t) = D\varepsilon_1^*(t)$$

and

$$(4)\qquad Dz(t) = A_{22}z(t) + \varepsilon_2^*(t)$$

where $z_1(t)$ is the first element in the $n-1$ dimensional vector $z(t)$, the eigenvalues of A_{22} have negative real parts, and $\varepsilon^{*\prime} = [\varepsilon_1^*,\ \varepsilon_2^{*\prime}]$ is an n dimensional vector white noise with intensity matrix V_0^*.[5]

Note that Assumption 1 allows ε_1^* and ε_2^* to be correlated contemporaneously.

Using (4) and the results from Hansen and Sargent (1981d) and Chapter 8 to solve the prediction problem on the right side of (3), we obtain

$$(5)\quad DK(t) = -\rho K(t) - (1/2\alpha)uA_{22}[A_{22} - \rho I]^{-1}z(t) - (1/2\alpha)\varepsilon_1^*(t)$$

where u is the $n-1$ dimensional unit row vector given by $u = [1, 0]$. We let $\varepsilon' = [(-1/2\alpha)\varepsilon_1^*,\ \varepsilon_2^*]$, and we stack equations (4) and (5) into the vector first order differential equation system:

$$\begin{bmatrix} DK(t) \\ Dz(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} K(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{bmatrix}$$

or

$$Dx(t) = A_0\, x(t) + \varepsilon(t)\ .$$

The partitions of the A_0 matrix satisfy the restrictions

$$(6)\qquad \begin{aligned} A_{11} &= -\rho \\ A_{21} &= 0 \\ A_{12} &= (-1/2\alpha)uA_{22}\,[A_{22} - \rho I]^{-1}\ . \end{aligned}$$

While the restriction on A_{21} is a zero restriction, the restrictions linking A_{11}, A_{12}, and A_{22} are highly nonlinear. Phillips (1973) considered the impact on identification of the zero restriction on A_{21}.[6] It happens that this exclusion restriction by itself is not sufficient to identify the parameters of A_{22} and A_{12}. However, we shall show that once we add the nonlinear cross-equation restrictions implied by rational expectations, it is possible to identify ρ, α, A_{22}, and, consequently, A_{12} and A_{11}.

It was shown by Phillips (1973) that the discrete time process X obtained by sampling x at the integers has a first order autoregressive representation,[7]

$$X(t) = B_0X(t-1) + \eta(t)$$

where

$$B_0 = \exp A_0$$

$$\eta(t) = \int_0^1 \exp(A_0\tau)\,\varepsilon(t-\tau)d\tau\ .$$

By virtue of the fact that ε is a continuous time white noise, it follows that η is a discrete time white noise. The parameters of B_0 are identified from knowledge of the discrete time matrix covariogram of the $X = (K, z)$ process.

We pose the following identification question: given the matrix B_0, is it possible uniquely to determine the free parameters of the matrix A_0?[8] That is, does the matrix equation

$$\exp A^* = B_0 = \exp A_0 \tag{7}$$

imply that $A^* = A_0$? We shall prove that the answer is yes. To proceed, we make the additional assumption:

Assumption 2: The eigenvalues of A_0 are distinct.

Write the spectral decomposition of A_0 as

$$A_0 = T\Lambda_0 T^{-1}$$

where Λ_0 is a diagonal matrix of eigenvalues of A_0 and T is the matrix whose columns are eigenvectors of A_0. Partition the matrices T and Λ_0 in the eigenvalue decomposition of A_0 conformably with A_0 so that

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \quad \Lambda_0 = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}.$$

It is readily verified that $-\rho = \Lambda_1$ and $A_{22} = T_{22}\Lambda_2 T_{22}^{-1}$, so that Λ_2 is the diagonal matrix of the eigenvalues of A_{22}. Now let the first $n-1-2m$ eigenvalues of A_{22} be real and the remainder occur in complex conjugate pairs as $\lambda_{n-m} = \bar{\lambda}_{n-2m}, \ldots, \lambda_{n-1} = \bar{\lambda}_{n-1-m}$. For analytical convenience, we require

Assumption 3: The eigenvalues of A_0 do not differ by integer multiples of $2\pi i$.

Then if a matrix A^* is to satisfy (7), it must be related to A_0 by[9]

$$A^* = A_0 + 2\pi i T \begin{bmatrix} 0 & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & -P \end{bmatrix} T^{-1} \tag{8}$$

where P is any m dimensional diagonal matrix whose diagonal elements are arbitrary integers. In effect, (8) displays the class of perturbations of the complex eigenvalues of A_0 which leave the relation $B_0 = \exp A^*$ satisfied.

To show that the restrictions imposed on the model by rational expectations can be used to identify A_0 from B_0, we shall use the special nature of the perturbations of A_0 which are admissible under (8). Notice that all A^*'s that satisfy (8) must have identical matrices of eigenvectors—that is, T matrices—and can differ only in the imaginary parts of their complex eigenvalues. So the T matrix is identified, as are the real parts of the eigenvalues. Since ρ is a real eigenvalue, it is automatically identified. We shall indicate how the cross-equation restrictions imposed by rational expectations, in effect, link T, ρ, and the eigenvalues Λ_2. This will serve to establish the existence of a unique inverse of $B = \exp A^*$.

Using the partitioned inverse formula

$$T^{-1} = \begin{bmatrix} T_{11}^{-1} & -T_{11}^{-1}T_{12}T_{22}^{-1} \\ 0 & T_{22}^{-1} \end{bmatrix},$$

we obtain the version of the eigenvalue decomposition appropriate for our problem

$$A_0 = \begin{bmatrix} T_{11}\Lambda_1 T_{11}^{-1} & T_{12}\Lambda_2 T_{22}^{-1} - T_{11}\Lambda_1 T_{11}^{-1} T_{12} T_{22}^{-1} \\ 0 & T_{22}\Lambda_2 T_{22}^{-1} \end{bmatrix}.$$

It follows that

$$A_{12} = [T_{12}\Lambda_2 T_{22}^{-1} + \rho T_{12} T_{22}^{-1}]. \tag{9}$$

We use (6) and (9) to express the cross-equation restrictions implied by the model in the form

$$(-1/2\alpha) u A_{22} [A_{22} - \rho I]^{-1} = [T_{12}\Lambda_2 + \rho T_{12}] T_{22}^{-1}$$

or

$$(-1/2\alpha) u T_{22}\Lambda_2 [\Lambda_2 - \rho I]^{-1} = T_{12} [\Lambda_2 + \rho I].$$

Solving for T_{12}, we obtain

$$(-1/2\alpha) u T_{22}\Lambda_2 [\Lambda_2 - \rho I]^{-1} [\Lambda_2 + \rho I]^{-1} = T_{12}$$

or

$$T_{12} = \frac{-uT_{22}}{2} \ \mathrm{diag} \left[\frac{\lambda_j}{(\lambda_j^2 - \rho^2)\alpha} \right]. \tag{10}$$

Since T_{12} and T_{22} are identified because the eigenvectors of A_0 are identified, equation (10) implies that the quantities

$$d_j = \frac{\lambda_j}{(\lambda_j^2 - \rho^2)\alpha} \tag{11}$$

can be inferred from the discrete time statistics. The question which remains is whether, given knowledge of d_j, ρ, and the real part of λ_j, we can infer α and the imaginary part of λ_j. To find the answer, first suppose λ_1 is real. Then it follows that α can be inferred from (11) for $j=1$. Let j be some other index such that λ_j is complex and suppose that $\lambda_j^* = \lambda_j + 2\pi i p$ for some integer p that satisfies (11). Then we know that

$$\lambda_j(\lambda_j^{*2} - \rho^2) = \lambda_j^*(\lambda_j^2 - \rho^2) . \tag{12}$$

The value of λ_j^* distinct from λ_j that satisfies (12) is

$$\lambda_j^* = \frac{-\rho^2}{\lambda_j} . \tag{13}$$

Write $\lambda_j = \theta_1 + \theta_2 i$ where θ_1 and θ_2 are real with θ_1 less than zero. Equation (13) implies that

$$\theta_1 + (\theta_2 + 2\pi p)i = \frac{-\rho^2}{\theta_1 + \theta_2 i}$$

or

$$\begin{aligned} &\theta_1\theta_2 + \pi p \theta_1 = 0 \\ &\theta_1^2 - \theta_2^2 - 2\pi p \theta_2 = -\rho^2 . \end{aligned} \tag{14}$$

However, there are no values of $\{\theta_1, \theta_2, p\}$ with $\theta_1 < 0$ that satisfy both equations in (14). It follows that all of the parameters of the model are identifiable from discrete time data whenever there is at least one real eigenvalue of A_{22}.

Thus we have the following:

Proposition 1: Suppose Assumptions 1–3 are satisfied. If A_{22} has at least one real eigenvalue, then the parameters α and β (or, equivalently, α and ρ) and the parameters of A_{22} are identifiable from discrete time observations.

If there are only complex eigenvalues of A_{22}, then it can be proved, except for singular cases, that the free parameters of the continuous time model are identifiable.[10]

4. Identification with z Strictly Exogenous with Respect to K in Continuous Time

In the preceding section, the unobservable forcing variable $y(t)$ was allowed to be correlated contemporaneously with the observable forcing variables in $z(t)$. However, identification of the feedback parameter ρ used the fact that the disturbance term to the decision rule was known to be a white noise. We now wish to relax this assumption together with the assumption that the observable forcing variables can be represented as a first-order Markov process. We relax these assumptions at the cost of imposing a stronger condition about the covariance of y and z.[11]

Assumption 4: The joint process (y, z) is covariance stationary, linearly regular and satisfies the extensive orthogonality conditions $Ey(t)z(t-\tau) = 0$ for all τ.[12]

A fundamental moving average representation for (y, z) can be written in partitioned form

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} C_1(D) & 0 \\ 0 & C_2(D) \end{bmatrix} \begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{bmatrix} .$$

where $C_j(s)$ is the Laplace transform of a square integrable matrix function that is concentrated on the nonnegative numbers, and where $[\varepsilon_1, \varepsilon_2']'$ is a vector white noise with intensity matrix I. For the representation to be fundamental, we must require that $[\varepsilon_1(t), \varepsilon_2(t)']'$ lie in the space spanned by linear combinations of $\{y(\tau), z(\tau);\ \tau \le t\}$.[13]

In order to use convenient results from linear prediction theory for continuous time processes, we assume that best linear predictions and conditional expectations coincide. The forecasting problem on the right side of equation (3) can be solved using techniques developed in Chapter 8 to obtain,

$$\begin{aligned} DK(t) = &-\rho K(t) - \frac{u}{2\alpha}\left[\frac{DC_2(D) - \rho C_2(\rho)}{D-\rho}\right]\varepsilon_2(t) \\ &- \frac{1}{2\alpha}\left[\frac{C_1(D) - C_1(\rho)}{D-\rho}\right]\varepsilon_1(t) . \end{aligned}$$

Next we solve for $K(t)$ and determine that[14]

(16)

$$K(t) = \frac{-u[DC_2(D) - \rho C_2(\rho)]}{2\alpha(D+\rho)(D-\rho)}\varepsilon_2(t) - \frac{1}{2\alpha}\left[\frac{C_1(D) - C_1(\rho)}{(D+\rho)(D-\rho)}\right]\varepsilon_1(t) .$$

Since $\varepsilon_1(t)$ is orthogonal to $\varepsilon_2(s)$ for all t and s, equations (15) and (16) can readily be used to calculate the projection of $K(t)$ onto current, past, and future z's. This projection is given by

$$K(t) = \frac{-u[DC_2(D) - \rho C_2(\rho)]}{2\alpha(D+\rho)(D-\rho)} C_2(D)^{-1} z(t) + \xi(t)$$
$$= \frac{-u[DI - \rho C_2(\rho) C_2(D)^{-1}]}{2\alpha(D+\rho)(D-\rho)} z(t) + \xi(t)$$

where[15]

$$Ez(t)\xi(t-\tau) = 0 \text{ for all } \tau .$$

It is instructive to calculate the discrete time cross spectral density of K and z:

$$F_1(\omega) = \sum_{j=-\infty}^{+\infty} \left\{ -u \left[\frac{i(\omega+2\pi j)C_2(i\omega+2\pi ij) - \rho C_2(\rho)}{2\alpha[-(\omega+2\pi j)^2 - \rho^2]} \right] C_2(-i\omega - 2\pi ij)' \right\} . \tag{17}$$

From discrete time data we can identify the function F_1 together with the discrete spectral density of z which is given by

$$F_2(\omega) = \sum_{j=-\infty}^{+\infty} C_2(i\omega + 2\pi ij) C_2(-i\omega - 2\pi ij)' . \tag{18}$$

The cross-equation rational expectations restrictions are apparent in that the parameterization C_2 occurs in both the spectral density matrix F_2 and in the cross spectral density F_1. The identification question is whether the function C_2 and the parameters ρ and α can be inferred from F_1 and F_2 using relations (17) and (18). Without imposing additional restrictions on C_2, the answer to this question would appear to be no. However, once we restrict the admissible parameterizations of C_2 to be rational in the way described by Hansen and Sargent (1983b), we can achieve identification.

To achieve identification, we impose the following additional assumption. Define $\lambda_0 = -\rho$.

Assumption 5: $C_2(s)$ is of the form

$$C_2(s) = \frac{G_0 + G_1 s + \ldots G_{q-1} s^{q-1}}{(s-\lambda_1)(s-\lambda_2) \ \ldots \ (s-\lambda_q)}$$

where $G_0, G_1, \ldots G_{q-1}$ are $(n-1) \times (n-1)$ real matrices. The zeros of det $G(s)$ have negative real parts and $G(\rho) + G(-\rho)$ is nonsingular where $G(s) = G_0 + G_1 s + \ldots + G_{q-1} s^{q-1}$.[16] In addition, we assume that $G(\lambda_j) \neq 0$ and the real part of λ_j is strictly negative for $j = 1, 2, \ldots, q$. Finally, for each j (including $j = 0$), $\lambda_j = \bar{\lambda}_k$ for some index k, and any two λ's with the same real part do not have imaginary parts that differ by integer multiples of $2\pi i$.

The λ's (for $j = 1, 2, \ldots, q$) are called the poles of $C_2(s)$.

With this specification for $C_2(s)$, the spectral density of z is known to have the form

$$f_2(\omega) = \sum_{j=1}^{q} \left[\frac{Q_j}{i\omega - \lambda_j} + \frac{Q_j'}{-i\omega - \lambda_j} \right] , \tag{19}$$

where

$$Q_j = \lim_{s \to \lambda_j} (s - \lambda_j) C_2(s) C_2(-s)' ,$$

f_2 is the spectral density matrix of z, and the prime denotes transposition but not conjugation. See Hansen and Sargent (1983b) or A.W. Phillips (1959) for the details of this construction. From (16) we can deduce that the cross spectral density matrix is rational.[17] In particular, let

$$h_1(s) = \frac{-u[sC_2(s) - \rho C_2(\rho)] C_2(-s)'}{2\alpha(s^2 - \rho^2)} .$$

Then the cross spectral density of z and K is given by $f_1(\omega) = h_1(i\omega)$. We form a partial fractions representation of h_1 to obtain

$$h_1(s) = \sum_{j=0}^{q} \left[\frac{\hat{Q}_j}{s - \lambda_j} + \frac{\tilde{Q}_j}{-s - \lambda_j} \right]$$

where

$$\hat{Q}_j = \lim_{s \to \lambda_j} (s - \lambda_j) h_1(s)$$

and

$$\tilde{Q}_j = - \lim_{s \to -\lambda_j} (s + \lambda_j) h_1(s) .$$

Note that for $j = 1, 2, \ldots, q$

$$\hat{Q}_j = \frac{-u\lambda_j Q_j}{2\alpha(\lambda_j - \rho^2)} , \tag{20}$$

and

$$\hat{Q}_0 = \frac{u[-\rho C_2(-\rho) - \rho C_2(\rho)]\, C_2(\rho)'}{4\alpha\rho} . \tag{21}$$

Since $\hat{Q}_0$ is different from zero, ρ can be identified from the discrete-time cross spectral density F_1. To identify α and the imaginary parts of the poles of $C_2(s)$, we make use of the fact that Q_j, $\hat{Q}_j$, and the real parts of the poles of $C_2(s)$ are identifiable from discrete time data and that (20) holds. The matrices Q_j and $\hat{Q}_j$ can be inferred from the discrete-time spectral density of z and the cross spectral density of K and z, respectively. The real parts of the poles of $C_2(s)$ can be inferred from the discrete time spectral density F_2 of z (see Phillips 1959 and Hansen and Sargent 1983b). Equation (20) is a restriction across the spectral density of z and the cross spectral density of K and z. Using (19) and (20) we see that the quantities

$$d_j = \frac{\lambda_j}{\alpha(\lambda_j^2 - \rho^2)} \tag{22}$$

are identified from discrete time statistics. Equation (21) is identical with equation (11) derived for the first-order Markov case. We summarize these results in

Proposition 2: Suppose Assumptions 4 and 5 are satisfied. If there is at least one real pole of $C_2(s)$, then the parameters α and β (or, equivalently, α and ρ) and the continuous time spectral density matrix of z are identifiable from discrete time observations.

If there fail to be any real poles of C_2, then all of the parameters can still be identified except possibly for some singular cases.

5. Conclusions

The two propositions proved in this paper indicate how the cross-equation restrictions of rational expectations models can serve to identify the parameters of a continuous time model from discrete time observations. The basic idea is that where decisions reflect forecasting in continuous time, the discrete time data on the decision variable and the forcing variables contain adequate clues to permit us to infer the parameters of the joint continuous time process of decision and forcing variables.

The basic identification mechanism promises to carry over to more complicated specifications than the two that are formally analyzed in

this paper. Extensions to our two specifications can be imagined in a variety of directions. These include

- Higher order Markov schemes for the forcing process $z(t)$ in our first setup.
- Higher order processes for the unobservable $y(t)$ in our first setup.
- Multiple interrelated decision variables.
- Higher order adjustment costs.

A formula expressing the cross-equation restrictions for a multiple decision variable problem that is highly suggestive of identification, though falling short of providing a formal proof, is reported in the appendix.

This paper is intended as a prologue to Hansen and Sargent (1980b) that describes methods for estimating continuous time linear rational expectations models that generalize the models analyzed in this present paper. While formal identification theorems are not yet available for those more general models, a method of checking for the presence of an aliasing identification problem is readily available in any particular application.[18]

Appendix

In this appendix we consider a multiple decision variable version of the quadratic optimization problem considered in Section 2. We let $K(t)$ be a p dimensional decision vector, $z_1(t)$ be a p dimensional vector of forcing variables that are observed by the econometrician, and $y(t)$ be a p dimensional vector of forcing variables that are not observed by the econometrician. We consider a firm that maximizes over strategies for $K(t)$ the criterion

$$E_0 \int_0^\infty J[K(t), DK(t), t, z_1(t), y(t)]dt$$

where

$$\begin{aligned} &J[K(t), DK(t), t, z_1(t), y(t)] \\ &\quad = \{y(t)'K(t) - K(t)'\beta K(t) - z_1(t)'DK(t) \\ &\quad - [DK(t)]'\alpha[DK(t)]\}e^{-rt} . \end{aligned}$$

Here α and β are $p \times p$ positive definite matrices. We assume that

$$y(t) = D\varepsilon_1^*(t)$$

and

$$Dz(t) = A_{22}z(t) + \varepsilon_2^*(t) \tag{23}$$

where $z_1(t)$ is a p dimensional subvector of the $n-p$ dimensional vector $z(t)$, the eigenvalues of A_{22} have negative real parts, and $\varepsilon^{*\prime} = [\varepsilon_1^{*\prime}, \varepsilon_2^{*\prime}]$ is an n dimensional vector white noise with intensity matrix V_0^*. We factor the characteristic polynomial of the Euler equation

$$-\alpha s^2 + (r^2/4)\alpha + \beta = [a - bs]'[a + bs]$$

where a and b are each $p \times p$ matrices such that the zeros of $\det(a - bs)$ lie in the right-half plane while the zeros of $\det(a + bs)$ lie in the left-half plane. This factorization is unique up to a premultiplication of a and b by a common orthogonal matrix.

Using results in Hansen and Sargent (1981d) and in Chapter 7, we find that the solution to the maximization problem of the firm is

$$\begin{aligned} DK(t) = &-[b^{-1}a - \frac{r}{2}I]K(t) + \frac{1}{2}\sum_{j=1}^{p}\Big\{N_j u[A_{22} - rI] \\ &[A_{22} - (s_j + \frac{r}{2})I]^{-1} z(t)\Big\} - \frac{1}{2}\sum_{j=1}^{p} N_j \varepsilon_1^*(t) \end{aligned} \tag{24}$$

where

$$\det[a'b - sb'b] = s_0(s - s_1)\ \dots\ (s - s_m)\,,$$

$$N_j = \frac{\operatorname{adj}[a'b - b'bs_j]}{s_0 \Pi_{i\neq j}(s_i - s_j)}\,,$$

and u is a $p \times (n-p)$ matrix of the form $u = [I, 0]$. We can write (23) and (24) as the joint first-order differential equation

$$Dx(t) = A_0\, x(t) + \varepsilon(t)$$

where

$$A_0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\begin{aligned} A_{11} &= -[b^{-1}a - \frac{r}{2}I] \\ A_{12} &= \frac{1}{2}\sum_{j=1}^{p} N_j u[A_{22} - rI]\,[A_{22} - (s_j + \frac{r}{2})I]^{-1} \\ A_{21} &= 0 \\ x(t) &= \begin{bmatrix} K(t) \\ z(t) \end{bmatrix} \qquad \varepsilon(t) = \begin{bmatrix} -\frac{1}{2}\sum_{j=1}^{p} N_j \varepsilon_1^*(t) \\ \varepsilon_2^*(t) \end{bmatrix}. \end{aligned} \tag{25}$$

As in the third section, we ask whether the matrix equation

$$\exp A^* = B_0 = \exp A_0$$

implies that $A^* = A_0$. Assume that the eigenvalues of A_0 are distinct and that they do not differ by integer multiples of $2\pi i$. Write the spectral decomposition of A_0 as $T\Lambda_0 T^{-1}$ where Λ_0 is the diagonal matrix of eigenvalues and T is a matrix whose columns are eigenvectors of A_0. Partition the matrices T and Λ_0 conformably with A_0 so that

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \qquad \Lambda_0 = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}.$$

It follows that

$$A_{12} = T_{12}\Lambda_2 T_{22}^{-1} - A_{11}T_{12}T_{22}^{-1}\,.$$

Restriction (25) implies that

$$T_{12}\Lambda_2 - A_{11}T_{12} = \frac{1}{2}\sum_{j=1}^{p} N_j u[A_{22} - rI]\,[A_{22} - (s_j + \frac{r}{2})I]^{-1}T_{22}\,. \tag{26}$$

Let vec(·) represent the vector formed by taking the direct sums of the rows of a matrix, and let $\otimes$ denote the Kronecker product. We solve (26) for T_{12} to obtain

$$\operatorname{vec} T_{12} = [(-A_{11} \otimes I) + (I \otimes \Lambda_2)]^{-1} \operatorname{vec} c \tag{27}$$

where

$$c = \frac{1}{2}\sum_{j=1}^{p} N_j u[A_{22} - rI]\,[A_{22} - (s_j + \frac{r}{2})I]^{-1}T_{22}\,.$$

From our discussion in the third section, we know that the eigenvector matrix T and the real parts of the eigenvalues in Λ_0 can be inferred from discrete time data. The imaginary parts of the complex eigenvalues can be perturbed by adding integer multiples of $2\pi i$ such that the complex conjugate pairs remain intact to generate alternative choices of A^* that satisfy

$$\exp A^* = B_0\,.$$

However, (27) restricts the class of *admissible perturbations* of the eigenvalues further so that it appears that in most circumstances A_0 is identifiable from discrete time data as are α and β.

Notes

1. This class of models includes continuous time, linear stochastic versions of the models discussed by Gould (1968), Lucas (1967), Mortensen (1973), and Treadway (1969). Geweke (1977a) used a model of this kind to motivate interpretations of some discrete time regressions.

2. Our discussion could be modified in a straightforward way to accommodate situations in which r is specified *a priori* but is different from zero. When r is set to zero, we have to interpret the decision rule we investigate as a limit of decisions rules as r declines to zero.

3. Given that ρ and α are identified, β can be inferred from the relation $\beta = \rho^2/\alpha$.

4. The assumption that y is the derivative of the white noise ε_1^* is contrived to imply that the decision rule has a white noise disturbance. In our discussion, the means of all of the random variables have been implicitly set to zero.

5. For an introduction to continuous time, linear stochastic processes, see Kwakernaak and Sivan (1972). A continuous time vector white noise $\varepsilon(t)$ is said to have intensity matrix V if $E\varepsilon(t)\varepsilon(t-\tau) = \delta(\tau)V$ where δ is the Dirac delta generalized function.

6. Phillips (1973) has also studied cross-equation linear restrictions.

7. See Kwakernaak and Sivan (1972), Coddington and Levinson (1955), and Gantmacher (1959) for the definition and properties of the matrix exponential function $\exp(A)$.

8. Hansen and Sargent (1983b) showed that there is extra identifying information about A_0 contained in the expression linking the covariance matrix of η to the intensity matrix of ε. In our discussion below, we supply sufficient conditions for identification that do not exploit this extra information.

9. See Coddington and Levinson (1955) or Gantmacher (1959).

10. For example, if there is only one complex conjugate pair of eigenvalues of A_{22} and no real eigenvalues, then it can be shown that the imaginary part of one of these eigenvalues has to satisfy a cubic equation. Unless the cubic equation has solutions that differ by an integer multiple of 2π, identification of the continuous time parameters is achieved. Thus, identification will only be a problem in singular cases. The existence of multiple pairs of complex conjugate eigenvalues of A_{22} will make identification even less likely to be a problem.

11. For discrete time models, Hatanaka (1975) treated the identification of structural parameters from the projections of the endogenous on the exogenous variables without using prior information about the orders of serial correlation of disturbance processes.

12. See Rozanov (1967) for a definition of the term linearly regular.

13. See Hansen and Sargent (1983b), for a fuller technical description of the setup being used here.

14. Here we have implicitly assumed that the decision rule of the firm has been employed forever.

15. Here we have implicitly assumed that z has a continuous time autoregressive representation. We do not need to make this assumption in what follows.

16. This is one of the setups used by Hansen and Sargent (1983b). They provide more technical details.

17. Although the spectral density of z and the cross spectral density of K and z are rational, the spectral density of K is not necessarily rational and is not necessarily identifiable from discrete time data.

18. The method involves calculating the poles of the estimated stochastic process of the forcing variables and constructing an observationally equivalent continuous time model by perturbing the complex eigenvalues by integer multiples of $2\pi i$. It can then be checked whether the implied continuous time model for the joint process of decision variables and forcing variables is observationally equivalent with the estimated model.